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EXISTENCE AND L_∞ ESTIMATES FOR A CLASS OF SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

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We prove the existence of a positive solution to an equation of the form $\big(1/\Phi(t)\big)\big(\Phi(t)u'(t)\big)' = f\big(u(t)\big)$ with Dirichlet conditions where the friction term Φ'/Φ is increasing. Our method combines variational and topological arguments and provides an L_∞ estimate of the solution.

1. INTRODUCTION AND MAIN RESULT

We are interested in the existence and estimation of the L_∞-norm of a positive solution to the problem

$$\big(\Phi(t)u'(t)\big)' + \Phi(t)f\big(u(t)\big) = 0 \tag{1}$$

$$u(0) = u(1) = 0 \tag{2}$$

We suppose that $\Phi \in C^1(]0,1])$,

$$\Phi \geqslant m > 0 \text{ in }]0,1], \tag{3}$$

and

$$\frac{\Phi'(t)}{\Phi(t)} \text{ is an increasing function.} \tag{4}$$

Moreover we assume that f is locally Lipchitz,

$$f(0) = 0, \tag{5}$$

and that there exists $M_0 > 0$ such that

$$f(t) < 0 \text{ if } t < M_0, \text{ and } f(t) > 0 \text{ if } t > M_0. \tag{6}$$

Note that an equivalent formulation to problem (1)–(2) is

$$u''(t) + a(t)u'(t) + f\big(u(t)\big) = 0, \qquad u(0) = u(1) = 0$$

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where the friction term $a(t)$ is an increasing function. There is a vast literature dealing with existence of solutions of singular boundary value problems (see for instance [**1, 3, 4, 5**] and the references therein). In this work, however, by restricting ourselves to a particular class of equations, we manage to provide L_∞ estimates to the solutions even when the non-linear term $f(.)$ has arbitrary growth. Our method is inspired by the topological shooting method (see for instance [**2**]) and the classical mountain pass theorem of Ambrosetti and Rabinowitz [**6**] and it is settled in Section 2. In Section 3 we prove our main result:

Theorem 1. *Suppose that $\Phi \in C^1(]0,1])$, f is a locally Lipchitz continuous function and that conditions* (3)–(6) *are satisfied. Moreover suppose that*

$$\frac{\Phi(s)}{\Phi(t)} \leqslant K \text{ for some } K > 0 \text{ and every } 0 < t \leqslant s \leqslant 1, \tag{7}$$

and that there exists a nonnegative $v \in H_0^1(]0,1[)$ such that

$$\frac{1}{2}\int_0^1 \Phi(t)v'(t)^2\,dt < \int_0^1 \Phi(t)F\big(v(t)\big)\,dt < \infty, \tag{8}$$

where $F(v) = \int_0^v f(s)\,ds$.

Then Problem (1)–(2) *has a positive solution u such that $M_0 < \max u \leqslant \|v\|_\infty$.*

As motivating examples we may consider $\Phi(t) = t^{-\alpha}$ or $\Phi(t) = \exp\left(t^{-\alpha}\right)$ with $\alpha > 0$ (the reader may easily verify that these functions fulfill conditions (3), (4) and (7)).

2. Variational setting and auxiliary results

Throughout this section we assume that there exist $m, m^*, L > 0$ such that, for all $t \in]0,1[$,

$$0 < m \leqslant \Phi(t) \leqslant m^* \quad , \quad \left|\Phi'(t)\right| \leqslant L, \tag{9}$$

and

$$\frac{\Phi'(t)}{\Phi(t)} \text{ is a strictly increasing function.} \tag{10}$$

Since we are looking for positive solutions we assume that f is extended by zero in $]-\infty,0]$. The reader may easily verify that any non-trivial solution to (1)–(2) where f has been extended by zero should be positive in $]0,1[$ therefore being a solution of the initial problem. We shall consider the Sobolev space $H = H_0^1(]0,1[)$ consisting of absolutely continuous functions u such that

$$\|u\|^2 = \int_0^1 u'^2(t)\,dt < \infty\ ,\ u(0) = u(1) = 0.$$

Problem (1)–(2) may be viewed as the Euler-Lagrange equation of the functional $J : H \to \mathbb{R}$ defined by:

$$J(u) = \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)\,dt - \int_0^1 \Phi(t)F\big(u(t)\big)\,dt$$

where $F(u) = \displaystyle\int_0^u f(s)\,ds$. We shall suppose that J satisfies the following property:

$$\exists\, v \in H : J(v) < 0. \tag{11}$$

REMARK 1. Property (11) is trivially satisfied if, for some $\varepsilon > 0$, $f(u) \geqslant \varepsilon u^\alpha - C$ for all $u \geqslant 0$, where $\alpha > 1$ and $C > 0$.

