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Microlocal Vanishing Cycles and Ramified Cauchy Problems in the Nilsson Class

KIYOSHI TAKEUCHI
Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-0006, Japan; e-mail: takechan@abel.math.tsukuba.ac.jp

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Abstract. We will clarify the microlocal structure of the vanishing cycle of the solution complexes to $\mathcal{D}$-modules. In particular, we find that the object introduced by D'Agnolo and Schapira is a kind of the direct product (with a monodromy structure) of the sheaf of holomorphic microfunctions. By this result, a totally new proof (that does not involve the use of the theory of microlocal inverse image) of the theorem of D'Agnolo and Schapira will be given. We also give an application to the ramified Cauchy problems with growth conditions, i.e., the problems in the Nilsson class functions of Deligne.

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1. Introduction

In this paper, we will determine the microlocal structure of the nearby (vanishing) cycles of the holomorphic solution complexes to $\mathcal{D}$-modules. Let X be a complex manifold and $H \subset X$ a complex hypersurface. If we denote by $\mathcal{O}_{H|X}^{ram}$ the sheaf of holomorphic functions on $X - H$ (arbitrarily) ramified along H (see the definition in Section 2), the solution complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{H|X}^{ram})$ to a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is the nearby cycle sheaf of the holomorphic solution complex $\mathrm{Sol}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Deligne [6] defined the sheaf $\mathcal{O}_{H|X}^{ram}$ by $R\mathcal{H}om_{\mathbb{C}_X}(F^H, \mathcal{O}_X)$ for a sheaf F^H attached to H , weakly $\mathbb{R}$-constructible in the sense of Kashiwara and Schapira [16]. Here we are interested in the structure of its microlocal version $\mu hom(F^H, \mathcal{O}_X)$ introduced by D'Agnolo and Schapira [3]. This is a complex of sheaves on the cotangent bundle T^*X, whose restriction to the zero section $T_X^*X \simeq X$ coincides with $\mathcal{O}_{H|X}^{ram} = R\mathcal{H}om_{\mathbb{C}_X}(F^H, \mathcal{O}_X)$. However, to the best of our knowledge, it seems that the concrete structure of $\mu hom(F^H, \mathcal{O}_X)$ has not yet been studied and this object has been treated only using purely sheaf theoretical methods (see, for example, [3, 7, 23]) up to now.

First in Section 2, we will show that the complex $\mu hom(F^H, \mathcal{O}_X)$ is concentrated in the degree 0 and, locally on $\dot{T}_H^*X$, it is even isomorphic to a kind of direct product of the sheaf $\mathcal{C}_{H|X}^{\mathbb{R}}$ of holomorphic microfunctions. Since it is not a direct product in the usual sense, we will call it the (local) 'simultaneous direct product' structure of the sheaf $\mu hom(F^H, \mathcal{O}_X)$. In fact, globally on $\dot{T}^*_H X$, it is not the (simultaneous) direct product of $\mathcal{C}^{\mathbb{R}}_{H|X}$ and has a 'monodromy' structure. We will explain the precise meaning in Section 2.

These results enable us to study the microlocal nearby (vanishing) cycle $\mu hom(F^H, \mathrm{Sol}(\mathcal{M})) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu hom(F^H, \mathcal{O}_X))$. We believe that it will be the starting point of the calculation of the vanishing cycles of holonomic systems. In effect, Kashiwara [10] studied the structure of the solution complexes $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{H|X})$ to holonomic systems $\mathcal{M}$. Our results show that the vanishing cycles of holonomic systems $\mathcal{M}$ can be obtained by calculating the microlocal vanishing cycle $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu hom(F^H, \mathcal{O}_X))$, which is the simultaneous direct product of $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{H|X})$.

In Section 3, we give a totally new proof of a result of D'Agnolo and Schapira [3] on the generalization of the theorem of Hamada, Leray and Wagschal [9]. We prove this theorem without using the theory of microlocal inverse image of Kashiwara and Schapira [16].

Roughly speaking, the situation that we encounter is (microlocally) just the (simultaneous) direct product of that considered in [15], and our proof goes along the same lines as the proof of [15]. Finally in Section 4, we obtain the tempered version of the result of [3] as a simple application of these studies. Namely, we solve the ramified Cauchy problems for $\mathcal{D}$-modules with regular singularities (in the sense of [14]) for the initial data in the Nilsson class of Deligne [5]. Let us mention that the same problem was also tackled in [23], in which the general ramified initial data could not be treated. This difficulty arises from the fact that the functor $T\text{-}\mu hom(*, \mathcal{O}_X)$ of Andronikof [1] was defined only for $\mathbb{R}$-constructible sheaves. We overcame this difficulty by introducing a subsheaf $T\text{-}\mu hom(F^H, \mathcal{O}_X)$ of $\mu hom(F^H, \mathcal{O}_X)$. For another type of Cauchy problem with growth conditions, see Tonin [23]. In fact, in the case of initial logarithmic pole data, the ramified Cauchy problem for $\mathcal{D}_X$-modules with growth conditions was solved by Laurent [18] and Monteiro Fernandes [19].

2. Microlocal Structure of Nearby (Vanishing) Cycles

In this paper, we essentially employ the terminology of Kashiwara and Schapira [16]. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category of complexes of sheaves of $\mathbb{C}_X$-modules. First, we assume that X is a complex manifold and $H \subset X$ a complex hypersurface. We shall recall the construction of Deligne [6] of nearby (vanishing) cycles (see also [3] for an introduction). Let $p : \tilde{\mathbb{C}}^* \simeq \mathbb{C}_t \to \mathbb{C}_z$ be the universal covering map of $\mathbb{C}^*$ given by $t \mapsto z = \exp(2\pi i t)$. We choose a holomorphic defining function g of H s.t. $H = \{g = 0\}$. The fiber product $\tilde{X}^*_H = \tilde{\mathbb{C}} \times_{\mathbb{C}} X$, $p_H : \tilde{X}^*_H \to X$ induced from p and g fits well into the Cartesian square:

$$\begin{array}{ccc} \tilde{X}^*_H & \longrightarrow & \tilde{\mathbb{C}}^* \\ {\scriptstyle p_H}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ X & \underset{g}{\longrightarrow} & \mathbb{C}. \end{array}$$

DEFINITION 2.1 ([6]). We define the sheaf of holomorphic functions on $X - H$ (arbitrarily) ramified along H by the formula: $\mathcal{O}^{ram}_{H|X} = R\mathcal{H}om_{\mathbb{C}_X}(F^H, \mathcal{O}_X)$, where $F^H = p_{H!}\mathbb{C}_{\tilde{X}^*_H}$.

