Bull. Austral. Math. Soc. **78** (2008), 107–110
doi:10.1017/S0004972708000518

ON SUPERNILPOTENT NONSPECIAL RADICALS

HALINA FRANCE-JACKSON

(Received 17 September 2007)

Abstract

Let ρ be a supernilpotent radical. Let ρ^* be the class of all rings A such that either A is a simple ring in ρ or the factor ring A/I is in ρ for every nonzero ideal I of A and every minimal ideal M of A is in ρ. Let $\mathcal{L}(\rho^*)$ be the lower radical determined by ρ^* and let ρ_φ denote the upper radical determined by the class of all subdirectly irreducible rings with ρ-semisimple hearts. Le Roux and Heyman proved that $\mathcal{L}(\rho^*)$ is a supernilpotent radical with $\rho \subseteq \mathcal{L}(\rho^*) \subseteq \rho_\varphi$ and they asked whether $\mathcal{L}(\rho^*) = \rho_\varphi$ if ρ is replaced by β, $\mathcal{L}$, $\mathcal{N}$ or $\mathcal{J}$, where β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ denote the Baer, the Levitzki, the Koethe and the Jacobson radical, respectively. In the present paper we will give a negative answer to this question by showing that if ρ is a supernilpotent radical whose semisimple class contains a nonzero nonsimple $*$-ring without minimal ideals, then $\mathcal{L}(\rho^*)$ is a nonspecial radical and consequently $\mathcal{L}(\rho^*) \neq \rho_\varphi$. We recall that a prime ring A is a $*$-ring if A/I is in β for every $0 \neq I \lhd A$.

2000 *Mathematics subject classification*: 16N80.
Keywords and phrases: special radical, supernilpotent radical, weakly special class, special class, prime essential ring, $*$-ring, subdirectly irreducible ring, essential ideal.

1. Introduction

All rings in this paper are associative and all classes of rings contain the one-element ring 0. All undefined radical theoretic terms and facts can be found in [1] and [4]. An ideal I of a ring A is called essential if $I \cap J \neq 0$ whenever J is a nonzero ideal of A. A class μ of rings is hereditary if μ is closed under ideals. A hereditary class μ of semiprime rings is called weakly special if μ is essentially closed, that is, whenever $I \in \mu$ is an essential ideal of a ring A, then $A \in \mu$ also holds. Throughout this paper, for a class μ of rings, $\mathcal{U}(\mu)$ will denote the class of all rings which have no nonzero homomorphic image in μ, $\mathcal{L}(\mu)$ will denote the lower radical class determined by μ and $\mathcal{S}(\mu)$ will stand for the class of all rings without nonzero ideals in μ. Moreover, μ^* [5] will denote the class of all rings A such that either A is a simple ring in μ or the factor ring A/I is in μ for every nonzero ideal I of A and every minimal ideal M of A is in μ. A supernilpotent radical is a hereditary radical class which contains all nilpotent rings. It is well known [1, 4] that ρ is a supernilpotent radical if and only if

$\rho = \mathcal{U}(\mu)$ for some weakly special class μ of rings. Le Roux and Heyman [5] proved that if ρ is a supernilpotent radical, then so is $\mathcal{L}(\rho^*)$ and $\rho \subseteq \mathcal{L}(\rho^*) \subseteq \rho_\varphi$, where ρ_φ denotes the upper radical determined by the class of all subdirectly irreducible rings with ρ-semisimple hearts. Moreover, $\mathcal{L}(\mathcal{G}^*) = \mathcal{G}_\varphi$, where $\mathcal{G}$ is the Brown–McCoy radical. They asked whether $\mathcal{L}(\rho^*) = \rho_\varphi$ if ρ is replaced by β, $\mathcal{L}$, $\mathcal{N}$ or $\mathcal{J}$, where β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ denote the Baer, the Levitzki, the Koethe and the Jacobson radical, respectively. In the present paper we will give a negative answer to this question.

2. Main results

We start with a few preliminary results which are interesting in their own right.

LEMMA 1. *If ρ is any radical class, then for any $A \in \rho^*$, either $A \in \rho$ or $A \in \mathcal{S}(\rho)$.*

PROOF. Let $A \in \rho^*$ and suppose that the ρ-radical $\rho(A)$ of A is nonzero. Then $A/\rho(A) \in \rho$ and, since $\rho(A) \in \rho$ and ρ is closed under extensions, it follows that $A \in \rho$. □

COROLLARY 2. *If ρ is a supernilpotent radical, then for any $A \in \rho^*$, either $A \in \rho$ or A is a prime ring.*