Let $\overline{M} = \|v\|_\infty$. Since $J(w) \geqslant 0$ for every $w \in H$ with $\|w\|_\infty \leqslant M_0$, we have $\overline{M} > M_0$. Given $M \in [M_0, \overline{M}]$, we shall consider the following subset of H:

$$\mathfrak{C}_M = \{u \in H : \max u \geqslant M\},$$

and the truncated functional $J_M : H \to \mathbb{R}$,

$$J_M(u) = \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)\,dt - \int_0^1 \Phi(t)F_M\big(u(t)\big)\,dt$$

where

$$F_M(u) = \begin{cases} F(u) & \text{if } u \leqslant M \\ F(M) & \text{if } u > M \end{cases}.$$

REMARK 2. From the compact injection of $H_0^1\big(]0,1[\big)$ in $C\big([0,1]\big)$ we conclude that $\mathfrak{C}_M$ is weakly sequentially closed and that J_M is coercive and weakly lower semi-continuous.

The main result of this section is the following proposition whose proof will become clear after a sequence of lemmas:

PROPOSITION 2. *Let $f \in C\big([0,+\infty[\big)$ and $\Phi \in C^1\big([0,1]\big)$ satisfy respectively properties (5)–(6) and (9)–(10). Moreover, suppose that J satisfies property (11) and let $\overline{M} = \|v\|_\infty$. Then there exists a classical positive solution u to problem (1)–(2) with $M_0 < \max u \leqslant \overline{M}$.*

We shall be interested in the family of minimisers u_M of J_M in $\mathfrak{C}_M$ where $M \in [M_0, \overline{M}]$. By Remark 2 we know that u_M exists for every $M \in [M_0, \overline{M}]$. We also know that:

LEMMA 3. *Let u_M be a minimiser of J_M in $\mathfrak{C}_M$. Then $\max u = M$.*

PROOF: Given $w \in \mathfrak{C}_M$ define

$$\overline{w}(t) = \min\{w(t), M\}.$$

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If $\overline{w} \neq w$ then,

$$\int_0^1 \Phi \overline{w}'^2 < \int_0^1 \Phi w'^2$$

and

$$\int_0^1 \Phi F_M(\overline{w}) = \int_0^1 \Phi F_M(w).$$

Therefore $J_M(\overline{w}) < J_M(w)$ and the lemma follows. ∎

Given $M \in [M_0, \overline{M}]$, we define two types of minimisers of J_M in $\mathfrak{C}_M$:

DEFINITION: Let u_M be a minimiser of J_M in $\mathfrak{C}_M$.

(i) We say that u_M is a minimiser of *type A* if there exists a unique $t_0 \in]0,1[$ such that $u(t_0) = M$.

(ii) We say that u_M is a minimiser of *type B* if, given t_α(respectively t_β) $= \min$(respectively $\max$)$\{t : u_M(t) = M\}$, we have

$$u'_-(t_\alpha) = u'_+(t_\beta) = 0.$$

REMARK 3. If u_M is a minimiser of type A then u satisfies equation (1) in $]0, t_0[\cup]t_0, 1[$ since $J'_M(u)v = J'(u)v$ for every $v \in C_0^\infty(]0, t_0[\cup]t_0, 1[)$. For the same reason, if v_M is a type B minimiser, it satisfies equation (1) in $]0, t_\alpha[\cup]t_\beta, 1[$. If w is simultaneously of type A and B, then w is a classical solution to problem (1)–(2).

LEMMA 4. *Let u be a minimiser of J_M in $\mathfrak{C}_M$. Then u is of type A or B (possibly both).*

PROOF: We may rephrase the lemma as:

Let t_α (t_β) $= \min(\max)\{t : u(t) = M\}$. Then either $u'_-(t_\alpha) = u'_+(t_\beta) = 0$ or $t_\alpha = t_\beta$.

