Canad. Math. Bull. Vol. **60** (4), 2017 pp. 872–878
http://dx.doi.org/10.4153/CMB-2016-060-0

Rational Function Operators from Poisson Integrals

Xu Xu and Laiyi Zhu

Abstract. In this paper, we construct two classes of rational function operators using the Poisson integrals of the function on the whole real axis. The convergence rates of the uniform and mean approximation of such rational function operators on the whole real axis are studied.

1 Introduction

Let R be the whole real axis. The collection of all continuous functions on R is denoted by C(R). Let $C_c(R)$ consist of all $f \in C(R)$ whose support is compact. We denote by $C^*(R)$ the space consisting of all functions $f \in C(R)$ with finite limit $\lim_{x\to\infty} f(x)$ and norm $\|f\|_{C(R)} = \sup_{x\in R} |f(x)|$, and we denote by $C_0(R)$ the space that consist of all functions $f \in C^*(R)$ with $\lim_{x\to\infty} f(x) = 0$. If $0 < p < +\infty$ and $f(x)$ is measurable on R, let $L^p(R)$ consist of all $f(x)$ for which

$$\|f\|_p = \left\{ \int_R |f(x)|^p dx \right\}^{\frac{1}{p}},$$

is finite, and call $\|f\|_p$ the *norm* of $f \in L^p(R)$ (although $\|f\|_p = 0$ only implies that $f = 0$ almost everywhere, not that $f = 0$).

For $f \in C^*(R)$, we define the *modulus of continuity* of $f(x)$ as

$$\omega(f,t) = \sup_{\substack{|x'-x''|\le t\\ x',x''\in R}} |f(x') - f(x'')|,$$

and for $f \in L^p(R)$ $(1 \le p < +\infty)$, we define the *integral modulus of continuity* of $f(x)$ as

$$\omega(f,t)_p = \sup_{|h|\le t} \left\{ \int_R |f(x+h) - f(x)|^p dx \right\}^{\frac{1}{p}}.$$

It follows from the uniform continuity of $f \in C^*(R)$ that the modulus of continuity of $f(x)$ has the following properties (see[1, p. 41]):

(a) $\lim_{t\to 0^+} \omega(f,t) = \omega(f,0) = 0$;
(b) $\omega(f,t)$ is non-negative and non-decreasing on $[0,+\infty)$;

Received by the editors June 13, 2016; revised August 19, 2016.
Published electronically September 29, 2016.
This research was supported by the National Nature Science Foundation of China (No.11571362).
AMS subject classification: **41A20**, 41A25, 41A35.
Keywords: rational function operators, Poisson integrals, convergence rate, uniform approximation, mean approximation.

(c)

$$\begin{aligned} \omega(f, t_1+t_2) &\le \omega(f,t_1)+\omega(f,t_2), && t_1,t_2\in[0,+\infty),\\ \omega(f,\lambda t) &\le (1+\lambda)\omega(f,t), && \lambda, t\in[0,+\infty); \end{aligned} \tag{1.1}$$

(d) $\omega(f,t)$ is continuous on $[0,+\infty)$.

By Riesz's theorem, if $f\in L^p(R)$ $(1\le p<+\infty)$, then $\omega(f,t)_p$, the integral modulus of continuity of $f(x)$, also has the properties (1.1).

In this paper we consider problems about uniform and mean rational approximation of functions on R. A complete survey of results on rational approximation can be found in [3–5]. As far as we know, there has been little discussion of rational approximation on the whole real axis.

In [4], there are the following two theorems (see [4, Theorem 1.6.1, p. 11 and Theorem 1.9.1, p. 20]).

Theorem 1.1 *Assume $R_n(x)\in C^*(R)$ is a sequence of rational functions*

$$R_n(x)=\frac{P_n(x)}{Q_n(x)}=\frac{\sum_{k=0}^n a_kx^k}{\sum_{k=0}^n b_kx^k}, \tag{1.2}$$

where a_k, b_k are real numbers, $k=0,1,\dots,n$, and $b_n\ne 0$, $Q_n(x)$ has no real zeros. If this sequence converges uniformly on R, then the limit function is in $C^*(R)$, and for any $f(x)\in C^*(R)$, there exists a sequence of the rational functions (1.2) such that

$$\lim_{n\to\infty}\|f(x)-R_n(x)\|_{C(R)}=0.$$

Theorem 1.2 *If $0<p<+\infty$, then the class $L^p(R)$ consists of those functions and only those that are the limits of sequences, convergent in mean, of the rational functions (1.2) in $L^p(R)$*, i.e., *sequence of rational functions such that*

$$\int_R |R_m(x)-R_n(x)|^p dx\to 0 \quad (m,n\to\infty),$$
$$\|R_n\|_{L^p(R)}<+\infty.$$

Furthermore, the ideas used in the proofs of Theorem 1.1 and Theorem 1.2 resemble those of Lebesgue. Thus it is natural to construct a class of rational function operators approximating well the functions on R. Motivated by the convergence properties of the Poisson integral of the functions on R (see [2, Theorem 3.1, p. 15]), we construct the following two classes of rational function operators.

