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ON THE CONSTRUCTION PROBLEM FOR SINGLE-EXIT MARKOV CHAINS

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I shall consider the following problem: given a stable, conservative, single-exit q-matrix, Q, over an irreducible state-space S and a μ-subinvariant measure, m, for Q, determine all Q-processes for which m is a μ-invariant measure. I shall provide necessary and sufficient conditions for the existence and uniqueness of such a process.

1. INTRODUCTION

The problem of constructing a Markov chain from its q-matrix of transition rates can be traced back to the work of Doob [4] in the late nineteen-forties. Since then, the problem has been considered by a number of authors. The major work was carried out in the fifties and early sixties by Feller ([5, 6]), Chung ([1, 2]), Reuter ([16, 17, 18]) and Williams ([24, 25]) (see also [3, 4, 10, 11, 12 and 20]). This work culminated in the solution, by Williams [25], of the classical construction problem formulated by Feller in [6]. The problem is as follows : given a stable, conservative q-matrix, $Q = (q_{ij}, i, j \in S)$, over a countable state-space S, construct all Q-processes, that is identify all standard, time-homogeneous, continuous-time Markov chains taking values in S, with transition rates Q. The Feller minimal process provides an example of one such process. But, it is the possibility that this process might explode by performing infinitely many jumps in a finite time that creates interest in the construction problem, for, as Doob [4] showed, certain simple rules for restarting the process after an explosion give rise to an infinity of Q-processes.

The Feller minimal process is the unique Q-process if and only if Q is regular, that is the equations

$$(1) \qquad \sum_{j \in S} q_{ij} x_j = \xi x_i, \qquad i \in S,$$

have no bounded, non-trivial solution (equivalently, non-negative solution), x, for some (and then for all) $\xi > 0$ ([16]). When this condition fails there are infinitely many

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Q-processes, including infinitely many honest ones ([16]), and the dimension, d, of the space of bounded vectors, x, on S satisfying (1) (a quantity which does not depend on ξ), determines the number of "escape routes to infinity" available to the process. Williams [25] was able to provide a construction of all Q-processes under the assumption that d is finite, following on from the work of Reuter ([17, 18]) who considered the single-exit case, $d = 1$.

If d is not assumed to be finite, little is known and the problem of finding all Q-processes appears to be very difficult, and remains unsolved. However, recently the problem has re-emerged and now attention is focused on finding *one* Q-process which satisfies a prescribed set of conditions. For example, it is of interest to know whether or not there exists an *honest* Q-process and, then, whether or not it is the *unique* honest Q-process. This question was first considered by Kendall [7] (see also Kendall and Reuter [8]) who used elegant but simple arguments based on the Hille-Yosida theorem from functional analysis. The most recent work centres on the assumption that one is given an *invariant measure* for the q-matrix. The problem is then to construct a process with m as its invariant measure. It has particular significance if $\sum m_i < \infty$, for then one is looking for a process, which of necessity is *honest*, whose *stationary distribution* has been specified in advance. In this paper I shall provide necessary and sufficient conditions for there to exist a *single-exit* process for which a given measure, m, is μ-invariant. Thus, although I shall deal with only a restricted class of processes, the invariance condition shall be weakened to μ-invariance. The important special case of when $\mu = 0$ is subsumed by the present study, although it was considered earlier in some detail (see [15]).

I hope that this work will provide some insight into how one should proceed in the more general setting, where the assumption that Q be a single-exit Q-matrix is relaxed. I shall begin by collecting together various results on continuous-time Markov chains.

2. Preliminaries

I shall refer to a set $P(\cdot) = (p_{ij}(\cdot), i, j \in S)$, of real-valued functions defined on $[0, \infty)$, where S is a countable set, as a *standard transition function* if

(2) $$p_{ij}(t) \geqslant 0, \qquad i, j \in S, t \geqslant 0,$$

(3) $$\sum_{j \in S} p_{ij}(t) \leqslant 1, \qquad i \in S, t \geqslant 0,$$

(4) $$p_{ij}(s + t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \qquad i, j \in S, s, t \geqslant 0,$$

(5) $$p_{ij}(0) = \delta_{ij} = \lim_{t \downarrow 0} p_{ij}(t), \qquad i, j \in S.$$