Integrating equation (1) between t_1 and t_2 and letting $t_1, t_2 \to t_\alpha$ (t_β), we conclude that $u'_-(t_\alpha)$ $(u'_+(t_\beta))$ is well defined. Suppose, in view of a contradiction, that $t_\alpha < t_\beta$ and $u'_-(t_\alpha) > 0$ (the case $u'_+(t_\beta) < 0$ is treated with similar arguments). Choose $\theta, \varepsilon > 0$ such that $u'(t) \geqslant \theta$ for every $t \in]t_\alpha - \varepsilon, t_\alpha[$ (we may suppose $\varepsilon < t_\beta - t_\alpha$) and consider:

$$v_\varepsilon(t) = -\big(|t - t_\alpha| - \varepsilon\big)_- . \tag{12}$$

We assert that, for a small ε,

$$\lim_{s \to 0} \frac{J_M(u + sv_\varepsilon) - J_M(u)}{s} < 0. \tag{13}$$

If (13) holds, then for a sufficiently small $s^* > 0$, we have $u + s^*v_\varepsilon \in \mathfrak{C}_M$ (since $(u + s^*v_\varepsilon)(t_\beta) = M$) and $J_M(u + s^*v_\varepsilon) < J_M(u)$ which contradicts the assumption that u is a minimiser of J_M in $\mathfrak{C}_M$.

In fact, Lemma 3 and (12) imply $u + s^* v_\varepsilon \leqslant M$. Therefore

$$\begin{aligned}\lim_{s\to 0} & \frac{J_M(u+sv_\varepsilon)-J_M(u)}{s}\\ &= \lim_{s\to 0}\frac{J(u+sv_\varepsilon)-J(u)}{s}\\ &= \int_0^1 \Phi(t)u'(t)v_\varepsilon'(t)\,dt - \int_0^1 \Phi(t)f\big(u(t)\big)v_\varepsilon(t)\,dt\\ &\leqslant -\theta\int_{t_\alpha-\varepsilon}^{t_\alpha}\Phi(t) + \int_{t_\alpha}^{t_\alpha+\varepsilon}\Phi(t)u'(t)\,dt - \int_{t_\alpha-\varepsilon}^{t_\alpha+\varepsilon}\Phi(t)f\big(u(t)\big)v_\varepsilon(t)\,dt.\end{aligned}$$

We observe that, by (9),

$$-\theta\int_{t_\alpha-\varepsilon}^{t_\alpha}\Phi(t) \leqslant -m\theta\varepsilon \tag{14}$$

and for some $C > 0$ independent of ε,

$$\int_{t_\alpha-\varepsilon}^{t_\alpha+\varepsilon}\Phi(t)f\big(u(t)\big)v_\varepsilon(t)\,dt \leqslant C\varepsilon^2. \tag{15}$$

Also, by Holder's inequality, (9) and Lemma 3, for $\gamma(t)\in]0,1[$,

$$\begin{aligned}\int_{t_\alpha}^{t_\alpha+\varepsilon}\Phi(t)u'(t)\,dt &= \int_{t_\alpha}^{t_\alpha+\varepsilon}\big(\Phi(t_\alpha)+\Phi'(t_\alpha+\gamma(t)(t-t_\alpha))(t-t_\alpha)\big)u'(t)\,dt\\ &\leqslant \Phi(t_\alpha)(u(t_\alpha+\varepsilon)-M) + L\|u\|\varepsilon^{3/2} \leqslant L\|u\|\varepsilon^{3/2}.\end{aligned} \tag{16}$$

Therefore, by (14), (15) and (16) we have

$$\lim_{s\to 0}\frac{J_M(u+sv_\varepsilon)-J_M(u)}{s} \leqslant -m\theta\varepsilon + C\varepsilon^2 + L\|u\|\varepsilon^{3/2},$$

and the assertion follows for sufficiently small ε. ∎

In the next lemma we provide a sharper characterisation of a type A minimiser.

Lemma 5. *Let u be a minimiser of J_M in $\mathfrak{C}_M$ of type A. Then*

(i) $u'_-(t_0) > 0$ *and* $u'_+(t_0) < 0$ *or*

(ii) $u'(t_0) = 0$.