PROOF. Let $A \in \rho^*$. Then by Lemma 1 either $A \in \rho$ or $A \in \mathcal{S}(\rho)$. If $A \in \rho$, then we are done. So assume that $A \in \mathcal{S}(\rho)$. Then, since ρ is a supernilpotent radical, A is a semiprime ring. We will now show that A is, in fact, a prime ring. Let I and J be ideals of A and suppose that $IJ = 0$ and $I \neq 0$. We will show that $J = 0$. Since $(I \cap J)^2 \subseteq IJ = 0$ and A is a semiprime ring, it follows that $I \cap J = 0$. But $(I + J)/I$ is an ideal of A/I and $A/I \in \rho$ because I is a nonzero ideal of A and $A \in \rho^*$. Thus, since ρ being a supernilpotent radical is hereditary, it follows that $(I + J)/I \in \rho$. But $(I + J)/I \simeq J/(I \cap J) \simeq J$ since $I \cap J = 0$. Thus $J \in \rho$. On the other hand, since $\mathcal{S}(\rho)$ is hereditary and $J \lhd A \in \mathcal{S}(\rho)$, it follows that $J \in \mathcal{S}(\rho)$. Thus $J \in \rho \cap \mathcal{S}(\rho) = \{0\}$ which implies that $J = 0$. □

A ring A is prime essential [3] if and only if A is semiprime and no nonzero ideal of A is a prime ring. In what follows the class of all prime essential rings will be denoted by $\mathcal{E}$.

Before we prove our main preliminary result, we will need the following construction.

THEOREM 3 [3]. *Let A be a nonzero semiprime ring, let $\kappa > 1$ be a cardinal number greater than the cardinality of A and let $W(\kappa)$ be the set of all finite words made from a (well-ordered) alphabet of cardinality κ, lexicographically ordered. Then $W(\kappa)$ is a semi-group with multiplication defined by $xy = \max\{x, y\}$ and the following results hold.*

https://doi.org/10.1017/S0004972708000518 Published online by Cambridge University Press

(1) *The semigroup ring $A(W(\kappa))$ is a subdirect sum of copies of A.*
(2) *$A(W(\kappa))$ is prime essential.*
(3) *Every prime homomorphic image $A(W(\kappa))/Q$ of $A(W(\kappa))$ is isomorphic to some prime homomorphic image A/P of A.*

We recall that a prime ring A is called a $*$-ring [2] if $A/I \in \beta$ for every $0 \neq I \lhd A$. Also, a special radical is an upper radical class $\mathcal{U}(\mu)$ determined by a special class μ of rings, that is, a hereditary and essentially closed class μ of prime rings.

THEOREM 4 [3]. *A supernilpotent radical ρ is a special radical if and only if every prime essential ρ-semisimple ring is a subdirect sum of prime ρ-semisimple rings.*

THEOREM 5. *If ρ is a supernilpotent radical whose semisimple class $\mathcal{S}(\rho)$ contains a nonzero nonsimple $*$-ring without minimal ideals, then $\mathcal{L}(\rho^*)$ is a nonspecial radical and consequently $\mathcal{L}(\rho^*) \neq \rho_\varphi$.*

PROOF. Let ρ be a supernilpotent radical and let a nonzero nonsimple $*$-ring A without minimal ideals be in $\mathcal{S}(\rho)$. Then $A \in \rho^* \cap \mathcal{S}(\rho)$.

Let $\kappa > 1$ be a cardinal number greater than the cardinality of A and let $A(W(\kappa))$ be the semigroup ring constructed in Theorem 3. Then, by Theorem 3, $A(W(\kappa))$ is prime essential and $A(W(\kappa))$ is a subdirect sum of copies of A. But, since $A \in \mathcal{S}(\rho)$, it follows that $A(W(\kappa)) \in \mathcal{S}(\rho)$ because $\mathcal{S}(\rho)$ is closed under subdirect sums. Thus $A(W(\kappa)) \in \mathcal{S}(\rho) \cap \mathcal{E}$. We will now show that $A(W(\kappa)) \in \mathcal{S}(\mathcal{L}(\rho^*))$.

It follows from [5, Theorem 2] that $\mathcal{L}(\rho^*) = \mathcal{U}(\sigma)$, where σ is the class of all rings without nonzero ideals in ρ^*. Since ρ is a supernilpotent radical, it follows from [5, Lemma 3] that ρ^* is hereditary and contains all the nilpotent rings. Then it follows from [5, Theorem 1] that σ is a weakly special class. Thus $\sigma \subseteq \mathcal{S}(\mathcal{U}(\sigma))$. It therefore suffices to show that $A(W(\kappa))$ has no nonzero ideals in ρ^*. Suppose that $0 \neq I \lhd A(W(\kappa))$ and $I \in \rho^*$. Then it follows from Corollary 2 that either $I \in \rho$ or I is a prime ring. But neither of the two cases can occur because $0 \neq I \lhd A(W(\kappa))$ and $A(W(\kappa)) \in \mathcal{S}(\rho) \cap \mathcal{E}$. Thus $A(W(\kappa)) \in \sigma$ and consequently $A(W(\kappa)) \in \mathcal{S}(\mathcal{L}(\rho^*)) \cap \mathcal{E}$.