Let us recall that the object

$$R\mathcal{H}om_{\mathbb{C}_X}(F^H, \mathcal{O}_X) \simeq Rp_{H*}\mathcal{O}_{\tilde{X}^*_H} \in \mathbf{D}^b(X)$$

is concentrated in the degree 0 and can be regarded as a sheaf on X. If we apply the functor

$$\mu hom(*, *)\colon\ \mathbf{D}^b(X)^{op} \times \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(T^*X)$$

of [16] to the sheaf F^H, we get the microlocal version $\mu hom(F^H, \mathcal{O}_X) \in \mathbf{D}^b(T^*X)$ of $\mathcal{O}^{ram}_{H|X}$ satisfying $R\pi_{X*}\mu hom(F^H, \mathcal{O}_X) \simeq \mathcal{O}^{ram}_{H|X}$ for the projection $\pi_X\colon T^*X \to X$. This microlocal object was first introduced in [3] and used to generalize the theorem of Hamada, Leray and Wagschal [9] into the case of D_X-modules. By the following proposition, one can understand the precise structure of the complex $\mu hom(F^H, \mathcal{O}_X)$. We endow the set $\mathbb{Z}$ of rational integers with the discrete topology and consider the first projection $\tau\colon X^\infty = X \times \mathbb{Z} \to X$. For an object $F \in \mathbf{D}^b(X)$, we set $\bigoplus^\infty F := \tau_!\tau^{-1}F \in \mathbf{D}^b(X)$.

PROPOSITION 2.2. (i) *The object* $\mu hom(F^H, \mathcal{O}_X)$ *is concentrated in the degree* 0.

(ii) *For any* $p \in \dot{T}^*_H X$, *we have an isomorphism:*

$$\mu hom(F^H, \mathcal{O}_X) \xrightarrow{\sim} \mu hom\Big(\bigoplus^\infty \mathbb{C}_H, \mathcal{O}_X\Big)[1]$$

on an open neighborhood of p.

(iii) *The stalk of* $\mu hom(F^H, \mathcal{O}_X)$ *at* $p \in \dot{T}^*_H X$ *is described as follows:*

$$\begin{aligned} \mu hom(F^H, \mathcal{O}_X)_p &\simeq \varinjlim_{S} H^1 R\mathcal{H}om_{\mathbb{C}_X}\Big(\bigoplus^\infty \mathbb{C}_S, \mathcal{O}_X\Big)_{\pi_X(p)} \\ &\simeq \varinjlim_{S,U}\Big[\prod^\infty \mathcal{O}_X(U - S)\Big/\prod^\infty \mathcal{O}_X(U)\Big], \end{aligned}$$

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where S ranges over the family of closed subsets of X s.t.

$$C_H(S)_{\pi_X(p)} \subset \{v \in (T_H X)_{\pi_X(p)};\ < v, p >> 0\} \cup \{0\}$$

and U ranges over the open neighborhoods of $\pi_X(p)$ *in X.*

Proof. (ii) Locally we may assume that

$$H = \{z_1 = 0\} \subset X, \qquad z_1 = x_1 + iy_1 \quad \text{and} \quad p = \mathrm{d}x_1 \in \dot{T}^*_H X.$$

We take a locally closed subset $K := \{x_1 < 0, y_1 = 0\}$ of X st. $\partial K = H$ and an open subset $\Omega := X - \bar{K} \subset X$. It follows from the isomorphism $F^H \simeq (F^H)_{X-H}$ that we have the exact sequence of sheaves:

$$0 \to (F^H)_\Omega \to F^H \to (F^H)_K \to 0.$$

Since on Ω and K the monodromy of the sheaf F^H disappears, we get $(F^H)_\Omega \simeq \bigoplus^\infty \mathbb{C}_\Omega$ and $(F^H)_K \simeq \bigoplus^\infty \mathbb{C}_K$. Now consider the isomorphism

$$\bigoplus^\infty \mathbb{C}_\Omega = \tau_! \tau^{-1} \mathbb{C}_\Omega \xrightarrow{\sim} [\tau_! \mathbb{C}_{X^\infty}] \otimes \mathbb{C}_\Omega = \left[\bigoplus^\infty \mathbb{C}_X\right] \otimes \mathbb{C}_\Omega.$$

Then we get the estimation $p \notin \mathrm{SS}(\bigoplus^\infty \mathbb{C}_\Omega)$ (by Proposition 5.4.14(i) of [16]) and the isomorphism $F^H \simeq \bigoplus^\infty \mathbb{C}_K$ in the localized category $\mathbf{D}^b(X; p)$ of $\mathbf{D}^b(X)$ at p. As in the argument of [21], one can associate a distinguished triangle

$$D'(\mathbb{C}_H) \to D'(\mathbb{C}_{\bar{K}}) \to D'(\mathbb{C}_K) \to +1$$

to the exact sequence

$$0 \to \mathbb{C}_K \to \mathbb{C}_{\bar{K}} \to \mathbb{C}_H \to 0,$$

where $D'(*) = R\mathcal{H}om_{\mathbb{C}_X}(*, \mathbb{C}_X)$ is the stupid dual functor on X. This distinguished triangle is reduced to:

$$\mathbb{C}_H[-2] \to \mathbb{C}_K[-1] \to \mathbb{C}_{\bar{K}}[-1] \to +1$$

and we obtain a morphism $\mathbb{C}_H[-1] \to \mathbb{C}_K$ which is isomorphic in $\mathbf{D}^b(X; p)$. Thus, by tensorizing $\bigoplus^\infty \mathbb{C}_X$, we have the isomorphism $\bigoplus^\infty \mathbb{C}_H[-1] \simeq \bigoplus^\infty \mathbb{C}_K$ in $\mathbf{D}^b(X; p)$. Summarizing, we have got a microlocal ismorphism $F^H \simeq \bigoplus^\infty \mathbb{C}_H[-1]$ which induces an isomorphism:

$$\mu hom(F^H, \mathcal{O}_X) \xrightarrow{\sim} \mu hom\left(\bigoplus^\infty \mathbb{C}_H, \mathcal{O}_X\right)[1]$$

on an open neighborhood of $p \in \dot{T}^*_H X$.

(i), (iii) At the zero section $T^*_X X$ of $T^* X$, the object $\mu hom(F^H, \mathcal{O}_X)$ coincides with the sheaf $\mathcal{O}^{ram}_{H|X}$. Hence, there is nothing to prove. Since

$$\mathrm{supp}\ \mu hom(F^H, \mathcal{O}_X) \subset \mathrm{SS}(F^H) = T^*_X X \cup T^*_H X$$

it is enough to consider the stalks at $p \in \dot{T}^*_H X$. By a simple calculation and (ii), we have the stalk formula

$$H^i\mu hom(F^H,\mathcal{O}_X)_p \simeq \varinjlim_{S} H^{i+1}R\mathcal{H}om_{\mathbb{C}_X}\left(\bigoplus^{\infty}\mathbb{C}_S,\mathcal{O}_X\right)_{\pi_X(p)}$$

at p for $^\forall i\in\mathbb{Z}$, where S ranges over the family of closed subsets of X s.t.

$$C_H(S)_{\pi_X(p)}\subset\{v\in(T_HX)_{\pi_X(p)};\ <v,p>>0\}\cup\{0\}.$$

The right-hand side vanishes except for $i=0$, and for $i=0$ it is isomorphic to

$$\varinjlim_{S,U}\left[\prod^{\infty}\mathcal{O}_X(U-S)\Big/\prod^{\infty}\mathcal{O}_X(U)\right],$$

where U ranges over the open neighborhoods of $\pi_X(p)$ in X. This completes the proof. □

Remark 2.3. The stalk at $p\in\dot{T}^*_HX$ of $\mu hom(F^H,\mathcal{O}_X)$ is very similar to that of the infinite direct product $\prod^{\infty}\mathcal{C}^{\mathbb{R}}_{H|X}$ of the sheaf of holomorphic microfunctions. However, the stalk at p of $\prod^{\infty}\mathcal{C}^{\mathbb{R}}_{H|X}$ is equal to

$$\varinjlim_{S}\left[\prod^{\infty}(\Gamma_{X-S}(\mathcal{O}_X)_z/(\mathcal{O}_X)_z\right],\qquad z=\pi_X(p)\in X,$$

where S ranges over the family of closed subsets of X s.t.

$$C_H(S)_{\pi_X(p)}\subset\{v\in(T_HX)_{\pi_X(p)};\ \langle v,p\rangle>0\}\cup\{0\}.$$

This is not isomorphic to the stalk of $\mu hom(F^H,\mathcal{O}_X)$.