Proof: In view of a contradiction, suppose that $u'_-(t_0) = 0$ and $u'_+(t_0) < 0$ (the reversed case is proved with similar arguments). Consider the following perturbation function:

$$w_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant t_0-\varepsilon\\ 4(t-t_0+\varepsilon) & \text{if } t_0-\varepsilon < t \leqslant t_0-\varepsilon/2\\ -6(t-t_0+\varepsilon/2)+2\varepsilon & \text{if } t_0-\varepsilon/2 < t \leqslant t_0\\ t-t_0-\varepsilon & \text{if } t_0 < t \leqslant t_0+\varepsilon\\ 0 & \text{if } t_0+\varepsilon < t \leqslant 1\end{cases}.$$

Trivially, for sufficiently small ε, $w_\varepsilon \in H$. Given $\lambda > 0$, since F_M is a Lipchitz function, we have for some $C_1 > 0$ independent of ε,

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t)F_M(u+\lambda w_\varepsilon)(t)\,dt \geqslant \int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t)F_M\big(u(t)\big)\,dt - C_1\varepsilon^2\lambda. \tag{17}$$

Also

$$\begin{aligned}\frac{1}{2}\int_{t_0-\varepsilon}^{t_0+\varepsilon} &\Phi(t)(u'+\lambda w'_\varepsilon)^2(t)\,dt \\ &= \frac{1}{2}\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t){u'}^2(t)\,dt + \lambda\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t)u'(t)w'_\varepsilon(t)\,dt + \frac{\lambda^2}{2}\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t){w'_\varepsilon}^2(t)\,dt \\ &\leqslant \frac{1}{2}\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t){u'}^2(t)\,dt + \lambda\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t)u'(t)w'_\varepsilon(t)\,dt + C_2\lambda^2\varepsilon, \qquad (18)\end{aligned}$$

where $C_2 = 36m^*$. Note that, by (17) and (18), we have

$$J_M(u+\lambda w_\varepsilon) \leqslant J_M(u) + \Psi(\lambda,\varepsilon) \tag{19}$$

with

$$\Psi(\lambda,\varepsilon) = \lambda\int_{t_0-\varepsilon}^{t_0+\varepsilon} \Phi(t)u'(t)w'_\varepsilon(t)\,dt + C(\lambda^2\varepsilon + \varepsilon^2\lambda),$$

where $C = \max\{C_1, C_2\}$. Our purpose is to show the existence of $\lambda, \varepsilon > 0$ such that $\Psi(\lambda,\varepsilon) < 0$ and $u + \lambda w_\varepsilon \in \mathfrak{C}_M$, obtaining a contradiction from (19). Since $u'_-(t_0) = 0$ and $u'_+(t_0) < 0$, by (9) we may take $\varepsilon_0, \theta > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$,

$$\big|\Phi(s)u'(s){w_\varepsilon}'(s)\big| < \theta/2 \text{ if } s \in [t_0-\varepsilon, t_0]$$

and

$$\Phi(s)u'(s) \leqslant -\theta \text{ if } s \in [t_0, t_0+\varepsilon].$$

Then

$$\Psi(\lambda,\varepsilon) \leqslant -\frac{\theta}{2}\lambda\varepsilon + C\lambda\varepsilon(\varepsilon+\lambda),$$

and, if we fix $\lambda = \theta/(4C)$, we have, for $\varepsilon < \min\{\theta/(4C), \varepsilon_0\}$,

$$\Psi(\lambda,\varepsilon) < 0. \tag{20}$$

In order to insure that $u + \lambda w_\varepsilon \in \mathfrak{C}_M$, take ε_1 such that if $s \in]t_0-\varepsilon_1, t_0[$, then

$$6\lambda - u'(s) > 2\lambda. \tag{21}$$

We have, for $\varepsilon < \min\{\theta/(4C), \varepsilon_1, \varepsilon_0\}$,

$$\begin{aligned}(u+\lambda w_\varepsilon)\Big(t_0 - \frac{\varepsilon}{2}\Big) &= (u+\lambda w_\varepsilon)(t_0) + \int_{t_0}^{t_0-(\varepsilon/2)} (u'+\lambda w'_\varepsilon)(s)\,ds \\ &= M - \lambda\varepsilon + \int_{t_0-(\varepsilon/2)}^{t_0} (6\lambda - u'(s))\,ds,\end{aligned}$$

