Divisibility properties for Fibonacci and related numbers

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Introduction

Although it is an old one, the fascinating world of Fibonacci numbers and Lucas numbers continues to provide rich areas of investigation for professional and amateur mathematicians. We revisit divisibility properties for those numbers along with the closely related Pell numbers and Pell-Lucas numbers by providing a unified approach for our investigation.

For non-negative integers \( n \), the recurrence relation defined by

\[
X_{n+2} = cX_{n+1} + X_n
\]

with initial conditions

\[
x_0 = a \quad \text{and} \quad x_1 = b
\]

can be used to study the Pell \((P_n)\), Fibonacci \((F_n)\), Lucas \((L_n)\), and Pell-Lucas \((Q_n)\) numbers in a unified way. In particular, if \( a = 0, b = 1 \) and \( c = 1 \), then (1) defines the Fibonacci numbers \( x_n = F_n \). If \( a = 2, b = 1 \) and \( c = 1 \), then \( x_n = L_n \). If \( a = 0, b = 1 \) and \( c = 2 \), then \( x_n = P_n \). If \( a = b = c = 2 \), then \( x_n = Q_n \) [1].

Divisibility properties involving \( P_n, F_n, L_n \) or \( Q_n \) are well known. For positive integers \( m \) and \( n \), the standard method of proving \( P_m | P_{mn} \) is by first establishing the relation \( F_{n+m} = F_{n-1}F_m + F_nF_{m+1} \), then using induction (see [2, p. 196]). An 'unusual' proof that involves the use of hyperbolic functions was published in [3]. The purpose of this paper is to give new elementary proofs of divisibility properties involving these numbers. We will show that, for non-zero integers \( m \) and \( n \), \( P_n | P_{mn} \) and \( F_n | F_{mn} \), and if \( m \) is odd, \( L_n | L_{mn} \) and \( Q_n | Q_{mn} \). We will prove our results assuming \( m \) and \( n \) are positive integers. Since, for \( n \geq 1 \), \( F_n = (-1)^{n+1}F_n \), \( L_n = (-1)^nL_n \) [2, p. 84], \( P_n = (-1)^{n+1}P_n \) [2, p. 444], and \( Q_n = (-1)^nQ_n \) by an induction argument, our results hold for all non-zero integers. The proofs use similar techniques that utilise the Binet forms and the binomial theorem.

Following the standard procedures for solving second order homogeneous recurrence relations with constant coefficients [2, pp. 143-145], the Binet formula for the integer family \( \{x_n\} \) defined by (1) is

\[
x_n = \frac{1}{u - v} \left[ (b - av)u^n - (b - au)v^n \right],
\]

where \( u = \frac{1}{2} \{ c + \sqrt{c^2 + 4} \}, \ v = \frac{1}{2} \{ c - \sqrt{c^2 + 4} \} \) and \( a, b, c \) are non-negative integers.

Notice that for \( F_n \) and \( L_n \), \( u = \alpha = \frac{1}{2}(1 + \sqrt{5}) \) and \( v = \beta = \frac{1}{2}(1 - \sqrt{5}) \). For \( P_n \) and \( Q_n \), \( u = \gamma = 1 + \sqrt{2} \) and \( v = \delta = 1 - \sqrt{2} \). For the four special cases we consider, the Binet formulas are as follows:
Our approach depends on relating \(x_{mn}\) to \(x_n\) using (2). In fact (2) implies

\[
u^n = \frac{(u - v)x_n + (b - au)v^n}{b - av}
\]

and so

\[
x_{mn} = \frac{1}{u - v} \left\{ (b - av)u^{mn} - (b - au)v^{mn} \right\}.
\]

Raising \(u^n\) to the \(m\)th power and substituting into \(x^{mn}\) gives

\[
(u - v)x_{mn} = (b - av)\left(\frac{(u - v)x_n + (b - au)v^n}{b - av}\right) - (b - au)v^{mn}.
\]

Now using the binomial theorem and simplifying we get

\[
(u - v)(b - av)^{m-1}x_{mn} = \left[ (u - v)x_n + (b - au)v^n \right]^{m} - (b - av)^{m-1}(b - au)v^{mn}
\]

\[
= x_n \sum_{k=0}^{m-1} \binom{m}{k} (u - v)^{m-k}(b - au)^k x_n^{m-k-1}v^k +
\]

\[
(b - au)^m v^{mn} - (b - av)^{m-1}(b - au)v^{mn}.
\] (3)

We will use the fact that the set \(Z[\sqrt{t}] = \{ x + y\sqrt{t} \mid x \in Z, y \in Z \}\), where \(t\) is not a perfect square, is closed under multiplication and addition. For example, \(k_1(x_1 + y_1\sqrt{t})^n + k_2(x_2 + y_2\sqrt{t})^m\), where \(x_i, y_i\) and \(k_i\) are integers, is of the form \(M + N\sqrt{t}\), where \(M\) and \(N\) are integers. This follows by the binomial theorem. We also recall that, if \(t\) is not a perfect square, \(x_1 + y_1\sqrt{t} = x_2 + y_2\sqrt{t}\) if, and only if, \(x_1 = x_2\) and \(y_1 = y_2\).