I shall refer to P as being *honest* if equality holds in (3) for all $i \in S$. Condition (5) guarantees that, for all $i,j \in S$, p_{ij} is uniformly continuous, as well as guaranteeing the existence of right-hand derivatives

$$q_{ij} = p'_{ij}(0) = \lim_{t\downarrow 0} \frac{p_{ij}(t) - \delta_{ij}}{t},$$

with the property that $\quad 0 \leqslant q_{ij} < \infty, \qquad j \neq i, i,j \in S,$

and $\quad \sum_{j\neq i} q_{ij} \leqslant -q_{ii} \leqslant \infty, \qquad i \in S,$

the set $Q = (q_{ij}, i,j \in S)$ being called a q-matrix.

Henceforth I shall suppose that Q is specified and I shall assume that Q is stable, that is

$$q_i := -q_{ii} < \infty, \qquad i \in S,$$

and conservative, that is $\quad \sum_{j\in S} q_{ij} = 0, \qquad i \in S.$

For simplicity, any standard transition function, P, that satisfies

$$p'_{ij}(0) = q_{ij}, \qquad i,j \in S,$$

will be called a Q-function. Under the conditions I have imposed, any Q-function, P, satisfies the backward differential equations,

$$p'_{ij}(t) = \sum_{k\in S} q_{ik} p_{kj}(t),$$

for all $i,j \in S$ and $t \geqslant 0$. The so-called Feller construction provides for the existence of a *minimal* solution, $F(\cdot) = (f_{ij}(\cdot), i,j \in S)$, to these equations, minimal in the sense that $f_{ij}(t) \leqslant p_{ij}(t)$ for all $t > 0$ and all $i,j \in S$, where $P(\cdot) = (p_{ij}(\cdot), i,j \in S)$ is any Q-function. F is also a Q-function and it satisfies the forward differential equations,

$$p'_{ij}(t) = \sum_{k\in S} p_{ik}(t) q_{kj},$$

for all $i,j \in S$ and $t \geqslant 0$.

3. The construction problem

As mentioned in the introduction, I shall restrict my attention to the case where Q is a single-exit q-matrix and so, henceforth, I shall suppose that the space of bounded, non-trivial, non-negative solutions to (1) has dimension 1. Under this condition, Reuter [17] identified all transition functions with a specified conservative q-matrix; for the non-conservative case see [18] and [26]. The problem is to determine for which of these transition functions is a specified measure μ-invariant. In particular, I shall suppose that $m = (m_j,\ j \in S)$ is a specified μ-subinvariant measure for Q, that is a collection of strictly positive numbers which satisfy

$$\sum_{i \in S} m_i q_{ij} \leqslant -\mu m_j, \qquad j \in S.$$

For simplicity, I shall suppose that S is irreducible for the minimal process, and, hence, for any other Q-process. For a μ-subinvariant measure to exist, one must have that $0 \leqslant \mu \leqslant \lambda_F$, where λ_F is the decay parameter of S for F, the minimal Q-function (see [23] and [14]). The main result of the paper establishes necessary and sufficient conditions for there to exist a unique Q-function, P, such that m is μ-invariant for P, that is

$$\sum_{i \in S} m_i p_{ij}(t) = e^{-\mu t} m_j,$$

for all $j \in S$ and $t \geqslant 0$; note that m is said to be μ-subinvariant for P if

$$\sum_{i \in S} m_i p_{ij}(t) \leqslant e^{-\mu t} m_j,$$

for all $j \in S$ and $t \geqslant 0$.

It will be convenient to present my results using Laplace transforms. Let P be an arbitrary standard transition function and define the *resolvent*, $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$, of P by

$$\psi_{ij}(\alpha) = \int_0^\infty e^{-\alpha t} p_{ij}(t)\,dt, \qquad i,j \in S; \tag{6}$$

this integral converges for all $\alpha > -\lambda_P$, where λ_P is the decay parameter of S for P (see [9]). Analogous to (2)–(5), Ψ satisfies

$$\psi_{ij}(\alpha) \geqslant 0, \qquad i,j \in S, \alpha > 0, \tag{7}$$

$$\sum_{j \in S} \alpha \psi_{ij}(\alpha) \leqslant 1, \qquad i \in S, \alpha > 0, \tag{8}$$