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and by (21),

$$(u+\lambda w_\varepsilon)\Big(t_0-\frac{\varepsilon}{2}\Big)\geqslant M,$$

then $u+\lambda w_\varepsilon\in\mathfrak{C}_M$ and the proof is concluded. ∎

In the next lemma we establish an important fact concerning the coexistence of type A and type B minimisers of J_M in $\mathfrak{C}_M$:

Lemma 6. *Suppose that for a certain $M\in[M_0,\overline{M}]$ there exist minimisers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then u is of type B (therefore being a classical solution to problem* (1)–(2)*).*

Proof: Since u is of type A let t_0 be the point where u equals M. Since v is of type B, let t_α(respectively t_β)= min(respectively max)$\{t:v(t)=M\}$ and

$$v'_-(t_\alpha)=v'_+(t_\beta)=0.$$

We have $t_0\leqslant t_\beta$ or $t_0\geqslant t_\alpha$. Suppose, in view of a contradiction, that $t_0\leqslant t_\beta$ and u is not of type B (the other case is proved with similar arguments). Then by Lemma 5 we have $u'_+(t_0)<0$.

Claim. For every $t\in]t_\beta,1[$, $u(t)<v(t)$. In particular, $u'(1)>v'(1)$.

Suppose that for some $t^*\in]t_\beta,1[$ we had $u(t^*)=v(t^*)$ and $u'(t^*)>v'(t^*)$ (the case $u'(t^*)=v'(t^*)$ is excluded by the existence and uniqueness theorem). Moreover, suppose that

$$\frac12\int_{t^*}^1\Phi u'^2-\int_{t^*}^1\Phi F_M(u)\leqslant\frac12\int_{t^*}^1\Phi v'^2\,dt-\int_{t^*}^1\Phi F_M(v),\tag{22}$$

and let

$$v^*(t)=\begin{cases}v(t)&\text{if }\ 0\leqslant t\leqslant t^*\\ u(t)&\text{if }\ t^*<t\leqslant 1\end{cases}.$$

Then $v^*\in H$ and

$$J_M(v^*)\leqslant J_M(v),$$

therefore v^* is also a minimiser in $\mathfrak{C}_M$. This is absurd since v^* is not differentiable at t^* (see remark 3). In case where, instead of (22), we had the reversed inequality we get the same contradiction by considering:

$$u^*(t)=\begin{cases}u(t)&\text{if }\ 0\leqslant t\leqslant t^*\\ v(t)&\text{if }\ t^*<t\leqslant 1\end{cases}.$$

The strict inequality $u'(1)>v'(1)$ is a consequence of the existence and uniqueness theorem and the claim is proved.

Let

$$\hat t=\sup\big\{t:\ t_0\leqslant t\leqslant 1:\ u'(s)\leqslant 0\ \forall s\in[t_0,t]\big\}.$$

We assert that
$$u(\widehat{t}) < M_0 \text{ and } u'(\widehat{t}) > v'(1).$$
In fact, if $\widehat{t} = 1$ the assertion simply follows from the previous claim. If $\widehat{t} < 1$, we may conclude from equation (1) and our assumptions on f that $u(\widehat{t}) < M_0$ and $u'(\widehat{t}) = 0$ (in fact, $u(\widehat{t})$ is a local minimum of u).