Motivated by the above remark, we shall regard the sheaf $\mu hom(F^H,\mathcal{O}_X)$ (restricted to an open neighborhood of a point $p\in\dot{T}^*_HX$) as a kind of the direct product of $\mathcal{C}^{\mathbb{R}}_{H|X}$ and call it the 'simultaneous direct product' of $\mathcal{C}^{\mathbb{R}}_{H|X}$. For every $k\in\mathbb{Z}$, consider the embedding

$$j_k: X=X\times\{k\}\longrightarrow X\times\mathbb{Z}=X^{\infty}$$

and the natural morphism

$$\bigoplus^{\infty}\mathbb{C}_H=\tau_!\tau^{-1}\mathbb{C}_H\to\tau_!j_{k*}j_k^{-1}\tau^{-1}\mathbb{C}_H\simeq\mathbb{C}_H,$$

from which we have an injective sheaf homomorphism

$$\mathcal{C}^{\mathbb{R}}_{H|X}=\mu hom(\mathbb{C}_H,\mathcal{O}_X)[1]\hookrightarrow\mu hom\left(\bigoplus^{\infty}\mathbb{C}_H,\mathcal{O}_X\right)[1]\simeq\mu hom(F^H,\mathcal{O}_X)$$

on an open neighborhood of $p\in\dot{T}^*_HX$. Hence we can consider that (locally on $\dot{T}^*_HX$) the sheaf $\mathcal{C}^{\mathbb{R}}_{H|X}$ is a subsheaf of the sheaf $\mu hom(F^H,\mathcal{O}_X)$. We call this subsheaf the kth component of $\mu hom(F^H, \mathcal{O}_X)$. Note that this embedding of $\mathcal{C}^{\mathbb{R}}_{H|X}$ is defined only on an open neighborhood of $p \in \dot{T}^*_H X$. As for the global structure of the sheaf $\mu hom(F^H, \mathcal{O}_X)$, we have the following results:

PROPOSITION 2.4. (i) *The restriction of the sheaf* $\mu hom(F^H, \mathcal{O}_X)$ *to* $\dot{T}^*_H X$ *has a 'monodromy' structure. Namely, if we turn (once) around* H *in* $\dot{T}^*_H X$ *in the clockwise direction, the* kth *component of* $\mu hom(F^H, \mathcal{O}_X)$ *is connected to the* $(k+1)$th *component.*

(ii) $R\pi_{X!}\mu hom(F^H, \mathcal{O}_X) \simeq \mathcal{O}_X|_H$ *on* H.

Proof. (i) In order to see the monodromy structure, we will give another proof of Proposition 2.2 (ii). In an open neighborhood of $p \in \dot{T}^*_H X$, $SS(F^H) \subset T^*_H X$ so, by Proposition 6.6.1 of [16] there exists an object $M \in \mathbf{D}^b(\{pt\})$ such that $F^H \simeq M_H$ in the localized derived category $\mathbf{D}^b(X; p)$. If $H = \{z_1 = 0\}$, $z_1 = x_1 + iy_1$ and $p = (0; \mathrm{d}x_1) \in \dot{T}^*_H X$, by the definition of $\mathbf{D}^b(X; p)$, we have

$$R\Gamma_{\{x_1 \geqslant 0\}}(F^H)_0 \simeq R\Gamma_{\{x_1 \geqslant 0\}}(M_H)_0 = M.$$

It follows from the distinguished triangle

$$R\Gamma_{\{x_1 \geqslant 0\}}(F^H)_0 \longrightarrow (F^H)_0 \longrightarrow R\Gamma_{\{x_1 < 0\}}(F^H)_0 \longrightarrow +1$$

and the isomorphisms (obtained from the definition of $F^H = p_{H!}\mathbb{C}_{\tilde{X}^*_H}$):

$$R\Gamma_{\{x_1 < 0\}}(F^H)_0 \simeq \bigoplus^{\infty} \mathbb{C} \quad \text{and} \quad (F^H)_0 \simeq 0$$

that we have an isomorphism $\bigoplus^{\infty} \mathbb{C} \simeq R\Gamma_{\{x_1 \geqslant 0\}}(F^H)_0[1]$. Therefore, $M \simeq \bigoplus^{\infty} \mathbb{C}[-1]$ and we got the desired isomorphism $F^H \simeq \bigoplus^{\infty} \mathbb{C}_H[-1]$ in $\mathbf{D}^b(X; p)$. If we rotate the covector $p \in \dot{T}^*_H X$ in the clockwise direction, the $(k+1)$th component of $F^H \simeq \bigoplus^{\infty} \mathbb{C}_H[-1]$ is continued to the k-th component from the above proof ($F^H = p_{H!}\mathbb{C}_{\tilde{X}^*_H}$). Since $\mu hom(*, \mathcal{O}_X)$ is a contravariant functor, part (i) follows.

(ii) First we recall that we have the morphism of adjunction

$$F^H = p_{H!}p^!_H \mathbb{C}_X \to \mathbb{C}_X$$

from which one can deduce a morphism

$$\mathcal{O}_X|_H \simeq [R\pi_{X!}\mu hom(\mathbb{C}_X, \mathcal{O}_X)]_H \longrightarrow [R\pi_{X!}\mu hom(F^H, \mathcal{O}_X)]_H.$$

Let us prove that it is in fact an isomorphism. For a point $z \in H \subset X$, the stalk of $R\pi_{X!}\mu hom(F^H, \mathcal{O}_X)$ at z is expressed by

$$R\pi_{X!}\mu hom(F^H, \mathcal{O}_X)_z \simeq [R\mathcal{H}om(q_2^{-1}F^H, q_1^!\mathcal{O}_X)|_{\Delta_X}[-\dim^{\mathbb{R}} X]]_z,$$

where q_1 (resp. q_2) is the first (resp. second) projection from $X \times X$ to X and $\Delta_X \subset X \times X$ is the diagonal set. Thus, for any $i \in \mathbb{Z}$, by taking the inductive limit