For fixed $y>0$ and positive integer n, we define the rational function operators $R_n(f,y;x)$ and $R_n^*(f,y;x)$ as follows:

$$R_n(f,y;x)=\frac{1}{\pi}\sum_{k=-n+1}^{n}\sum_{i=1}^{n}\frac{y}{(x-t_{k,i})^2+y^2}f(t_{k,i}),\quad \text{for } f(x)\in C_0(R), \tag{1.3}$$

$$R_n^*(f,y;x)=\frac{1}{\pi}\sum_{k=-n+1}^{n}\sum_{i=1}^{n}\frac{y}{(x-t_{k,i})^2+y^2}\int_{a_{k,i}}^{b_{k,i}} f(t)dt, \tag{1.4}$$
$$\text{for } f(x)\in L^p(R), p\in[1,+\infty),$$

where

$$a_{k,i} = k - 1 + \frac{i-1}{n} \quad b_{k,i} = k - 1 + \frac{i}{n}, \quad t_{k,i} = k - 1 + \frac{2i-1}{2n},$$
$$i = 1, 2, \ldots, n \quad k = -n+1, \ldots, -1, 0, 1, 2, \ldots, n.$$

Our main results are as follows.

Theorem 1.3 *If $f(x) \in C^*(R)$ with $\lim_{x\to\infty} f(x) = c$, then for any decreasing sequence $\{\delta_n\}$ with $\lim_{n\to\infty} \delta_n = 0$, there is an increasing positive integer sequence*

$$N_n = \max\{[\delta_n^{-3}] + 1, [X_n] + 1\}, \tag{1.5}$$

where $[x]$ denotes the greatest integer $\le x$, and

$$X_n = inf\{X_0 > 0 \big| |f(x) - c| \le \omega(f, \delta_n), \text{ as } |x| > X_0\} \tag{1.6}$$

such that

$$\big| R_{N_n}(f - c, \delta_n^2; x) - f(x) + c \big| \le C\big[\delta_n + \omega(f, \delta_n)\big]$$

holds for every $x \in R$ and some $C > 0$ that is independent of n.

Theorem 1.4 *If $f(x) \in L^p(R)$, $p \in [1, +\infty)$, then for any positive integer n, we have*

$$\big\| R_n^*(f, n^{-\frac{2}{3}}; x) - f(x) \big\|_{L^p(R)} \le C_p\big[\delta_{n,p} + \omega(f, \delta_{n,p})_p\big],$$

where

$$\delta_{n,p} = \max\Big\{ n^{-\frac{1}{3}}, \Big(\int_{|x|\ge n} |f(x)|^p dx\Big)^{\frac{1}{p}}\Big\}, \tag{1.7}$$

and C_p is a positive constant depending on p, independent of n.

2 The Proof of Theorem 1.3

Let

$$P_y(t) = \frac{1}{\pi} \frac{y}{t^2 + y^2}$$

be the Poisson kernel for the upper half plane, where $y > 0$, $t \in R$. The Poisson kernel $P_y(t) \ge 0$ and satisfies (see [2, p. 11])

$$\int_R P_y(t) dt = 1. \tag{2.1}$$

If $f(x) \in C^*(R)$, we define its Poisson integral as

$$\int_R P_y(x - t) f(t) dt = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-t)^2 + y^2} f(t) dt. \tag{2.2}$$

Proof of Theorem 1.3 If $f(x) \in C^*(R)$ and $F(x) := f(x) - c \in C_0(R)$, then

$$\omega(F, t) = \omega(f, t)$$

https://doi.org/10.4153/CMB-2016-060-0 Published online by Cambridge University Press

Without loss of generality, we can assume that $f(x) \in C_0(R)$. It follows from (1.3) and (2.2) that for fixed $y > 0$,