Now, if $\mathcal{L}(\rho^*)$ were a special radical, then by Theorem 4, $A(W(\kappa))$ would contain a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals I_λ such that $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$ and $A(W(\kappa))/I_\lambda \in \mathcal{S}(\mathcal{L}(\rho^*)) \cap \pi$, where π denotes the class of all prime rings. Consequently, $A(W(\kappa))/I_\lambda$ would be a nonzero prime homomorphic image of $A(W(\kappa))$ for at least one I_λ. Then it follows from the third part of Theorem 3 that $A(W(\kappa))/I_\lambda \simeq A/P$ for some ideal P of A. Thus $0 \neq A/P \in \pi$ and, as A is a nonzero $*$-ring, it follows that $P = 0$. Thus $A(W(\kappa))/I_\lambda \simeq A$ and consequently $A \in \mathcal{S}(\mathcal{L}(\rho^*))$. On the other hand, $A \in \rho^* \subseteq \mathcal{L}(\rho^*)$. Thus $0 \neq A \in \mathcal{L}(\rho^*) \cap \mathcal{S}(\mathcal{L}(\rho^*)) = \{0\}$ and we have a contradiction. Thus $\mathcal{L}(\rho^*)$ is a nonspecial radical.

Now, since ρ_φ is a special radical [1], it follows that $\mathcal{L}(\rho^*) \neq \rho_\varphi$, which concludes the proof. □

We are now ready to answer the question of Le Roux and Heyman. To do so we need the following result.

https://doi.org/10.1017/S0004972708000518 Published online by Cambridge University Press

EXAMPLE 6 [1, Example 13, pp. 113–115]. Let F be a field of characteristic 0 which has an automorphism S such that no integral power of S is the identity automorphism. For example, F might be a field generated by the real numbers and an infinite number of independent variables labelled $\ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ and S the automorphism which leaves the real numbers alone and which sends x_i into x_{i+1} for every i. Let R be the set of all polynomials in an indeterminate z of the form $a_0 + za_1 + z^2a_2 + \cdots + z^n a_n$, where the coefficients a_i belong to F. Addition and multiplication of such polynomials are defined in the usual way except that z does not commute with the coefficients a. We define $az = zS(a)$, where $S(a)$ is the image of a under the automorphism S. Then $az^m = zS^m(a)$ for any positive integer m. Then this definition, together with the distributive law, makes R into a ring denoted by $F[z, S]$. Then $F[z, S]$ is a noncommutative integral domain and its every ideal I is of the form $I = z^k R = Rz^k$ for some positive integer k. Moreover, $F[z, S]$ is a primitive ring whose subring $T = zR$ does not contain simple prime ideals and every proper homomorphic image of T is a nilpotent ring.

COROLLARY 7. *If ρ is replaced by β, $\mathcal{L}$, $\mathcal{N}$ or $\mathcal{J}$, then $\rho \subsetneq \mathcal{L}(\rho^*) \subsetneq \rho_\varphi$.*

PROOF. It is well known [1, 4] that β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ are special radicals and $\beta \subseteq \mathcal{L} \subseteq \mathcal{N} \subseteq \mathcal{J}$. Let T be the ring of Example 6. Clearly, T is a nonzero nonsimple $*$-ring without minimal ideals. Moreover, since T is an ideal of the primitive ring $F[z, S]$ and the class of all primitive rings is hereditary, it follows that T is primitive and so $T \in \mathcal{S}(\mathcal{J}) \subseteq \mathcal{S}(\mathcal{N}) \subseteq \mathcal{S}(\mathcal{L}) \subseteq \mathcal{S}(\beta)$. Now the result follows directly from Theorem 5. □

References

[1] N. Divinsky, *Rings and Radicals* (Allen & Unwin, London, 1965).
[2] H. France-Jackson, '$*$-rings and their radicals', *Quaest. Math.* **8**(3) (1985), 231–239.
[3] B. J. Gardner and P. N. Stewart, 'Prime essential rings', *Proc. Edinb. Math. Soc.* **34** (1991), 241–250.
[4] B. J. Gardner and R. Wiegandt, *Radical Theory of Rings* (Marcel Dekker, New York, 2004).
[5] H. J. Le Roux and G. A. P. Heyman, 'A question on the characterization of certain upper radical classes', *Boll. Unione Mat. Ital. Sez. A* **17**(5) (1980), 67–72.

HALINA FRANCE-JACKSON, Department of Mathematics and Applied Mathematics, Nelson Mandela Metropolitan University,
Summerstrand Campus (South), PO Box 77000, Port Elizabeth 6031, South Africa
e-mail: cbf@easterncape.co.uk