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for open neighborhoods U of z in X, we have

$$\begin{aligned}
&H^i[R\pi_{X!}\mu hom(F^H, \mathcal{O}_X)]_z \\
&\quad\simeq \varinjlim_{z\in U} H^i R\Gamma(U \times U; R\mathcal{H}om(q_2^{-1}F^H[\dim^{\mathbb{R}} X], q_1^!\mathcal{O}_X)) \\
&\quad\simeq \varinjlim_{z\in U} H^i R\Gamma(U; Rq_{1*}R\mathcal{H}om(q_2^{-1}(F^H)_U[\dim^{\mathbb{R}} X], q_1^!\mathcal{O}_X)) \qquad (1)\\
&\quad\simeq \varinjlim_{z\in U} H^i R\Gamma(U; R\mathcal{H}om(Rq_{1!}q_2^{-1}(F^H)_U[\dim^{\mathbb{R}} X], \mathcal{O}_X)).
\end{aligned}$$

Now for any point $w \in U$

$$\begin{aligned}
&[Rq_{1!}q_2^{-1}(F^H)_U[\dim^{\mathbb{R}} X]]_w \\
&\quad\simeq R\Gamma_c(U; F^H)[\dim^{\mathbb{R}} X] = R\Gamma_c(p_H^{-1}(U); \mathbb{C}_{\tilde{X}_H^*})[\dim^{\mathbb{R}} X],
\end{aligned}$$

and if we take U so that $p_H^{-1}(U)$ is homeomorphic to $\mathbb{R}^{2n}$, $2n = \dim^{\mathbb{R}} X$, the right-hand side is isomorphic to $\mathbb{C}$. Replacing F^H with $\mathbb{C}_X$, we also have $[Rq_{1!}q_2^{-1}(\mathbb{C}_X)_U[\dim^{\mathbb{R}} X]]_w \simeq \mathbb{C}$ for $w \in U$ and we obtain the required isomorphism:

$$R\pi_{X!}\mu hom(\mathbb{C}_X, \mathcal{O}_X)_z \xrightarrow{\sim} R\pi_{X!}\mu hom(F^H, \mathcal{O}_X)_z.$$

This completes the proof. □

We set $\dot{\pi}_X: \dot{T}^*X \to X$ as usual. By applying Sato's distinguished triangle $R\pi_{X!} \to R\pi_{X*} \to R\dot{\pi}_{X*} \to +1$ to the sheaf $\mu hom(F^H, \mathcal{O}_X)$ we obtain the following corollary:

COROLLARY 2.5. *There exists an exact sequence on H:*

$$0 \to \mathcal{O}_X|_H \longrightarrow \mathcal{O}_{X|H}^{ram}|_H \to [\dot{\pi}_{X*}\mu hom(F^H, \mathcal{O}_X)]_H \to 0.$$

Now for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, set $F = \mathrm{Sol}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Then the complex

$$R\mathcal{H}om(F^H, F)_H \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X|H}^{ram})_H \in \mathbf{D}^b(H)$$

is the so-called nearby cycle of F and, by the above corollary the complex $[R\dot{\pi}_{X*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu hom(F^H, \mathcal{O}_X))]_H$ is the vanishing cycle of F. Hence, $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu hom(F^H, \mathcal{O}_X))_{\dot{T}_H^*X}$ is the microlocal object of the vanishing cycle of F. On the other hand, Kashiwara [10] obtained a (local) isomorphism $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{H|X}^{\mathbb{R}}) \simeq \mathbb{C}_{T_H^*X}^m$ for holonomic $\mathcal{D}_X$-modules satisfying the condition $char\mathcal{M} = T_H^*X$ for the characteristic varieties on an open subset of $\dot{T}_H^*X$. Here m stands for the multiplicity of $\mathcal{M}$ along T_H^*X. Since the sheaf $\mu hom(F^H, \mathcal{O}_X)$ is (locally on $\dot{T}_H^*X$) a simultaneous direct product of $\mathcal{C}_{H|X}^{\mathbb{R}}$ with a monodromy structure, it would be very interesting to study the microlocal vanishing cycle $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu hom(F^H, \mathcal{O}_X))_{\dot{T}_H^*X}$ by using the result of [10]. Note also that

Kashiwara and Kawai [13] obtained the canonical form of the holonomic $\mathcal{E}_X$-modules $\mathcal{M}$ satisfying the condition $char\mathcal{M} \subset T^*_H X$ for a complex hypersurface $H \subset X$. We will study the vanishing cycles for holonomic systems by these observations in a forthcoming paper.

Remark 2.6. Proposition 2.4(i) (monodromy structure) can be shown also by calculating the stalks of $\mu hom(F^H, \mathcal{O}_X)$ using Proposition 4.4.4 *of* [16], which was the starting point of this study. However to do so, we have to pay much more attention to the geometric settings, and we will not present this calculation here.

3. Microlocal Equivalence of Two Ramified Cauchy Problems

Roughly speaking, there are two types of ramified Cauchy problems for $\mathcal{D}_X$-modules, the one for logarithmic pole-type initial data as in [15] and the other for general ramified initial data due to D'Agnolo and Schapira [3]. The methods of the proof employed in these papers are completely different. That is, the proof of the former case [15] is essentially based on the result of [2] concerning the Cauchy–Kowalevski type theorem for pseudodifferential equations. To the contrary, the proof of the latter case of [3] is purely algebraic and makes use of the sophisticated theory of the microlocal inverse image. Here we will show that these two problems are microlocally (almost) equivalent. Outside the zero section of the cotangent bundle T^*X, the latter case turns out to be just the 'simultaneous direct product' of the former case. From now on, we will give a totally new proof of the work [3] which does not involve the use of a microlocal inverse image.

Until the end of this paper, X is a complex manifold of dimension n and $Y \subset X$ is a complex hypersurface. We also take a complex hypersurface Z of Y to consider the Cauchy problem with initial data in $\mathcal{O}^{ram}_{Z|Y}$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is noncharacteristic. We shall use the natural morphisms

$$T^*Y \underset{\rho}{\longleftarrow} Y \times_X T^*X \underset{\varpi}{\longrightarrow} T^*X$$

associated to the inclusion $f\colon Y \to X$. Assume that $\mathrm{char}\mathcal{M} \cap \rho^{-1}(\dot{T}^*_Z Y)$ is a disjoint union $\bigsqcup_{i=1}^r \Lambda_i$ of complex manifolds $\{\Lambda_i\}_{i=1}^r$ such that $\rho\,|_{\Lambda_i}\colon \Lambda_i \longrightarrow \dot{T}^*_Z Y$ are complex diffeomorphisms. For the sake of simplicity, we assume that the characteristic variety $V = \mathrm{char}\mathcal{M}$ is purely of codimension one in T^*X and V is smooth on each Λ_i $(i = 1, 2, \ldots, r)$. We choose local holomorphic defining functions g_i of V in an open neighborhood of Λ_i $(i = 1, 2, \ldots, r)$. Then the union of the integral curves of the complex Hamiltonian vector field H_{g_i} passing through Λ_i is Lagrangian, and can be expressed as $\dot{T}^*_{Z_i} X$ by a complex hypersurface Z_i of X satisfying $Z_i \cap Y = Z$. Finally, as in [15], define the sheaf $\sum_{i=1}^r \mathcal{O}^{ram}_{Z_i|X}$ by the exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{O}_X^{r-1} \underset{\lambda}{\longrightarrow} \bigoplus_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X} \longrightarrow \sum_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X} \longrightarrow 0,$$

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where the morphism λ is given by

$$(f_1, f_2, \ldots, f_{r-1}) \mapsto (f_1, f_2 - f_1, f_3 - f_2, \cdots, -f_{r-1}).$$

Under these assumptions, we can state the following theorem [3], which generalizes the theorem of [9] to the case of $\mathcal{D}_X$-modules.