We also recall the well-known equalities \(\alpha + \beta = 1, 1 - 2\alpha = -\sqrt{5}\) and \(1 - 2\beta = \sqrt{5}\). These equalities will be used in the computations involved in Theorems 2 and 3. In Theorem 4, we use \(\gamma + \delta = 2, \gamma - 1 = \sqrt{2}\) and \(\delta - 1 = -\sqrt{2}\). We are now ready to state and prove our results. We start with *Pell numbers* and prove the following theorem.

**Theorem 1:** \(P_n \mid P_{mn}\).

**Proof:** For the Pell numbers, equation (3) and the fact that \(Z[\sqrt{2}]\) is closed under multiplication and addition imply

\[
P_{mn} = P_n \sum_{k=0}^{m-1} \binom{m}{k} (2\sqrt{2})^{m-k-1} P_n^{m-k-1} \delta^{kn}
\]
where \( P \) and \( Q \) are integers. This implies \( Q = 0 \) and the desired conclusion follows.

Next, we turn our attention to the Fibonacci numbers and prove the following theorem.

**Theorem 2:** \( F_n \mid F_{mn} \).

**Proof:** For the Fibonacci numbers, equation (3) and the aforementioned closure properties of \( Z[\sqrt{5}] \) imply

\[
F_{mn} = F_n \sum_{k=0}^{m-1} \binom{m}{k} (\sqrt{5}F_n)^{m-k-1} \beta^{kn}
\]

\[
= F_n \sum_{k=0}^{m-1} \binom{m}{k} (\sqrt{5}F_n)^{m-k-1} \left( \frac{1}{2}L_{kn} - \frac{1}{2}F_{kn}\sqrt{5} \right)
\]

\[
= F_n \left( \frac{X}{2} + \frac{Y}{2}\sqrt{5} \right),
\]

where \( X \) and \( Y \) are integers. Using the fact that \( X, Y, F_{mn} \) and \( F_n \) are integers, (5) implies \( Y = 0 \). There are two cases:

1. Suppose \( F_n \) is odd. The equality \( F_{mn} = F_n \left( \frac{X}{2} \right) \) implies \( \frac{X}{2} \) has to be an integer. The desired conclusion follows.

2. Suppose \( F_n \) is even. \( F_k \) and \( L_k \) are both even if, and only if, \( k \) is a multiple of three (see [2, p.209] for a proof). It follows that \( L_{kn} \) and \( F_{kn} \) in (4) are both even for any positive integer \( k \). Thus \( F_{mn} = F_n (F + G\sqrt{5}) \) for some integers \( F \) and \( G \) and so \( F_{mn} = F_n F \). The desired conclusion follows.

Our third result concerns the Lucas numbers.

**Theorem 3:** If \( m \) is odd, then \( L_n \mid L_{mn} \).

**Proof:** Since \( m \) is odd, the last two terms in (3) cancel out and so after simplification (3) becomes

\[
L_{mn} = L_n \sum_{k=0}^{m-1} \binom{m}{k} L_n^{m-k-1} (-1)^k \beta^{kn}
\]

\[
= L_n \sum_{k=0}^{m-1} \binom{m}{k} L_n^{m-k-1} (-1)^k \left( \frac{1}{2}L_{kn} - \frac{1}{2}F_{kn}\sqrt{5} \right).
\]

As in Theorem 2, we consider the cases when \( L_n \) is odd or even to show that (6) is of the form

\[
L_{mn} = L_n \left( L + M\sqrt{5} \right),
\]
where \( L \) and \( M \) are integers. This implies that \( M = 0 \) and the desired conclusion follows.

Note that the condition on \( m \) in Theorem 3 is necessary since \( L_2 = 3 \), \( L_4 = 7 \) and \( L_2 \nmid L_{2,2} \).

Finally, we consider the Pell-Lucas numbers.

**Theorem 4:** If \( m \) is odd, then \( Q_n \mid Q_{mn} \).

**Proof:** Since \( m \) is odd, the last two terms in (3) cancel out and so (3) becomes

\[
Q_{mn} = Q_n \sum_{k=0}^{m-1} \binom{m}{k} Q_n^{m-k-1} (-1)^k \delta^{kn}
\]

where \( A \) and \( B \) are integers. This implies \( B = 0 \) and the result follows.

**Notes**

1. The condition on \( m \) in Theorem 4 is necessary since \( Q_2 = 6 \), \( Q_4 = 34 \) and \( Q_2 \nmid Q_{2,2} \).
2. Since the Pell-Lucas numbers are all even, by factoring out a 2, the divisibility property proved in Theorem 4 also holds for the sequence 1, 1, 3, 7, 17, 41, 99,

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**References**


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