Similarly, if we define
$$\bar{t} = \inf\{t : \ t_\beta \leqslant t \leqslant 1 \ v'(s) \leqslant 0 \ \forall s \in [t, 1]\},$$
we have that
$$v'(\bar{t}) = 0 \text{ and } v(\bar{t}) > M_0$$
(in fact, $v(\widehat{t})$ is a local maximum of v). Then, if we consider in the phase plane (x, x') the curves U and V corresponding to $u|_{[t_0,\widehat{t}]}$ and $v|_{[\bar{t},1]}$, they must intersect at some point $P = (\mu, \mu')$ in the fourth quadrant. That is, for some $t_1 < t_2$,
$$u(t_1) = v(t_2) = \mu \text{ and } u'(t_1) = u'(t_2) = \mu' < 0.$$
Moreover we may suppose that P is such that μ is minimal. Let $T = t_2 - t_1$ and consider $v_T(t) = v(t + T)$. This translate of v satisfies, for every $t \in]t_1, 1 - t_2 + t_1[$,
$$\big(\Phi(t+T)v_T{}'(t)\big)' = -\Phi(t+T)f\big(v_T(t)\big),$$
or equivalently
$$v_T{}'' = -f\big(v_T(t)\big) - v_T{}'(t)\frac{\Phi'(t+T)}{\Phi(t+T)},$$
with initial conditions $v_T(t_1) = u(t_1)$ and $v_T'(t_1) = u'(t_1)$. However, u is a solution to
$$u'' = -f\big(u(t)\big) - u'(t)\frac{\Phi'(t)}{\Phi(t)}.$$
Since Φ'/Φ is strictly increasing and $-u'(t_1) = -v_T'(t_1) > 0$, we obtain

(23) $$u''(t_1) < v_T''(t_1).$$

Again, by considering in the phase plane (x, x') the curves corresponding to $v_T|_{[t_1,1-t_2+t_1]}$ and $u|_{[t_1,1]}$, we conclude from (23) the existence of $(\widehat{\mu}, \widehat{\mu}')$ in the fourth quadrant, with $\widehat{\mu} < \mu$, such that, for some $t_1 < t' < 1 - t_1 + t_2$,
$$u(t_1) = v_T(t') = \widehat{\mu} \text{ and } u'(t_1) = v_T(t') = \widehat{\mu}',$$
or, for $t_3 = t' + t_2 - t_1$,
$$u(t_1) = v(t_3) = \widehat{\mu} \text{ and } u'(t_1) = v'(t_3) = \widehat{\mu}' < 0.$$

But this contradicts our assumption that μ is minimal. We may conclude that if $t_0 \leqslant t_\beta$, u must be of type B.

If $t_0 \geqslant t_\alpha$ the proof is identical and we shall just sketch it. By Lemma 5 we have $u'(t_0) > 0$. With a similar reasoning to the one in the claim we may prove that $v'(0) > u'(0) > 0$. We define

$$\widehat{t^*} = \inf\{t : \ 0 \leqslant t \leqslant t_0 \ , \ u'(s) \geqslant 0 \ \forall s \in [t, t_0]\}$$

and

$$\overline{t^*} = \sup\{t : \ 0 \leqslant t \leqslant t_\alpha \ , \ v'(s) \geqslant 0 \ \forall s \in [0, t]\}.$$

Then $u(\widehat{t^*}) < M_0$ and $u'(\widehat{t^*}) < v'(0)$. Also $v(\overline{t^*}) > M_0$ and $v'(\overline{t^*}) = 0$. Then the curves $u|_{[\widehat{t^*}, t_0]}$ and $v|_{[0,\overline{t^*}]}$ must intersect at some point $P = (\nu, \nu')$ in the first quadrant of the phase plane (x, x'). Let us assume that ν is minimal and consider v_{-T} the translate of the left branch of v that, at some point t_1, coincides with u with same image and same positive derivative. Our assumption on the term Φ'/Φ implies that $u''(t_1) < v''_{-T}(t_1)$ and we get the same contradiction. $\square$

We are now in a position to prove proposition 2.

Proof of Proposition 2: Let $I = [M_0, \overline{M}]$ and consider the following subsets I_A and I_B:

$$I_A\,(I_B) = \{M \in [M_0, \overline{M}] : J_M \text{ has a minimiser in } \mathfrak{C}_M \text{ of type A (B)}\}.$$