THEOREM 3.1 ([3]). *There exists an isomorphism on* Y

$$R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X}\right)_Y \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{ram}_{Z|Y}),$$

where $\mathcal{M}_Y$ *is the induced system of* $\mathcal{M}$ *to* Y.

Proof. We shall give a new proof of this theorem. As in Chapter III, 2 of [22], consider the commutative diagram of the morphisms of sheaves on $Z \subset X$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{r-1} & \longrightarrow & \mathcal{O}_X^{r-1} & \longrightarrow & 0 & & \\
 & & \downarrow\lambda & & \downarrow\lambda & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{r} & \longrightarrow & \bigoplus_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X} & \longrightarrow & \bigoplus_{i=1}^{r} \dot{\pi}_{X_*}\mu hom(F^{Z_i}, \mathcal{O}_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \sum_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X} & \longrightarrow & \bigoplus_{i=1}^{r} \dot{\pi}_{X_*}\mu hom(F^{Z_i}, \mathcal{O}_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all columns and the top and middle rows are exact. Therefore by the nine lemma, we get an exact sequence of sheaves on Z:

$$0 \longrightarrow \mathcal{O}_X|_Z \longrightarrow \sum_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X}|_Z \longrightarrow \bigoplus_{i=1}^{r} [\dot{\pi}_{X*}\mu hom(F^{Z_i}, \mathcal{O}_X)]_Z \longrightarrow 0.$$

Note that the complex hypersurfaces Z_i's are transversal to each other and the supports of $\mu hom(F^{Z_i}, \mathcal{O}_X)$ $(i = 1, 2, \ldots, r)$ are disjoint in $\dot{T}^*X$. The following lemma, which is almost trivial by a simple calculation, is necessary to prove the theorem.

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LEMMA 3.2. *For any $i = 1, 2, \ldots, r$, there exist isomorphisms*

$$R\rho_*\left[\mathcal{E}_{Y\to X} \underset{\varpi^{-1}\mathcal{E}_X}{\overset{L}{\bigotimes}} \varpi^{-1}\mathcal{C}^{\mathbb{R}}_{Z_i|X}\right] \simeq \mathcal{C}^{\mathbb{R}}_{Z|Y},$$

$$R\rho_*\left[\mathcal{E}_{Y\to X} \underset{\varpi^{-1}\mathcal{E}_X}{\overset{L}{\bigotimes}} \varpi^{-1}\mu hom(F^{Z_i}, \mathcal{O}_X)\right] \simeq \mu hom(F^Z, \mathcal{O}_Y). \tag{2}$$

Proof. The first isomorphism is well known. The second one can be shown as follows:

$$\begin{aligned}
&R\rho_*\left[\mathcal{E}_{Y\to X} \underset{\varpi^{-1}\mathcal{E}_X}{\overset{L}{\bigotimes}} \varpi^{-1}\mu hom(F^{Z_i}, \mathcal{O}_X)\right] \\
&\quad\simeq R\rho_*\varpi^{-1}\mu hom\left(F^{Z_i}, f_*\left(\mathcal{D}_{Y\to X} \underset{f^{-1}\mathcal{D}_X}{\overset{L}{\bigotimes}} f^{-1}\mathcal{O}_X\right)\right) \simeq R\rho_*\varpi^{-1}\mu hom(F^{Z_i}, f_*\mathcal{O}_Y) \\
&\quad\simeq \mu hom(f^{-1}F^{Z_i}, \mathcal{O}_Y) = \mu hom(F^Z, \mathcal{O}_Y),
\end{aligned} \tag{3}$$

where we have used Corollary 6.7.6 of [16]. □

Let us continue the proof of the theorem. Setting $\pi_Y\colon T^*Y \to Y$ and $\dot{\pi}_Y\colon \dot{T}^*Y \to Y$, consider the natural morphism of distinguished triangles in $\mathbf{D}^b(Z)$:

$$\begin{array}{ccccccc}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_Z & \to & R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_{i=1}^{r} \mathcal{O}^{ram}_{Z_i|X}\right) & \to & R\dot{\pi}_{X_*} R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \bigoplus_{i=1}^{r} \mu hom(F^{Z_i}, \mathcal{O}_X)\right) & \overset{+1}{\to} \\
\downarrow & & \downarrow & & \downarrow & \\
R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)_Z & \to & R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{ram}_{Z|Y}) & \to & R\dot{\pi}_{Y_*} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mu hom(F^Z, \mathcal{O}_Y)) & \overset{+1}{\to}
\end{array}$$

where all vertical arrows are induced from the morphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, *)\,|_Y \to R\mathcal{H}om_{\mathcal{D}_Y}\left(\mathcal{D}_{Y\to X} \underset{f^{-1}\mathcal{D}_X}{\overset{L}{\bigotimes}} f^{-1}\mathcal{M}, \mathcal{D}_{Y\to X} \underset{f^{-1}\mathcal{D}_X}{\overset{L}{\bigotimes}} f^{-1}(*)\right)$$

and the left vertical arrow is an isomorphism by the Cauchy–Kowalevski–Kashiwara theorem. So, to complete the proof, it remains to show that the canonical morphism

$$\begin{aligned}
&R\rho_*\varpi^{-1}R\mathcal{H}om_{\mathcal{E}_X}\left(\tilde{\mathcal{M}}, \bigoplus_{i=1}^{r} \mu hom(F^{Z_i}, \mathcal{O}_X)\right) \longrightarrow \\
&R\rho_* R\mathcal{H}om_{\rho^{-1}\mathcal{E}_Y}\left(\mathcal{E}_{Y\to X} \underset{\varpi^{-1}\mathcal{E}_X}{\overset{L}{\bigotimes}} \varpi^{-1}\tilde{\mathcal{M}}, \bigoplus_{i=1}^{r} \mathcal{E}_{Y\to X} \overset{L}{\bigotimes}_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mu hom(F^{Z_i}, \mathcal{O}_X)\right) \\
&\quad\simeq R\mathcal{H}om_{\mathcal{E}_Y}(\tilde{\mathcal{M}}_Y, \mu hom(F^Z, \mathcal{O}_Y))
\end{aligned} \tag{4}$$

