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ON THE TOTAL TORSION OF CERTAIN NON-CLOSED SPHERE CURVES

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In this note, we establish some results concerning the total torsion and the total absolute torsion of certain non-closed stereographically projected analytic curves. The method of proof involves only elementary techniques of integration, a periodicity argument and Liouville's Theorem.

1. Introduction

Many authors [3, 4, 5] have considered the total torsion of closed curves on the unit sphere. It is well-known that the total torsion of a closed unit speed sphere curve is zero. (See [2, p.170], for example.) More generally, we have the following result:

THEOREM. *(Santaló, [4]) Let* κ *be the curvature and* τ *the torsion of a closed curve* C *of class* C^3 *. Let* s *be the arc length of* C *and* $f(\kappa,\tau)$ *a function of* κ *and* τ *. If the relation*

$$* \qquad \int_C f(\kappa,\tau)\, ds = 0$$

holds for every closed spherical curve C *, then* $f = \phi(\kappa)\tau$*, where* $\phi(\kappa)$

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is an arbitrary function of κ . *Conversely, if the function* $f(\kappa,\tau)$ *has the form* $f = \phi(\kappa)\tau$ *, then* * *holds for every closed spherical curve such that* $\kappa^{-1} \neq 0$ *at every point.*

Also other authors [1, 6] have considered the total absolute torsion of closed sphere curves.

In this note, we shall prove analogous results for certain non-closed stereographically projected analytic arcs on the unit sphere. For such an arc γ , we shall show that

$$\int_{\gamma} \kappa^{z}\tau \, ds = 0$$

for all complex z whenever

$$\int_{\gamma} \tau \, ds = 0 \ .$$

It is interesting to note that in establishing this result, the method of proof differs considerably from the usual methods. All that will be required are elementary integration techniques, a periodicity argument and Liouville's Theorem. We shall also easily deduce a total absolute torsion result for non-closed sphere curves.

2. Background Information

Let $C(R_1;R_2)$ denote the class of conformal mappings of the annulus $A(R_1;R_2) = \{z:R_1 < |z| < R_2\}$, and let $f \in C(R_1;R_2)$. Let Π denote the stereographic projection of the image plane of f onto S^2 , the unit sphere in $\mathbb{R}^3$. For fixed r, $R_1 < r < R_2$, we let $C_r = \{z:|z| = r\}$, $C_r' = f(C_r)$ and $C_r'' = \Pi(C_r')$. For fixed θ, $-\pi < \theta \le \pi$, we let $L_\theta = \{z:\arg z = \theta\}$, $L_\theta' = f(L_\theta)$ and $L_\theta'' = \Pi(L_\theta')$. A parameterization of the curves C_r'' and L_θ'' is easily prescribed. Indeed, if we write

$$f(z) = f(r,\theta) = (u(r,\theta),\ v(r,\theta)) \ , \qquad (z = re^{i\theta})$$