By Lemma 4 we have $I = I_A \cup I_B$. We assert that I_A and I_B are non-empty. We have $M_0 \in I_A$. In fact, let u_{M_0} be a minimiser of J_{M_0} in $\mathfrak{C}_{M_0}$. If u_{M_0} is of type B then by the existence and uniqueness theorem, $u_{M_0} \equiv M_0$, an obvious contradiction. Now, in order to prove that I_B is non-empty, we have $\overline{M} \in I_B$ or $\overline{M} \notin I_B$. Suppose that $\overline{M} \notin I_B$ and let $u_{\overline{M}}$ be a minimiser of $J_{\overline{M}}$ in $\mathfrak{C}_{\overline{M}}$. By Lemma 5 there exists t_0 such that $u_{\overline{M}}(t_0) = \overline{M}$, $u'_{\overline{M}}(t_0)_- > 0$ and $u'_{\overline{M}}(t_0)_+ < 0$. Then, for sufficiently small $\lambda, \varepsilon > 0$ we have

$$J_{\overline{M}}(u_{\overline{M}} + \lambda v_\varepsilon) < J_{\overline{M}}(u_{\overline{M}}),$$

where v_ε was defined in (12) (see Lemma 4 for details). Then, by (11),

$$\min J_{\overline{M}} < \min J_{\overline{M}}|_{\mathfrak{C}_{\overline{M}}} < 0,$$

and the continuous embedding of H in $C([0,1])$ implies that a minimum of $J_{\overline{M}}$ is a local minimum of J. It is therefore a nontrivial classical solution to (1)–(2) with $M_0 < \max u \leqslant \overline{M}$. In particular, it implies that I_B is nonempty.

Finally, we state that I_A and I_B are closed subsets of I. We shall only consider I_A since the other case is identical. Let (M_n) be a sequence in I_A such that $M_n \to M$. Let u_n be a corresponding sequence of type A minimisers of J_{M_n} in $\mathfrak{C}_{M_n}$. Since (u_n) is trivially bounded we may extract a weakly convergent subsequence (still denoted by u_n)

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such that $u_n \rightharpoonup u$. Since the weak convergence implies L_∞ convergence one gets that $u \in \mathfrak{C}_M$ and u is of type A. It remains to show that u is a minimiser of J_M in $\mathfrak{C}_M$. Since

$$\lim_{n\to\infty} \int_0^1 \Phi F_{M_n}(u_n) = \int_0^1 \Phi F_M(u)$$

and

$$\int_0^1 \Phi u'^2 \leqslant \liminf_{n\to\infty} \int_0^1 \Phi {u_n'}^2,$$

we have

$$J_M(u) \leqslant \liminf J_{M_n}(u_n).$$

Moreover, if we consider the sequence $w_n = (M_n/M)u$, we have $w_n \to u$ in H and $w_n \in \mathfrak{C}_{M_n}$, for all $n \in \mathbb{N}$. Then

$$J_M(u) = \lim_{n\to\infty} J_{M_n}(w_n)$$

and

$$J_{M_n}(w_n) \geqslant J_{M_n}(u_n),$$

for all $n \in \mathbb{N}$. Consequently,

$$J_M(u) \geqslant \limsup_{n\to\infty} J_{M_n}(u_n) \geqslant \liminf_{n\to\infty} J_{M_n}(u_n) \geqslant J_M(u),$$

or

$$\lim_{n\to\infty} J_{M_n}(u_n) = J_M(u).$$

If, for a certain u^* in $\mathfrak{C}_M$, $J_M(u^*) < J_M(u)$, then, for sufficiently large n, we would have

$$J_{M_n}(w_n^*) < J_{M_n}(u_n),$$

where $w_n^* = (M_n/M)u^*$, a contradiction. Then $M \in I_A$ and I_A is closed. With a similar reasoning one proves that I_B is closed. Then, by connectedness we have $I_A \cap I_B \neq \emptyset$ and by Lemma 6 we conclude the existence of a classical solution u with $\max u \in I_A \cap I_B$. $\square$

REMARK 4. Note that the existence of a solution to problem (1)–(2) under the assumptions of proposition 2 could have been proved with the use of the Mountain Pass theorem of Ambrosetti and Rabinowitz. However this theorem does not provide sharp estimates on the L_∞ norm of the solutions.

3. Proof of the main result

In this section we extend proposition 2 to the case where Φ may have a singularity at zero. Our technique relies in a simple approximation procedure to problem (1)–(2). The results established in the previous section obviously remain true when initial conditions imposed to equation (1) are $u(a) = u(b) = 0$ (where $a < b$).