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(for $\tilde{\mathcal{M}} = \mathcal{E}_X \bigotimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$ and $\tilde{\mathcal{M}}_Y = \mathcal{E}_Y \bigotimes_{\pi_Y^{-1}\mathcal{D}_Y} \pi_Y^{-1}\mathcal{M}_Y$) is an isomorphism on $\dot{T}^*_Z Y$. Here the last equality follows from Lemma 3.2. If we consider the resolution $\bigoplus_{j=1}^N \mathcal{E}_X/\mathcal{E}_X P_j \to \tilde{\mathcal{M}} \to 0$ of $\tilde{\mathcal{M}}$ (by pseudodifferential operators P_j which are non-microcharacteristic for Y along $T^*_{Z_i}X$) of [15], by a standard argument, we only have to show the isomorphism:

$$\varpi^{-1} R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P_j, \mu hom(F^{Z_i}, \mathcal{O}_X)) \\ \xrightarrow{\sim} R\mathcal{H}om_{\rho^{-1}\mathcal{E}_Y}\left(\mathcal{E}_{Y\to X} \bigotimes^L_{\varpi^{-1}\mathcal{E}_X} \mathcal{E}_X/\mathcal{E}_X P_j, \mathcal{E}_{Y\to X} \bigotimes^L_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mu hom(F^{Z_i}, \mathcal{O}_X)\right) \quad (5)$$

at any point ${}^\forall p \in \Lambda_i$ for ${}^\forall j = 1, 2, \ldots, N$. Now recall that Kashiwara–Schapira [15] proved the isomorphism:

$$\varpi^{-1} R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P_j, \mathcal{C}^{\mathbb{R}}_{Z_i|X}) \\ \xrightarrow{\sim} R\mathcal{H}om_{\rho^{-1}\mathcal{E}_Y}\left(\mathcal{E}_{Y\to X} \bigotimes^L_{\varpi^{-1}\mathcal{E}_X} \mathcal{E}_X/\mathcal{E}_X P_j, \mathcal{E}_{Y\to X} \bigotimes^L_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mathcal{C}^{\mathbb{R}}_{Z_i|X}\right) \quad (6)$$

at p by applying a result of Bony and Schapira [2] on the Cauchy–Kowalevski-type theorem for pseudodifferential equations. Since $\mu hom(F^{Z_i}, \mathcal{O}_X)$ is the simultaneous direct product of $\mathcal{C}^{\mathbb{R}}_{Z_i|X}$, the isomorphism (5) follows from (the proof of) (6). It is enough to simultaneously solve the holomorphic Cauchy problems for infinitely many open subsets in X. This completes the proof. □

Let us summarize what we found in the course of the proof. The microlocal object that we encounter treating the case of logrithmic poles was $\bigoplus_{i=1}^r \mathcal{C}^{\mathbb{R}}_{Z_i|X}$. On the other hand, the corresponding object for the case of general ramified holomorphic functions was $\bigoplus_{i=1}^r \mu hom(F^{Z_i}, \mathcal{O}_X)$, namely, the simultaneous direct product of $\bigoplus_{i=1}^r \mathcal{C}^{\mathbb{R}}_{Z_i|X}$ (with a twisting by the monodromy structure). This observation enabled us to prove the theorem of D'Agnolo and Schapira [3] in the lines of Kashiwara and Schapira [15].

4. Ramified Cauchy Problems for Nilsson Class Functions

In this section, we show how our results in previous sections allow us to treat the ramified Cauchy problems for initial data with growth conditions. Here we consider the growth condition of the so-called Nilsson class, which has been introduced by Deligne [5]. We inherit the notations of two previous sections and recall the definition of the Nilsson class. Let $H \subset X$ be a complex hypersurface and consider the sheaf $\mathcal{O}^{ram}_{H|X}$ of holomorphic functions ramified along $H = \{t = 0\}$. We take a coordinate system (t, z') of X.

DEFINITION 4.1. We say a section $f \in \mathcal{O}^{ram}_{H|X} \simeq p_{H*}(\mathcal{O}_{\tilde{X}^*_H})$ is in the Nilsson class iff the restriction of f to the angular sector

$$S_{[a,b]} = \{(t, z') \in X;\ t \neq 0, arg(t) \in [a, b]\}$$

for $^{\forall}[a, b] \subset\subset \mathbb{R}$ satisfies the condition

$$^{\exists}N \gg 0 \quad s.t. \quad |\, f(t, z')\,| \times |\, t\,|^N \quad \text{is bounded on} \quad S_{[a,b]}.$$

We denote by $\mathcal{O}^{ram}_{[H|X]} \subset \mathcal{O}^{ram}_{H|X}$ the subsheaf consisting of sections in the Nilsson class.

Our main theorem in this section is as follows:

THEOREM 4.2. *We assume the conditions in Theorem 3.1. Assume, moreover, that the $\mathcal{D}_X$-module $\mathcal{M}$ has regular singularities (in the sense of [14]) along $V = char\mathcal{M}$ on each $\Lambda_i \subset V \cap \rho^{-1}(\dot{T}^*_Z Y)$ $(i = 1, 2, \ldots, r)$. Then we have an isomorphism:*

$$R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_{i=1}^{r} \mathcal{O}^{ram}_{[Z_i|X]}\right)_Y \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{ram}_{[Z|Y]}).$$

Proof. As is calculated in Proposition 2.2(iii), the stalk at $p \in \dot{T}^*_H X$ of $\mu hom(F^H, \mathcal{O}_X)$ for a complex hypersurface $H \subset X$ is equal to

$$\varinjlim_{S,U}\left[\prod^{\infty} \mathcal{O}_X(U - S) \Big/ \prod^{\infty} \mathcal{O}_X(U)\right].$$

where S ranges over the family of closed subsets of X s.t.

$$C_H(S)_{\pi_X(p)} \subset \{v \in (T_H X)_{\pi_X(p)};\ \langle v, p\rangle > 0\} \cup \{0\}$$

and U is an open neighborhood of $\pi_X(p) \in X$. If we replace $\mathcal{O}_X(U - S)$ (and $\mathcal{O}_X(U)$) by the tempered sections $\mathcal{O}^t_X(U - S)$ (and $\mathcal{O}^t_X(U)$), which are extendible as distributions to the whole X, we can construct the subsheaf $\mathrm{T} - \mu hom(F^H, \mathcal{O}_X)$ of $\mu hom(F^H, \mathcal{O}_X)$ on T^*X whose stalk at p is represented by

$$\varinjlim_{S,U}\left[\prod^{\infty} \mathcal{O}^t_X(U - S) \Big/ \prod^{\infty} \mathcal{O}^t_X(U)\right].$$

Though we cannot apply Andronikof's functor $\mathrm{T} - \mu hom(*, \mathcal{O}_X)$ in [1] to $F^H \in \mathbf{D}^b(X)$ (F^H is not $\mathbb{R}$-constructible), we adopt here the same notation. In fact, it would be possible to define $\mathrm{T} - \mu hom(G, \mathcal{O}_X)$ for arbitrary sheaves $G \in \mathbf{D}^b(X)$ if we use the very recent theory of Kashiwara and Schapira [17] on ind-sheaves.