(2.3)

$$\begin{aligned} &\left|R_{N_n}(f,y;x) - f(x)\right| \\ &\le \frac{1}{\pi}\sum_{k=-N_n+1}^{N_n}\sum_{i=1}^{N_n}\int_{\frac{i-1}{N_n}}^{\frac{i}{N_n}}\left|\frac{y}{(x-t_{k,i})^2+y^2}f(t_{k,i}) - \frac{y}{(x-k+1-t)^2+y^2}f(k-1+t)\right|dt \\ &\quad + \frac{1}{\pi}\int_{|t|\ge N_n}\frac{y}{(x-t)^2+y^2}|f(t)|dt + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y}{t^2+y^2}\left|f(x+t)-f(x)\right|dt \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

From the definitions (1.5) and (1.6), if $|t| > N_n$, then $|f(t)| \le \omega(f,\delta_n)$. Therefore, using (2.1), we have

$$S_2 \le \omega(f,\delta_n). \tag{2.4}$$

We shall estimate S_3. Setting $M = \sup_{x\in R}|f(x)|$, by (2.1), we have

$$\begin{aligned} S_3 &= \frac{1}{\pi}\int_{|t|\ge\sqrt{y}}\frac{y}{t^2+y^2}\left|f(x+t)-f(t)\right|dt \\ &\quad + \frac{1}{\pi}\int_{|t|<\sqrt{y}}\frac{y}{t^2+y^2}\left|f(x+t)-f(t)\right|dt \\ &\le \frac{4M}{\pi}\Big(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{y}}\Big) + \omega(f,\sqrt{y}) \\ &\le \frac{4M}{\pi}\sqrt{y} + \omega(f,\sqrt{y}). \end{aligned} \tag{2.5}$$

To estimate S_1, we write

$$\begin{aligned} I_{k,i} &= \left|\frac{y}{(x-t_{k,i})^2+y^2}f(t_{k,i}) - \frac{y}{(x-k+1-t)^2+y^2}f(k-1+t)\right| \\ &\le \frac{y}{(x-k+1-t)^2+y^2}\left|f(k-1+t)-f(t_{k,i})\right| \\ &\quad + \left|f(t_{k,i})\right|\frac{y|t+k-1-t_{k,i}||(x-k+1-t)+(x-t_{k,i})|}{[(x-t_{k,i})^2+y^2][(x-k+1-t)^2+y^2]}. \end{aligned}$$

Since $t \in [\frac{i-1}{N_n}, \frac{i}{N_n}]$, $t_{k,i} = k-1+\frac{2i-1}{2N_n}$, we have

$$\begin{aligned} I_{k,i} &\le \omega\Big(f,\frac{1}{2N_n}\Big)\frac{y}{(x-k+1-t)^2+y^2} \\ &\qquad + \frac{My}{2N_n}\frac{|(x-k+1-t)+(x-t_{k,i})|}{[(x-t_{k,i})^2+y^2][(x-k+1-t)^2+y^2]}. \end{aligned}$$

Noting that for any $t \in [\frac{i-1}{N_n}, \frac{i}{N_n}]$,

$$\begin{aligned} \frac{|(x-k+1-t)+(x-t_{k,i})|}{(x-t_{k,i})^2+y^2} &\le \frac{2|x-t_{k,i}|+|t_{k,i}-k+1-t|}{(x-t_{k,i})^2+y^2} \\ &\le \frac{1}{y} + \frac{1}{2N_ny^2}, \end{aligned} \tag{2.6}$$

https://doi.org/10.4153/CMB-2016-060-0 Published online by Cambridge University Press

we obtain

$$I_{k,i} \le \Big[\omega\Big(f, \frac{1}{2N_n}\Big) + \frac{2N_n y + 1}{2N_n^2 y^2} M\Big] \frac{y}{(x-k+1-t)^2 + y^2}.$$

Therefore,

$$\begin{aligned}(2.7)\quad S_1 &\le \Big[\omega\Big(f, \frac{1}{2N_n}\Big) + \frac{2N_n y + 1}{4N_n^2 y^2} M\Big] \frac{1}{\pi} \sum_{k=-N_n+1}^{N_n} \sum_{i=1}^{N_n} \int_{\frac{i-1}{N_n}}^{\frac{i}{N_n}} \frac{y}{(x-k+1-t)^2 + y^2} dt \\ &= \Big[\omega\Big(f, \frac{1}{2N_n}\Big) + \frac{2N_n y + 1}{4N_n^2 y^2} M\Big] \frac{1}{\pi} \int_{-N_n}^{N_n} \frac{y}{(x-t)^2 + y^2} dt \\ &\le \omega\Big(f, \frac{1}{2N_n}\Big) + \frac{2N_n y + 1}{4N_n^2 y^2} M.\end{aligned}$$