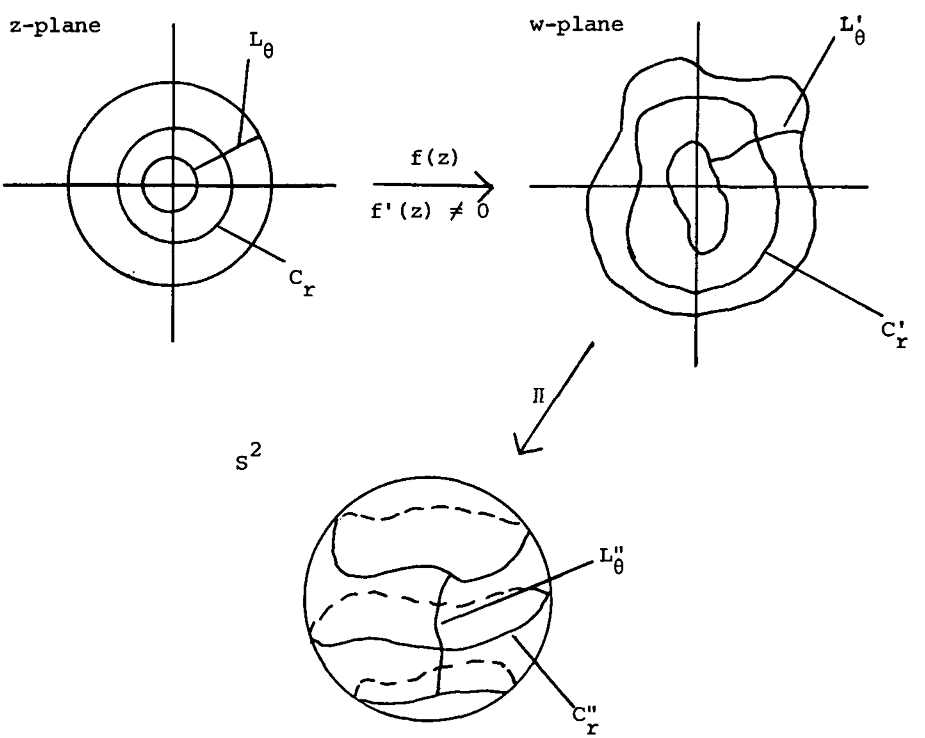
then we have

$$C''_r = \{X(r,\theta) : -\pi < \theta \le \pi\}$$

and
$$L''_\theta = \{X(r,\theta) : R_1 < r < R_2\} ,$$

where

(1) $$X(r,\theta) = \left(\frac{2u(r,\theta)}{1 + |f|^2} , \frac{2v(r,\theta)}{1 + |f|^2} , \frac{|f|^2 - 1}{1 + |f|^2} \right) .$$



In all results to follow, the quantity

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the <u>spherical derivative of $f(z)$</u> , and the quantity

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

denotes the <u>Schwarzian derivative of $f(z)$</u> .

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PROPOSITION. *Let* $f \in C(R_1;R_2)$. *Then, at the point* $\Pi(f(r,\theta))$ *on the curve* C''_r , *the curvature* $\kappa_r(f;\theta)$ *is given by*

$$(2) \qquad \kappa_r(f;\theta) = \left\{1 + \left[\frac{1}{2}\, r\, \frac{\partial}{\partial r}\left(\frac{1}{rf^{\#}}\right)\right]^2\right\}^{1/2}$$

and the torsion $\tau_r(f;\theta)$ *may be determined from the relationship*

$$(3) \qquad 4r^2 f^{\#2} \kappa_r^2(f;\theta)\,\tau_r(f;\theta) = Im[z^2\{f,z\}] .$$

At the point $\Pi(f(r,\theta))$ *on the curve* L''_θ , *the curvature* $\kappa_\theta(f;r)$ *is given by*

$$(4) \qquad \kappa_\theta(f;r) = \left\{1 + \left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\left(\frac{1}{rf^{\#}}\right)\right]^2\right\}^{1/2}$$

and the torsion $\tau_\theta(f;r)$ *may be determined from the relationship*

$$(5) \qquad 4r^2 f^{\#2} \kappa_\theta^2(f;r)\,\tau_\theta(f;r) = -\, Im[z^2\{f,z\}] .$$

Furthermore, since

$$(6) \qquad r\,\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{rf^{\#}}\right) = \frac{1}{rf^{\#}}\, Im\, [z^2\{f,z\}] ,$$

we may write

$$\tau_r(f;\theta) = \frac{r\,\dfrac{\partial^2}{\partial r\partial\theta}\left(\dfrac{1}{rf^{\#}}\right)}{4rf^{\#}\kappa_r^2(f;\theta)} \quad \text{and} \quad \tau_\theta(f;r) = -\,\frac{r\,\dfrac{\partial^2}{\partial r\partial\theta}\left(\dfrac{1}{rf^{\#}}\right)}{4rf^{\#}\kappa_\theta^2(f;r)}$$

The proof of this proposition consists of lengthy but straightforward elementary computations and will appear elsewhere. We wish to note in passing that (6) establishes an explicit relationship between the spherical and Schwarzian derivatives. We also wish to point out that relations (3) and (5) establish an interesting connection between these classical derivatives and the geometrical quantities under consideration.

3. Results

We first introduce the auxiliary function $\alpha(f;r,\theta)$ defined by the relationship

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$$\text{Tan}\ \alpha(f;r,\theta) = \frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{1}{rf^{\#}}\right), \tag{7}$$

where we require that $-\pi/2 < \alpha(f;r,\theta) < \pi/2$.