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This sheaf $\mathrm{T}-\mu hom(F^H, \mathcal{O}_X)$ on $\dot{T}^*_H X$ fits into the exact sequence on H:

$$0 \to \mathcal{O}_X|_H \to \mathcal{O}^{ram}_{[H|X]}|_H \to \dot{\pi}_{X_*}\mathrm{T}-\mu hom(F^H, \mathcal{O}_X) \to 0.$$

The exactitude can be verified by calculating the complex $R\dot{\pi}_{X_*}\mathrm{T}-\mu hom(F^H, \mathcal{O}_X)$. Hence, as in the proof of Theorem 3.1, there exist an exact sequence.:

$$0 \to \mathcal{O}_X|_Z \to \sum_{i=1}^{r} \mathcal{O}^{ram}_{[Z_i|X]}|_Z \to \bigoplus_{i=1}^{r}[\dot{\pi}_{X_*}\mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X)]_Z \to 0$$

and a morphism of distinguished triangles in $\mathbf{D}^b(Z)$:

$$\begin{array}{ccccc} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_Z & \to & R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_{i=1}^{r}\mathcal{O}^{ram}_{[Z_i|X]}\right) & \to & R\dot{\pi}_{X_*}R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \bigoplus_{i=1}^{r}\mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X)\right) \\ \downarrow & & \downarrow & & \downarrow \\ R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)_Z & \to & R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{ram}_{[Z|Y]}) & \to & R\dot{\pi}_{Y_*}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathrm{T}-\mu hom(F^Z, \mathcal{O}_Y)) \quad \xrightarrow{+1} \end{array}$$

Therefore, to prove the theorem, it suffices to show the isomorphism:

$$\begin{aligned} &\varpi^{-1}R\mathcal{H}om_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X)) \\ &\quad \xrightarrow{\sim} R\mathcal{H}om_{\rho^{-1}\mathcal{E}_Y}\left(\mathcal{E}_{Y\to X}\overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\otimes}}\tilde{\mathcal{M}}, \mathcal{E}_{Y\to X}\overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\otimes}}\varpi^{-1}\mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X)\right) \end{aligned} \tag{7}$$

for $\tilde{\mathcal{M}} = \mathcal{E}_X \bigotimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$ at ${}^\forall p \in \Lambda_i \subset \dot{T}^*X$ $(i = 1, 2, \ldots, r)$. We need the next lemma to perform a quantized contact transformation.

LEMMA 4.3. *For* ${}^\forall p \in \Lambda_i \subset \dot{T}^*X$, *there exists a germ of complex contact transformation* $\chi\colon T^*X \xrightarrow{\sim} T^*\mathbb{C}^n_z$ *at* p *such that*

$$\begin{aligned} &\chi(Y \times_X T^*X) = \{z_1 = 0\}, \\ &\chi(char\mathcal{M}) = \{\zeta_1 = 0\}, \\ &\chi(T^*_{Z_i}X) = T^*_H\mathbb{C}^n \quad for \quad H = \{z_2 = 0\} \subset \mathbb{C}^n, \end{aligned} \tag{8}$$

where $(z; \zeta dz)$, $z = (z_1, z_2, \ldots, z_n)$ *is a local coordinate system of* $T^*\mathbb{C}^n$.

Proof. By Theorem A.4.4. of [22], there exists a contact transformation $\chi_0\colon T^*X \xrightarrow{\sim} T^*\mathbb{C}^n_z$ at p s.t.

$$\begin{aligned} &\chi_0(Y \times_X T^*X) = \{z_1 = 0\}, \\ &\chi_0(\mathrm{char}\mathcal{M}) = \{\zeta_1 = 0\}, \\ &\chi_0(\Lambda_i) \subset \{z_1 = \zeta_1 = 0\} = \{(0, 0)\} \times T^*\mathbb{C}^{n-1} \simeq T^*\mathbb{C}^{n-1}. \end{aligned} \tag{9}$$

Since Λ_i is isotropic in T^*X, $\chi_0(\Lambda_i)$ is also isotropic in $T^*\mathbb{C}^n$. If we regard $\chi_0(\Lambda_i)$ as a closed submanifold of $T^*\mathbb{C}^{n-1}$, it implies that $\chi_0(\Lambda_i)$ is a conic Lagrangian submanifold in $T^*\mathbb{C}^{n-1}$. Now let us choose a contact transformation $\chi_1\colon T^*\mathbb{C}^{n-1} \xrightarrow{\sim} T^*\mathbb{C}^{n-1}$ at $\chi_0(p) \in T^*\mathbb{C}^{n-1}$ which takes $\chi_0(\Lambda_i)$ to the conormal bundle $T^*_{H'}\mathbb{C}^{n-1}$ of $H' = \{z_2 = 0\} \subset \mathbb{C}^{n-1}$ (Theorem A.4.2 of [22]) and set

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$\chi = (\mathrm{id}_{T^*\mathbb{C}^1} \times \chi_1) \circ \chi_0 \colon T^*X \xrightarrow{\sim} T^*\mathbb{C}^n_z$. Then

$$\begin{aligned} &\chi(Y \times_X T^*X) = \{z_1 = 0\}, \\ &\chi(\mathrm{char}\mathcal{M}) = \{\zeta_1 = 0\}, \\ &\chi(\Lambda_i) = \{(0,0)\} \times T^*_{H'}\mathbb{C}^{n-1} \subset T^*\mathbb{C}^1 \times T^*\mathbb{C}^{n-1}. \end{aligned} \tag{10}$$

It follows from the construction of $T^*_{Z_i}X$ that $\chi(T^*_{Z_i}X)$ is the union of the integral curves of the complex Hamiltonian vector field $H_{\zeta_1} = \partial_{z_1}$ passing through $\chi(\Lambda_i) = \{(0,0)\} \times T^*_{H'}\mathbb{C}^{n-1}$, that is, $T^*_H\mathbb{C}^n$ $(H = \{z_2 = 0\} \subset \mathbb{C}^n_z)$. □

Let us continue the proof of the theorem. We shall use the contact transformation χ at ${}^\forall p \in \Lambda_i$ in the above lemma and denote the associated integral transformation of sheaves by Φ. Then

$$\Phi(F^{Z_i}) \simeq \Phi\left(\bigoplus^{\infty} \mathbb{C}_{Z_i}[-1]\right) \simeq \bigoplus^{\infty} \Phi(\mathbb{C}_{Z_i})[-1] \simeq \bigoplus^{\infty} \mathbb{C}_H[-1]$$

at $\chi(p) \in \dot{T}^*_H\mathbb{C}^n$ and we have an isomorphism:

$$\chi_*\mu hom(F^{Z_i}, \mathcal{O}_X) \simeq \mu hom\left(\bigoplus^{\infty} \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1].$$

We will show that the temperedness is preserved by this quantized contact transformation, i.e.

$$\chi_*\mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X) \simeq \mathrm{T}-\mu hom\left(\bigoplus^{\infty} \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1].$$

If we take the kth component $\mathcal{C}^{\mathbb{R}}_{Z_i|X} \subset \mu hom(F^{Z_i}, \mathcal{O}_X)$ and set

$$\mathcal{C}^{\mathbb{R},f}_{Z_i|X} := \mathcal{C}^{\mathbb{R}}_{Z_i|X} \cap \mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X),$$

it follows from the theory of Andronikof [1] that $\chi_*\mathcal{C}^{\mathbb{R},f}_{Z_i|X}$ is isomorphic to the tempered kth component

$$\mathcal{C}^{\mathbb{R},f}_{H|\mathbb{C}^n} = \mathcal{C}^{\mathbb{R}}_{H|\mathbb{C}^n} \cap \mathrm{T}-\mu hom\left(\bigoplus^{\infty} \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1].$$