The inequalities (2.4)–(2.7) when used in (2.3) yield

$$\begin{aligned}\big\| R_{N_n}(f, y; x) - f(x)\big\|_{C(R)} &\le \omega(f, \delta_n) + \omega\Big(f, \frac{1}{2N_n}\Big) \\ &\quad + \omega(f, \sqrt{y}) + \frac{4M}{\pi}\sqrt{y} + \frac{2N_n y + 1}{4N_n^2 y^2} M.\end{aligned}$$

Finally, we take $y = \delta_n^2$, and obtain, using (1.5),

$$\big\| R_{N_n}(f, \delta_n^2; x) - f(x)\big\|_{C(R)} \le C\big[\delta_n + \omega(f, \delta_n)\big],$$

where $C > 0$ is independent of n.

This completes the proof of Theorem 1.3. ■

Remark 2.1 For $f_0(x) = \frac{1}{1+|x|} \in C_0(R)$, we have $\omega(f_0, t) \le t$, and taking $\delta_n = \frac{1}{n}$, we obtain $N_n = n^3 + 1$ in (1.5), and

$$\Big\| R_{n^3+1}\Big(f_0, \frac{1}{n^2}; x\Big) - f_0(x)\Big\|_{C(R)} \le \frac{C}{n},$$

where $C > 0$ is independent of n.

Remark 2.2 If $f(x) \in C_c(R)$, for example the support of $f(x)$ is $[a, b]$, we have $X_n \le \max\{|a|, |b|\}$ in (1.6), and taking $\delta_n = n^{-\frac{1}{3}}$, we obtain $N_n = n + 1$ in (1.5), for large n and

$$\Big\| R_{n+1}\Big(f, n^{-\frac{2}{3}}; x\Big) - f(x)\Big\|_{C(R)} \le C[n^{-\frac{1}{3}} + \omega(f, n^{-\frac{1}{3}})],$$

where $C > 0$ is independent of n.

https://doi.org/10.4153/CMB-2016-060-0 Published online by Cambridge University Press

3 The Proof of Theorem 1.4

Proof of Theorem 1.4 If $f(x) \in L^p(R)$, $p \in [1,+\infty)$, it follows from (1.4) that for fixed $y > 0$,

$$
\begin{aligned}
(3.1)\quad & \left|R_n^*(f,y;x) - f(x)\right| \\
&\le \frac{1}{\pi}\sum_{k=-n+1}^{n}\sum_{i=1}^{n}\int_{a_{k,i}}^{b_{k,i}} |t-t_{k,i}| \frac{|x-t+x-t_{k,i}|}{(x-t_{k,i})^2+y^2}\frac{y}{(x-t)^2+y^2}|f(t)|dt \\
&\quad + \frac{1}{\pi}\int_{|t|\ge n}\frac{y}{(x-t)^2+y^2}|f(t)|dt + \left|f(x) - \frac{1}{\pi}\int_R \frac{y}{(x-t)^2+y^2}f(t)dt\right| \\
&=: S_{p_1}^* + S_{p_2}^* + S_{p_3}^*.
\end{aligned}
$$

For every $x \in R$,

$$
\begin{aligned}
(3.2)\quad S_{p_2}^* &= \frac{1}{\pi}\int_{|t|\ge n}\frac{y}{(x-t)^2+y^2}|f(t)|dt \\
&= \frac{1}{\pi}\int_{|t+x|\ge n}\frac{y}{t^2+y^2}|f(t+x)|dt = \frac{1}{\pi}\int_R \frac{y}{t^2+y^2}|f(t+x)|\chi_E(x,t)dt,
\end{aligned}
$$

where $E = \{(x,t) \in R^2 \big|\ |t+x| \ge n\}$ and $\chi_E(x,t)$ is the characteristic function of the set E.