LEMMA. *Let $f \in C(R_1;R_2)$ and let $\{T_\theta,N_\theta,B_\theta,\kappa_\theta,\tau_\theta\}$ be the Frenet-Serret apparatus of the curve L''_θ . If $\psi_{B_\theta X}(f;r,\theta)$ [respectively $\psi_{N_\theta X}(f;r,\theta)$] denotes the angle between the binormal B_θ [respectively normal N_θ] and the position vector $X(r,\theta)$ given by (1), then*

$$\text{Sin}\ \alpha(f;r,\theta) = -\ \text{Cos}\ \psi_{B_\theta X}(f;r,\theta)\ , \tag{8}$$

$$\text{Cos}\ \alpha(f;r,\theta) = -\ \text{Cos}\ \psi_{N_\theta X}(f;r,\theta)\ , \tag{9}$$

$$\text{Sec}\ \alpha(f;r,\theta) = \kappa_\theta(f;r)\ , \tag{10}$$

and

$$\left|\frac{\partial}{\partial\theta}\psi_{B_\theta X}(f;r,\theta)\right| = \left|\frac{\partial}{\partial\theta}\psi_{N_\theta X}(f;r,\theta)\right| = \left|\frac{\partial}{\partial\theta}\alpha(f;r,\theta)\right| . \tag{11}$$

Proof. Equations (8) and (9) follow from direct computation, since $\text{Cos}\ \psi_{B_\theta X} = (B_\theta\cdot X)$ and $\text{Cos}\ \psi_{N_\theta X} = (N_\theta\cdot X)$. Equation (10) follows directly from (4) and (7). In (11), the first equality holds since $\psi_{B_\theta X}$ and $\psi_{N_\theta X}$ differ by a constant. Equation (9) implies that $|\text{Sin}\ \psi_{N_\theta X}| = |\text{Sin}\ \alpha|$; thus, differentiating (9) yields the second equality in (11).

We now consider the total absolute torsion of the curves L''_θ . Since these curves are not closed, we shall integrate over closed sub-segments in order to guarantee the existence of the integrals involved.

THEOREM 1. *Let $f \in C(R_1;R_2)$ and suppose that $R_1 < \rho_1 < \rho_2 < R_2$. Then, for each $\theta \in (-\pi,+\pi]$, we have*

$$\int_{\rho_1}^{\rho_2} |\tau_\theta(f;r)|\ \ ds = V[\psi_{B_\theta X}(f;r,\theta);\rho_1,\rho_2]\ , \tag{12}$$

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where $ds = ||X_r||\, dr = 2f^{\#}\, dr$ *is the arc-length differential along the curve* L''_θ *and* $V[\cdot]$ *denotes the total absolute variation of the angle* $\psi_{B_\theta X}$.

Proof. By definition

$$\int_{\rho_1}^{\rho_2} |\tau_\theta(f;r)|\, ds = \int_{\rho_1}^{\rho_2} 2f^{\#} |\tau_\theta(f;r)|\, dr \,.$$

We shall show that

$$\text{(13)} \qquad \frac{\partial}{\partial r}\alpha(f;r,\theta) = -2f^{\#}\tau_\theta(f;r)\,.$$

This relation, together with (11), will yield the result. To establish (13), we first recall (10) and then observe that, on one hand, we have

$$\text{(14)} \qquad \frac{\partial}{\partial r}\kappa_\theta(f;r) = \operatorname{Sec}\alpha(f;r,\theta)\operatorname{Tan}\alpha(f;r,\theta)\,\frac{\partial}{\partial r}\alpha(f;r,\theta)\,.$$

On the other hand, from (4), (7), (10), (6) and (5), we get

$$\text{(15)} \qquad \frac{\partial}{\partial r}\kappa_\theta(f;r) = -\,2f^{\#}\tau_\theta(f;r)\operatorname{Sec}\alpha(f;r,\theta)\operatorname{Tan}\alpha(f;r,\theta)\,.$$

Comparing (14) with (15) establishes (13) and thereby completes the proof of the theorem.

We come to the main result of this paper. It states that if the total torsion over a certain analytic arc is equal to zero, then the total "curvature-weighted" torsion is also equal to zero on that arc.

THEOREM 2. *Let* $f \in C(R_1;R_2)$. *Fix* θ *and suppose that* $R_1 < \rho_1 < \rho_2 < R_2$. *Define the entire function*

$$I(z) = I_f(z;\theta,\rho_1,\rho_2) = \int_{\rho_1}^{\rho_2} \kappa_\theta^z(f;r)\,\tau_\theta(f;r)\, ds\,.$$

If $I(0) = 0$, *then* $I(z) \equiv 0$ *for every complex* z.