This shows that the temperedness is preserved through the contact transformation χ. Hence, after the quantized contact transformation, it remains to show the isomorphism:

$$\begin{aligned} &R\mathcal{H}\mathrm{om}_{\mathcal{E}_{\mathbb{C}^n}}\left(\tilde{\mathcal{N}}, \mathrm{T}-\mu hom\left(\bigoplus^{\infty} \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1]\right) \\ &\quad \xrightarrow{\sim} R\mathcal{H}\mathrm{om}_{\mathcal{E}_{\mathbb{C}^{n-1}}}\left(\mathcal{E}_{\mathbb{C}^{n-1}\to\mathbb{C}^n} \overset{L}{\underset{\mathcal{E}_{\mathbb{C}^n}}{\bigotimes}} \tilde{\mathcal{N}}, \mathcal{E}_{\mathbb{C}^{n-1}\to\mathbb{C}^n} \overset{L}{\underset{\mathcal{E}_{\mathbb{C}^n}}{\bigotimes}} \mathrm{T}-\mu hom\left(\bigoplus^{\infty} \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1]\right) \end{aligned} \tag{11}$$

at $\chi(p) \in \dot{T}^*_H\mathbb{C}^n$ for the $\mathcal{E}_{\mathbb{C}^n}$-module $\tilde{\mathcal{N}} = \chi(\tilde{\mathcal{M}})$. Since $\tilde{\mathcal{N}}$ has regular singularities along $\chi(\mathrm{char}\mathcal{M}) = \{\zeta_1 = 0\}$, $\tilde{\mathcal{N}}$ has a resolution of the type:

$$\to \mathcal{S}^{N_k} \to \mathcal{S}^{N_{k-1}} \cdots \to \mathcal{S}^{N_0} \to \tilde{\mathcal{N}} \to 0,$$

by the simple system $\mathcal{S} = \mathcal{E}_{\mathbb{C}^n}/\mathcal{E}_{\mathbb{C}^n}D_1$, whose existence is a consequence of [14]. By the standard argument (see, for example, [4]), it suffices to show (11) for $\tilde{\mathcal{N}} = \mathcal{S} = \mathcal{E}_{\mathbb{C}^n}/\mathcal{E}_{\mathbb{C}^n}D_1$. For $\tilde{\mathcal{N}} = \mathcal{S}$ the left-hand side of (11) is isomorphic to $\mathrm{T} - \mu hom(\bigoplus^\infty \mathbb{C}_{H\cap\{z_1=0\}}, \mathcal{O}_{\mathbb{C}^{n-1}})[1]$ at $\chi(p)$. Since we have, at $\chi(p)$, $\mathcal{E}_{\mathbb{C}^{n-1}\to\mathbb{C}^n} \otimes^L_{\mathcal{E}_{\mathbb{C}_n}} \mathcal{S} \simeq \mathcal{E}_{\mathbb{C}^{n-1}}$ and

$$\begin{aligned}\mathcal{E}_{\mathbb{C}^{n-1}\to\mathbb{C}^n} \overset{L}{\underset{\mathcal{E}_{\mathbb{C}^n}}{\bigotimes}} \mathrm{T} - \mu hom\left(\bigoplus^\infty \mathbb{C}_H, \mathcal{O}_{\mathbb{C}^n}\right)[1] \\ \simeq \mathrm{T} - \mu hom\left(\bigoplus^\infty \mathbb{C}_{H\cap\{z_1=0\}}, \mathcal{O}_{\mathbb{C}^{n-1}}\right)[1]\end{aligned}$$

holds by the lemma below, the right-hand side of (11) is equal to te left-hand side. This completes the proof. □

LEMMA 4.4. *For any $i = 1, 2, \ldots, r$, there exist isomorphisms*

$$\begin{aligned} R\rho_*\left[\mathcal{E}_{Y\to X} \overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\bigotimes}} \varpi^{-1}\mathcal{C}^{\mathbb{R},f}_{Z_i|X}\right] &\simeq \mathcal{C}^{\mathbb{R},f}_{Z|Y}, \\ R\rho_*\left[\mathcal{E}_{Y\to X} \overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\bigotimes}} \varpi^{-1}T - \mu hom(F^{Z_i}, \mathcal{O}_X)\right] &\simeq T - \mu hom(F^Z, \mathcal{O}_Y). \end{aligned} \tag{12}$$

Proof. We can construct a natural morphism from

$$R\rho_*\left[\mathcal{E}_{Y\to X} \overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\bigotimes}} \varpi^{-1}\mathrm{T} - \mu hom(F^{Z_i}, \mathcal{O}_X)\right]$$

to

$$R\rho_*\left[\mathcal{E}_{Y\to X} \overset{L}{\underset{\varpi^{-1}\mathcal{E}_X}{\bigotimes}} \varpi^{-1}\mu hom(F^{Z_i}, \mathcal{O}_X)\right] \simeq \mu hom(F^Z, \mathcal{O}_Y)$$

by Lemma 3.2. We will verify that its image is equal to $\mathrm{T} - \mu hom(F^Z, \mathcal{O}_Y)$. By the definition of $\mathrm{T} - \mu hom(F^{Z_i}, \mathcal{O}_X)$, its stalk at $p \in \Lambda_i \subset \dot{T}^*_{Z_i}X$ is

$$\varinjlim_{S,U}\left[\prod^\infty \mathcal{O}^t_X(U - S) \Big/ \prod^\infty \mathcal{O}^t_X(U)\right],$$

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where S ranges over the family of closed subsets of X s.t.

$$C_{Z_i}(S)_{\pi_X(p)} \subset \{v \in (T_{Z_i}X)_{\pi_X(p)};\ \langle v, p\rangle > 0\} \cup \{0\}$$

and U is an open neighborhood of $\pi_X(p) \in X$. If $Y = \{z_1 = 0\}$ in X, the restriction of $R\rho_*\Big[\mathcal{E}_{Y\to X} \otimes^L_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mathrm{T}\mu hom(F^{Z_i}, \mathcal{O}_X)\Big]$ to the point p is the complex:

$$0 \longrightarrow \varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big] \underset{z_1\times}{\longrightarrow} \varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big] \longrightarrow 0.$$

Now we can show by a simple calculation:

$$\mathrm{Ker}\Big[\varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big] \underset{z_1\times}{\longrightarrow} \varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big]\Big] \xrightarrow{\sim} 0.$$

$$\begin{aligned}&\mathrm{Coker}\Big[\varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big]\\ &\qquad \underset{z_1\times}{\longrightarrow} \varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_X(U-S)\Big/\prod^{\infty}\mathcal{O}^t_X(U)\Big]\Big]\\ &\qquad \simeq \varinjlim_{S,U}\Big[\prod^{\infty}\mathcal{O}^t_Y(\{U-S\}\cap Y)\Big/\prod^{\infty}\mathcal{O}^t_Y(U\cap Y)\Big],\end{aligned} \tag{13}$$

which implies the isomorphism

$$R\rho_*\Big[\mathcal{E}_{Y\to X} \bigotimes^L_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mathrm{T}-\mu hom(F^{Z_i}, \mathcal{O}_X)\Big] \simeq \mathrm{T}-\mu hom(F^{z}, \mathcal{O}_Y).$$

This completes the proof. □

Remark 4.5. In the case of single differential operators, the condition of regular singularities is the so-called Levi condition in the classical literatures. As for the examples of systems which satisfy this condition of theorem 4.2, one may consult section 6 of [4] . We can find examples of square matrix type systems there.

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