PROOF OF THEOREM 1: We may assume that the function v given by (8) has support contained in $[\varepsilon_0, 1]$, where $\varepsilon_0 > 0$ is sufficiently small (the more general assertion can be obtained as a limit case). Take $n_0 \in \mathbb{N}$ such that $0 < 1/n_0 < \varepsilon_0$, and for every $n > n_0$, define:

$$\Phi_n(t) = \exp\Big(\frac{t^2}{2n}\Big)\Phi(t).$$

Note that Φ_n'/Φ_n is strictly increasing and $\Phi_n \to \Phi$ uniformly. Then, taking $H = H_0^1\big(\,](1/n), 1[\,\big)$, (8) implies that (11) is proven for large n if we consider the restriction of v to $\big[(1/n), 1\big]$. We can therefore apply proposition 2 to the family of problems

$$\begin{aligned}\big(\Phi_n(t)u'(t)\big)' + \Phi_n(t)f\big(u(t)\big) &= 0\\ u\Big(\frac{1}{n}\Big) = u(1) &= 0\end{aligned}$$

obtaining a sequence of solutions (u_n) such that $M_0 < \max u_n \leqslant \overline{M}$. We may suppose that the u_n's are extended by zero in $]0,1]$.

CLAIM. The sequence (u_n) is equicontinuous.

Let $t_n \in \big](1/n), 1\big[$ be such that $u_n(t_n) = \max u_{n[(1/n),1]}$. For any $t \in \big[(1/n), 1\big]$, we have

$$\Phi_n(t)u_n'(t) = \int_t^{t_n} \Phi_n(s)f\big(u_n(s)\big)\,ds.$$

If $1/n \leqslant t \leqslant t_n$, assumption (7) implies

$$|u_n'(t)| \leqslant \int_t^{t_n} \frac{\Phi_n(s)}{\Phi_n(t)}\Big|f\big(u_n(s)\big)\Big|\,ds \leqslant \exp\Big(\frac{1}{2n}\Big)K\overline{f}. \tag{24}$$

where $\overline{f} = \max\limits_{[0,\overline{M}]}|f|$. Since $u_n(t_n) \geqslant M_0$ we may conclude the existence of $l > 0$ independent of n such that $t_n \geqslant l$. Then, for any $t_n \leqslant t \leqslant 1$, we have, by (3),

$$|u_n'(t)| \leqslant \int_{t_n}^1 \frac{\Phi_n(s)}{\Phi_n(t)}\Big|f\big(u_n(s)\big)\Big|\,ds \leqslant (1-l)\exp\Big(\frac{1}{2n}\Big)\frac{\overline{f}}{m}\max_{[l,1]}\Phi, \tag{25}$$

and the claim follows from (24)–(25).

We can therefore take a subsequence (still denoted by (u_n)) such that $u_n \to u$ uniformly and by standard arguments, u is a solution of (1)–(2). Since $M_0 < \max u_n \leqslant \overline{M}$ for all n, we conclude that u is nontrivial and $M_0 < \max u \leqslant \overline{M}$ (the case $\max u = M_0$ is excluded by the existence and uniqueness theorem). $\square$

REMARK 5. If we consider the change of variables $t = 1 - t'$ we may restate theorem 1 for a class of functions $\Phi \in C^1\big([0,1[\big)$ having a singularity at 1:

Suppose that f satisfies (5)–(6) and:

(1) $\Phi(t) \geqslant m > 0$ $\forall t \in [0,1[$,

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(2) Φ'/Φ is a decreasing function,

(3) $\big(\Phi(t)/\Phi(s)\big) \leqslant K$ for some $K > 0$ and every $0 \leqslant t \leqslant s < 1$,

(4) There exists a nonnegative $v \in H_0^1\big(]0,1[\big)$ such that

$$\frac{1}{2}\int_0^1 \Phi(t)v'(t)^2\,dt < \int_0^1 \Phi(t)F\big(v(t)\big)\,dt < \infty,$$

where $F(v) = \displaystyle\int_0^v f(s)\,ds$.

Then Problem (1)–(2) has a positive solution u such that $M_0 < \max u \leqslant \|v\|_\infty$.

REMARK 6. Note that the method only requires that $f > 0$ in $\big[M_0, \|v\|_\infty\big[$ where $\|v\|_\infty$ is given by (8). The behaviour of f in $\big[\|v\|_\infty, \infty\big[$ is not relevant for our existence-estimation result.

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