(9) the "resolvent equation"

$$\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in S} \psi_{ik}(\alpha) \psi_{kj}(\beta) = 0, \qquad i,j \in S, \alpha, \beta > 0,$$

$$\lim_{\alpha \to \infty} \alpha \psi_{ij}(\alpha) = \delta_{ij}, \qquad i,j \in S, \tag{10}$$

and, any Ψ which satisfies (7)–(10) is the resolvent of a standard transition function; for an elegant proof of this characterisation see [**17**] (see also [**19**]). Thus, there is a one-to-one correspondence between resolvents and standard transition functions. Further, (8) is satisfied with equality for all $i \in S$ and $\alpha > 0$ if and only if P is honest, in which case the *resolvent* is said to be honest. The q-matrix of P can be recovered from Ψ using the following identity:

$$q_{ij} = \lim_{\alpha \to \infty} \alpha(\alpha \psi_{ij}(\alpha) - \delta_{ij}). \tag{11}$$

And, a resolvent that satisfies (11) is called a Q-resolvent. Explicit analogues of the backward and the forward equations will not be needed here. It will suffice to note that there is a one-to-one correspondence between Q-resolvents and Q-functions and that the resolvent, $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$, of the minimal Q-function, F, has itself a minimal interpretation (see [**16**] and [**17**]); for this reason Φ is called the minimal Q-resolvent.

The following result summarises Reuter's [**17**] construction:

THEOREM 1. *If Q is a stable, conservative, single-exit q-matrix and if Ψ is the resolvent of an arbitrary Q-function, P, then either $\Psi = \Phi$, the minimal Q-resolvent, or otherwise Ψ must be of the form*

$$\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + z_i(\alpha) y_j(\alpha), \qquad i, j \in S, \alpha > 0, \tag{12}$$

where

$$z_i(\alpha) = 1 - \sum_{j \in S} \alpha \phi_{ij}(\alpha), \qquad i \in S, \alpha > 0.$$

The quantity $y(\alpha) = (y_j(\alpha), j \in S)$ must be of the form

$$y_j(\alpha) = \frac{\eta_j(\alpha)}{c + \sum_{k \in S} \alpha \eta_k(\alpha)}, \qquad j \in S, \alpha > 0, \tag{13}$$

where $c \geqslant 0$ and $\eta(\alpha) = (\eta_j(\alpha), j \in S)$ is a non-negative vector that satisfies

$$\sum_{k \in S} \eta_k(\alpha) < \infty, \qquad \alpha > 0, \tag{14}$$

and

$$\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in S} \eta_k(\alpha) \phi_{kj}(\beta) = 0, \qquad j \in S, \alpha, \beta > 0. \tag{15}$$

Ψ is honest if and only if $c = 0$.

REMARKS: The theorem states that the resolvents of all processes with q-matrix Q must be of the form (12). Indeed, once η is specified, a family of Q-processes (exactly one of which is honest) is obtained by varying c. Thus, the problem of determining those Q-processes which satisfy a specified criterion amounts to determining which choices of η and c are admissible.

Expression (12) specifies $\Psi(\alpha)$ for all $\alpha > 0$. However, the expression is valid for all α in the domain of Ψ, namely $\alpha > -\lambda_P$.

In order to identify which Q-functions have a given μ-invariant measure, it will be necessary to explain how μ-invariant and μ-subinvariant measures can be identified using resolvents. If P is an arbitrary Q-function with resolvent Ψ and $m = (m_j, j \in S)$ is a μ-subinvariant measure for P, where of necessity $\mu \leqslant \lambda_P$ (see Lemma 4.1 of [22]), then, since the integral (6) converges for all $\alpha > -\lambda_P$, we have that, for all j in S and $\alpha > 0$,

$$\sum_{i \in S} m_i \alpha \psi_{ij}(\alpha - \mu) \leqslant m_j, \tag{16}$$

with equality for all j and α if m is μ-invariant for P. One may, therefore, refer to m as being μ-subinvariant for Ψ if (16) is satisfied and μ-invariant if it is satisfied with equality. The following result establishes a characterisation of μ-invariance and μ-subinvariance for P in terms of Ψ.