Proof. Since $\kappa_\theta(f;r) \geq 1$, $\kappa_\theta^z(f;r)$ is single-valued. Hence, $I(z)$ is clearly entire.

Now let $\ell_x = \{x + it: -\infty < t < +\infty\}$. We first show that $I(z)$ is bounded on ℓ_x . Indeed, we have

(16) $$|I(x + it)| \le \left(\max_{r \in [\rho_1, \rho_2]} \kappa_\theta^x(f;r) \right) V[\psi_{g_\theta} \chi; \rho_1, \rho_2] ,$$

by using Theorem 1. Let $b(x)$ denote the right-hand side of (16). We have shown that $|I(x + it)| \le b(x)$ for each real x ; that is, that I is bounded on ℓ_x independently of t .

Next, we show that $I(z)$ satisfies the relation

(17) $$(z - 1)I(z) = (z - 2)I(z - 2) .$$

In view of (10) and (13), we may write, for $z \ne 1$,

$$I(z) = \int_{\rho_1}^{\rho_2} Sec^z \alpha(f;r,\theta) \frac{\partial}{\partial r} \alpha(f;r,\theta)\, dr.$$

$$= \frac{Sec^{z-2}\alpha(f;r,\theta) \operatorname{Tan} \alpha(f;r,\theta)}{z - 1} \Bigg|_{\rho_1}^{\rho_2}$$

$$+ \frac{z - 2}{z - 1} \int_{\rho_1}^{\rho_2} Sec^{z-2}\alpha(f;r,\theta) \frac{\partial}{\partial r} \alpha(f;r,\theta)\, dr,$$

by employing any standard table of integrals.

Using the hypothesis and equation (13), we get

$$I(0) = \int_{\rho_1}^{\rho_2} 2f^{\#}\tau_\theta(f;r)\, dr$$

$$= \int_{\rho_1}^{\rho_2} - \frac{\partial}{\partial r} \alpha(f\ r,\theta)\, dr$$

$$= \alpha(f;\rho_1,\theta) - \alpha(f;\rho_2,\theta)$$

$$= 0$$

Hence, the first term on the right-hand side of (18) vanishes and we obtain (17) for . This relation extends continuously to since

$I(z)$ is entire.

Replacing z by $z + 2$ in (17), we get

$$(z + 1)I(z + 2) = zI(z); \tag{19}$$

replacing z by $-z + 1$ in (17), we get

$$-zI(-z + 1) = (-z - 1)I(-z - 1) . \tag{20}$$

Multiplying (19) and (20), we obtain the relation

$$I(z + 2)I((-z - 1) + 2) = I(z)I(-z -1)$$

which shows that the function

$$J(z) = I(z)I(-z - 1)$$

is an entire, periodic function with period 2. Furthermore, $J(z)$ is bounded on each line ℓ_x, since $|J(x + it)| \leq b(x)b(-x - 1)$. Since $b(x)$ is continuous on the real axis, there must exist an absolute constant β such that $|J(x + it)| \leq \beta$ for all $x \in [0,2]$. Thus $|J(z)| \leq \beta$ for all z in the infinite strip $\{x + it: 0 \leq x \leq 2, -\infty < t < +\infty\}$. Since J has period 2, $|J(z)| \leq \beta$ in the plane. By Liouville's Theorem, J must be constant. But since $I(0) = 0$, we must have $J(z) \equiv 0$, and this readily implies that $I(z) \equiv 0$.

We state the next two theorems for the sake of completeness. The proofs are omitted since they are completely analogous to the proofs of the two previous theorems.

THEOREM 3. *Let* $f \in C(R_1;R_2)$. *Then, for each* $r \in (R_1,R_2)$,

$$\int_{C''_r} |\tau_r(f;\theta)|\, ds = V[\phi_{B_rX}(f;r,\theta);-\pi,+\pi]$$

where $ds = ||X_\theta||\, d\theta = 2rf^{\#}\, d\theta$ *is the arc-length differential along the curve* C''_r *and* $V[\cdot]$ *is the total absolute variation of the angle* ϕ_{B_rX} *between the position vector* X *and the binormal* B_r *to the curve* C''_r.

THEOREM 4. *Let* $f \in C(R_1;R_2)$. *Then, for each* $r \in (R_1,R_2)$,

$$I_f(z;r) = \int_{C''_r} \kappa_r^z(f;\theta)\tau_r(f;\theta)\, ds = 0$$

for every complex z.

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