Noting that for every $t \in R$,

$$
\begin{aligned}
\|f(t+x)\chi_E(x,t)\|_{L^p(dx)}^p &= \int_R |f(x+t)|^p[\chi_E(x,t)]^p dx \\
&= \int_{|x+t|\ge n}|f(x+t)|^p dx = \int_{|x|\ge n}|f(x)|^p dx.
\end{aligned}
$$

Thus, from the Minkowski inequality for integrals, by (2.1), (1.7), and (3.2), we find

$$
(3.3)\qquad \|S_{p_2}^*\|_{L^p(dx)} \le \frac{1}{\pi}\int_R \frac{y}{t^2+y^2}\left\|f(x+t)\chi_E(x,t)\right\|_{L^p(dx)}dt \le \delta_{n,p}.
$$

Next, the Minkowski inequality for integrals and (2.1) yield

$$
\begin{aligned}
(3.4)\qquad \|S_{p_3}^*\|_{L^p(dx)} &= \left\{\int_R \left|\frac{1}{\pi}\int_R \frac{y}{t^2+y^2}[f(x-t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}} \\
&\le \frac{1}{\pi}\int_R \left\{\int_R \left[\frac{y}{t^2+y^2}|f(x-t)-f(x)|\right]^p dx\right\}^{\frac{1}{p}} dt \\
&= \frac{1}{\pi}\int_R \frac{y}{t^2+y^2}\left[\int_R |f(x-t)-f(x)|^p dx\right]^{\frac{1}{p}} dt \\
&\le \frac{1}{\pi}\int_R \frac{y\omega(f,|t|)_p}{t^2+y^2}dt = \frac{2}{\pi}\int_0^{+\infty}\frac{y\omega(f,t)_p}{t^2+y^2}dt \\
&= \frac{2}{\pi}\int_0^{\sqrt{y}}\frac{y\omega(f,t)_p}{t^2+y^2}dt + \frac{2}{\pi}\int_{\sqrt{y}}^{+\infty}\frac{y\omega(f,t)_p}{t^2+y^2}dt \\
&\le \omega(f,\sqrt{y})_p + \frac{4}{\pi}\|f\|_{L^p(R)}\left(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{y}}\right) \\
&\le \omega(f,\sqrt{y})_p + \frac{4}{\pi}\|f\|_{L^p(R)}\sqrt{y}.
\end{aligned}
$$

https://doi.org/10.4153/CMB-2016-060-0 Published online by Cambridge University Press

It is similar to (2.6) that

$$S_{p_1}^* \le \frac{1}{2n}\Big(\frac{1}{y}+\frac{1}{2ny^2}\Big)\frac{1}{\pi}\int_R \frac{y}{(x-t)^2+y^2}|f(t)|dt \tag{3.5}$$
$$= \frac{1}{2n}\Big(\frac{1}{y}+\frac{1}{2ny^2}\Big)\frac{1}{\pi}\int_R \frac{y}{t^2+y^2}|f(x+t)|dt.$$

Therefore, (2.1), (3.5), and the Minkowski inequality for integrals yield

$$\|S_{p_1}^*\|_{L^p(dx)} \le \Big(\frac{1}{2ny}+\frac{1}{4n^2y^2}\Big)\|f\|_{L^p(R)}. \tag{3.6}$$

The inequalities (3.3), (3.4), and (3.6) when used in (3.1) imply

$$\big\|R_n^*(f,y;x)-f(x)\big\|_{L^p(R)} \le \Big(\frac{1}{2ny}+\frac{1}{4n^2y^2}\Big)\|f\|_{L^p(R)} + \frac{4}{\pi}\|f\|_{L^p(R)}\sqrt{y}+\delta_{n,p}+\omega(f,\sqrt{y})_p.$$

Finally, we take $y = n^{-\frac{2}{3}}$, and obtain, using (1.7),

$$\big\|R_n^*(f,n^{-\frac{2}{3}};x)-f(x)\big\|_{L^p(R)} \le C_p[\delta_{n,p}+\omega(f,\delta_{n,p})_p],$$

where C_p is a positive constant depending on p, independent of n.

This completes the proof of Theorem 1.4. ■

References

[1] R. A. DeVore and G. G. Lorentz, *Constructive approximations*. Grundlehren der Mathematischen Wissenschaften, 303, Springer-Verlag, Berlin, 1993.

[2] J. B. Garnett, *Bounded analytic functions*. Pure and Applied Mathematics, 96, Academic Press Inc., New York-London, 1981.

[3] P. P. Petrushev and V. A. Popov, *Rational approximation of real functions*. Encyclopedia of Mathematics and its Applications, 28, Cambridge University Press, Cambridge, 1987.

[4] A. F. Timan, *Theory of approximation of functions of a real variable (Russian)*. J. GIFML, Moscow; English translation: International Series of Monographs in Pure and Applied Mathematics, 34, Pergamon Press, The Macmillan Co., New York, 1963.

[5] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*. Colloquium Publications, 20, AMS, 1935.

School of Information, Renmin University of China, Beijing, 100872, China
e-mail: eunice0420@163.com zhulaiyi@ruc.edu.cn