LEMMA 1. *Let m be a measure on S and let P be a standard transition function with resolvent Ψ. Then, if m is μ-subinvariant for P, it is μ-subinvariant for Ψ and strictly μ-invariant for Ψ if it is μ-invariant for P. Conversely, if $\mu \leqslant \lambda_P$ and m is μ-subinvariant for Ψ, then m is μ-subinvariant for P and strictly μ-invariant for P if it is μ-invariant for Ψ.*

PROOF: We need only show that the μ-subinvariance and, then, μ-invariance of m for Ψ implies that the same is true for P. So, suppose that m is μ-subinvariant for Ψ, where $\mu \leqslant \lambda_P$, and define Ψ^* by

$$\psi^*_{ij}(\alpha) = \frac{m_j \psi_{ji}(\alpha - \mu)}{m_i}, \qquad i, j \in S, \alpha > 0.$$

Then, it is easy to verify that Ψ^* satisfies (7)–(10). Condition (10) is immediate. Conditions (7) and (9) hold because it is clear, from the definition of Ψ, that Ψ satisfies (7) for all $\alpha > -\lambda_P$ and (9) for all $\alpha, \beta > -\lambda_P$. And, Condition (8) is satisfied by virtue of (16). Thus, Ψ^* is the resolvent of a unique (standard) transition function, P^*. Now define $\widetilde{P}(\cdot) = (\widetilde{p}_{ij}(\cdot), i, j \in S)$ by

$$\widetilde{p}_{ij}(t) = e^{\mu t} \frac{m_j p_{ji}(t)}{m_i}, \qquad i, j \in S, t \geqslant 0,$$

and $\widetilde{\Psi}(\cdot) = \left(\widetilde{\psi}_{ij}(\cdot), i,j \in S\right)$ by

$$\widetilde{\psi}_{ij}(\alpha) = \int_0^\infty e^{-\alpha t}\widetilde{p}_{ij}(t)dt, \qquad i,j \in S, \alpha > 0.$$

Then, for all $i,j \in S$ and $\alpha > 0$,

$$\begin{aligned}\widetilde{\psi}_{ij}(\alpha) &= \int_0^\infty e^{-(\alpha-\mu)t}\frac{m_j p_{ji}(t)}{m_i}dt \\ &= \frac{m_j \psi_{ji}(\alpha-\mu)}{m_i} \\ &= \psi^*_{ij}(\alpha).\end{aligned}$$

Thus $\widetilde{\Psi} = \Psi^*$, and hence, from Reuter's characterisation, $\widetilde{P} = P^*$. Since P^* satisfies (3), it follows immediately that m is μ-subinvariant for P. Further, we see that m is μ-invariant for P if and only if P^* is honest. Thus, if m is μ-invariant for Ψ, then Ψ^* is honest and so the ensuing honesty of P^* implies that m is μ-invariant for P. □

I shall now suppose that m is a prescribed μ-subinvariant measure for Q and then, using Theorem 1, I shall determine for which Q-functions, P, other than F, can m be a μ-invariant measure; notice that if m is μ-invariant for a Q-function, then, by the minimality of F, it is μ-subinvariant for F and so, by Proposition 1 of [**21**], it *must* be μ-subinvariant for Q.

THEOREM 2. *Let Q be a stable, conservative, single-exit q-matrix over an irreducible state-space, S, and suppose that m is a μ-subinvariant measure on S for Q. Let $\Phi(\cdot) = (\phi_{ij}(\cdot), i,j \in S)$ be the resolvent of F, the minimal Q-function. Define $z(\cdot) = (z_i(\cdot), i \in S)$ by*

$$z_i(\alpha) = 1 - \sum_{j\in S}\alpha\phi_{ij}(\alpha), \qquad i \in S, \alpha > -\lambda_F,$$

and $d(\cdot) = (d_i(\cdot), i \in S)$ by

$$d_i(\alpha) = m_i - \sum_{j\in S} m_j(\alpha+\mu)\phi_{ji}(\alpha), \qquad i \in S, \alpha > -\mu. \tag{17}$$

Then there exists a Q-function, P, for which m is μ-invariant if and only if $d = 0$ or, otherwise,

$$\left(\frac{\alpha}{\alpha+\mu}\right)\sum_{i\in S} d_i(\alpha) \leqslant \sum_{i\in S} m_i z_i(\alpha) < \infty, \tag{18}$$

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for all $\alpha > -\mu$. *When such a* Q*-function exists it is unique and its resolvent,* $\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$, *is given by*

$$\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{z_i(\alpha)d_j(\alpha)}{(\alpha+\mu)\sum_{k\in S} m_k z_k(\alpha)}, \qquad i,j \in S. \tag{19}$$

It is then the unique honest Q*-function for which* m *is* μ*-invariant if and only if*

$$\left(\frac{\alpha}{\alpha+\mu}\right)\sum_{i\in S} d_i(\alpha) = \sum_{i\in S} m_i z_i(\alpha), \tag{20}$$

for all $\alpha > -\mu$.

REMARK: The condition $d = 0$ is essentially known (see [**21**] and [**13**]). If $d \neq 0$ then $d(\cdot) = (d_i(\cdot), i \in S)$ gives the deficit in the μ-subinvariance of m for Φ; notice that if m *is* μ-invariant for P then, by the minimality of F, it must be *strictly* μ-subinvariant for F and, hence, for Φ, and so $d_i(\alpha) > 0$ for all i and for all $\alpha > -\mu$.

PROOF: First observe that, since m is μ-subinvariant for Q, Proposition 2 of [**21**] implies that m is μ-subinvariant for F and so, by Lemma 1, it is μ-subinvariant for Φ. Thus $d_i(\alpha) \geqslant 0$ for all $i \in S$ and $\alpha > -\mu$. Further, since m is μ-subinvariant for F, it follows, from Lemma 4.1 of [**22**], that $\mu \leqslant \lambda_F$.

Let P be an arbitrary Q-function with resolvent Ψ specified by Theorem 1. I shall show that the stated condition is necessary for m to be μ-invariant for P. So, suppose that m is μ-invariant for P and, hence, for Ψ. If $P = F$, then m is μ-invariant for Φ and it follows immediately that $d = 0$. To deal with the case $P \neq F$, first observe that, by the minimality of F, m cannot be μ-invariant for F, and so $d_i(\alpha) > 0$ for all i and α. Too, neither z nor y in (12) is identically zero. If $\alpha > 0$ then $\alpha - \mu$ lies in the domain of Ψ since, of necessity, $\mu \leqslant \lambda_P$. Thus, on substituting $\alpha - \mu$ for α in (12), multiplying by αm_i and, then, summing over $i \in S$, we find that

$$\sum_{i\in S} m_i z_i(\alpha) < \infty,$$

for all $\alpha > -\mu$, and, further, that

$$m_j = \sum_{i\in S} m_i \alpha \phi_{ij}(\alpha-\mu) + \alpha y_j(\alpha-\mu)\sum_{i\in S} m_i z_i(\alpha-\mu),$$

for all $\alpha > 0$. Hence, in view of (13), we require

$$\frac{(\alpha+\mu)\eta_j(\alpha)}{c + \sum_{k\in S} \alpha\eta_k(\alpha)} = \frac{d_j(\alpha)}{\sum_{i\in S} m_i z_i(\alpha)} \tag{21}$$

for all $\alpha > -\mu$. But, since (14) must hold and because we require $c \geqslant 0$, we must have that

$$\left(\frac{\alpha}{\alpha+\mu}\right)\sum_{i\in S} d_i(\alpha) \leqslant \sum_{i\in S} m_i z_i(\alpha),$$

for all $\alpha > -\mu$. Thus, (18) is necessary for m to be μ-invariant for P when $P \neq F$.

Conversely, if $d = 0$ then m is μ-invariant for Φ and so, on recalling that $\mu \leqslant \lambda_F$, it follows, from Lemma 1, that m is μ-invariant for F; by the minimality of F, m is μ-invariant for no other Q-function. If $d \neq 0$ and (18) holds, then, in order to construct the resolvent of a Q-function, P, for which m is μ-invariant, define η by

$$\eta_j(\alpha) = d_j(\alpha), \qquad j \in S, \alpha > -\mu.$$

Clearly (14) is satisfied and, using the resolvent equation for Φ, it is easy to show that (15) holds. Thus, in order to specify a Q-resolvent, it remains only to determine a value of c so as to be consistent with (13). This can be done as follows:

Using the resolvent equation for Φ, it is easy to show that z and d satisfy

$$z_i(\alpha) - z_i(\beta) + (\alpha-\beta)\sum_{k\in S}\phi_{ik}(\alpha)z_k(\beta) = 0$$

for all $\alpha, \beta > -\lambda_F$, and

$$d_i(\alpha) - d_i(\beta) + (\alpha-\beta)\sum_{k\in S} d_k(\alpha)\phi_{ki}(\beta) = 0,$$

for all $\alpha, \beta > -\mu$. On multiplying the first equation by m_i and summing over i, we find that

$$(\alpha+\mu)\sum_{i\in S} m_i z_i(\alpha) - (\beta+\mu)\sum_{i\in S} m_i z_i(\beta) = (\alpha-\beta)\sum_{i\in S} d_i(\alpha) z_i(\beta),$$

for all $\alpha, \beta > -\mu$. Similarly, summing the second equation over i gives

$$\alpha\sum_{i\in S} d_i(\alpha) - \beta\sum_{i\in S} d_i(\beta) = (\alpha-\beta)\sum_{i\in S} d_i(\alpha) z_i(\beta), \qquad \alpha, \beta > -\mu.$$

Thus

$$(\alpha+\mu)\sum_{i\in S} m_i z_i(\alpha) - \alpha\sum_{i\in S} d_i(\alpha) = (\beta+\mu)\sum_{i\in S} m_i z_i(\beta) - \beta\sum_{i\in S} d_i(\beta), \qquad \alpha, \beta > -\mu,$$

and so, if the sums converge, then

$$(\alpha+\mu)\sum_{i\in S} m_i z_i(\alpha) - \alpha\sum_{i\in S} d_i(\alpha)$$

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is the same for all $\alpha > -\mu$. Thus, since (18) is satisfied, we may set c equal to this quantity and then arrive at the specification (19) of a Q-resolvent which is valid for all $\alpha > -\mu$. Multiplying (19) by $(\alpha+\mu)m_i$ and summing over i shows that m is μ-invariant for Ψ. Now, as the domain of Ψ must contain (μ, ∞) it follows that $\mu \leqslant \lambda_P$, where λ_P is the decay parameter of P, and, hence, that m is μ-invariant for P. To see that P is the *unique* Q-function for which m is μ-invariant, observe that if m is to be μ-invariant for an *arbitrary* Q-resolvent, $\widehat{\Psi}$, then, in view of (21), we must have (in an obvious notation) that $\widehat{\eta} = Kd$ for some positive scalar function K. Now, on substituting $\widehat{\eta}$ into (21) we find (again, using an obvious notation) that $K(\alpha)c = \widehat{c}$ for all α. Thus K is constant, and, moreover,

$$\frac{\widehat{\eta}_j(\alpha)}{\widehat{c} + \sum_{k\in S} \alpha\widehat{\eta}_k(\alpha)} = \frac{d_j(\alpha)}{(\alpha+\mu)\sum\limits_{i\in S} m_i z_i(\alpha)}.$$

Thus, Ψ is the unique Q-resolvent for which m is μ-invariant.

Finally, the condition for the existence of a unique *honest* Q-function follows on observing that Ψ is honest if and only if $c = 0$. □

I shall complete this section by looking at the important special case where m can be normalised to produce a probability distribution over S; under certain conditions m can then be interpreted as a quasistationary distribution (see, for example, [23]).

COROLLARY 1. *A sufficient condition for the existence of a unique Q-function for which m is μ-invariant is that*

$$\sum_{i\in S} m_i < \infty.$$

It is honest if and only if $\mu = 0$.

PROOF: First observe that, since $z_i(\alpha) \leqslant 1$, we have that

$$\sum_{i\in S} m_i z_i(\alpha) < \infty,$$

for all $\alpha > -\mu$. On summing over i in (17) we find that (18) is satisfied and, in particular, that

$$\left(\frac{\alpha}{\alpha+\mu}\right)\sum_{i\in S} d_i(\alpha) = \sum_{i\in S} m_i z_i(\alpha) - \left(\frac{\mu}{\alpha+\mu}\right)\sum_{i\in S} m_i,$$

for all $\alpha > -\mu$. Finally, (20) holds if and only if $\mu = 0$. □

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