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ON OPEN PROJECTIONS OF GCR ALGEBRAS

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Throughout this paper $\mathfrak{A}$ will denote a C^*-algebra and $\mathfrak{B}$ will denote its second dual, which is identified with the enveloping von Neumann algebra of $\mathfrak{A}$. A projection E in $\mathfrak{B}$ is said to be *open* if it supports a left ideal in $\mathfrak{A}$, that is, if $\mathfrak{B}E = \mathfrak{J}^-$ for some left ideal $\mathfrak{J}$ in $\mathfrak{A}$. Here the bar $^-$ means the *stong* closure. When $\mathfrak{A}$ has a unit, this definition coincides with the definition of Akemann [**1**, Definition II.1]. In the sequel, we shall solely be concerned with two-sided ideals, and consequently central projections [**4**, I, § 3, Corollary 3 of Theorem 2]. Our aim is to show that $\mathfrak{A}$ is CCR if and only if the open central projections are strongly dense in the set of central projections on $\mathfrak{B}$. If $\mathfrak{A}$ is a GCR algebra, this implies that the complete Boolean lattice generated by the open central projections of $\mathfrak{B}$ is the set of central projections of $\mathfrak{B}$. As a corollary we get the result that every ideal preserving automorphism of a GCR algebra is weakly inner. This was proved by Lance [**4**] for separable GCR algebras and recently extended to arbitrary GCR algebras by Elliott [**5**] using methods different from ours.

Before we begin we fix some notation. Let $\mathfrak{P}$ be the complete Boolean algebra of central projections in $\mathfrak{B}$, let $\mathfrak{P}_0$ be the set of all open central projections in $\mathfrak{B}$, and let $\langle\mathfrak{P}_0\rangle$ be the complete Boolean algebra generated by $\mathfrak{P}_0$ in $\mathfrak{P}$. For every $A \in \mathfrak{B}$ let C_A denote the orthogonal complement of the largest $P \in \mathfrak{P}$ such that $AP = 0$. If $A \in \mathfrak{A}$, then C_A is the support of the two-sided ideal in $\mathfrak{A}$ that is generated by A and thus the projection C_A is in $\mathfrak{P}_0$. Let $\mathfrak{J}_a$ be the closed two-sided ideal of $\mathfrak{B}$ generated by the abelian projections of $\mathfrak{B}$.

LEMMA 1. *Let π be a non-degenerate representation of the CCR algebra $\mathfrak{A}$ on a Hilbert space H, let $A \in \mathfrak{A}$, and let P be a central projection of the von Neumann algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$. Then there is a net $\{Q_n\}$ of central projections of $\pi(\mathfrak{A})''$ which converge strongly to* 1 *and a corresponding net $\{A_n\}$ in $\pi(\mathfrak{A})$ such that $\pi(A)PQ_n = A_nQ_n$ for every Q_n.*

Proof. There is a normal homomorphism ϕ of the enveloping von Neumann algebra $\mathfrak{B}$ of $\mathfrak{A}$ onto $\pi(\mathfrak{A})''$ such that $\phi(B) = \pi(B)$ for every $B \in \mathfrak{A}$ [**3**, 12.1.5]. Because ϕ maps the centre of $\mathfrak{B}$ onto the centre of $\pi(\mathfrak{A})''$ and since ϕ is strongly continuous on bounded subsets of $\mathfrak{B}$, there is no loss generality in the assumption that π is the canonical embedding of $\mathfrak{A}$ in $\mathfrak{B}$ and that H is the canonical Hilbert space of $\mathfrak{B}$.

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Now it is sufficient to find, for every finite subset $\mathfrak{X}$ of H and every $\epsilon > 0$, a projection $Q \in \mathfrak{P}$ and an element $A' \in \mathfrak{A}$ such that $APQ = A'Q$ and such that $||x - Qx|| \leqq \epsilon$ for every $x \in \mathfrak{X}$. Let $\{P_i | 0 \leqq i \leqq i_0\}$ be a subset of $\mathfrak{P}_0$ indexed by the ordinals such that

(i) $P_0 = 0, P_{i_0} = 1$;

(ii) $P_i < P_{i+1}$ $(i < i_0)$;

(iii) $\text{lub}\{P_i | i < j\} = P_j$ if j is a limit ordinal; and

(iv) $\mathfrak{B}(P_{i+1} - P_i)$ is the strong closure of a closed two-sided ideal in $\mathfrak{A}(1 - P_i)$ generated by elements in $\mathfrak{A}(1 - P_i)$ that are contained in $\mathfrak{J}_a$ [**6**, pp. 148–149].

Setting $P_i' = P_{i+1} - P_i$ for all $i < i_0$, we obtain a set of orthogonal central projections of sum 1. There are ordinals $\{i(j) | 1 \leqq j \leqq n\}$ with $i(n) < i(n - 1) < \ldots < i(1) < i_0$ such that

$$||\sum {}_j P_{i(j)}'x - x|| \leqq \epsilon/3$$

for every $x \in \mathfrak{X}$. There is also an orthogonal set $\{R_k\}$ of central projections of sum 1 and a corresponding bounded set $\{B_k\}$ of elements of $\mathfrak{J}_a$ such that $A = \sum B_k R_k$ [**6**, Corollary, Theorem 6]. Consequently, there is a finite subset of the R_k of sum R such that

$$||Rx - x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. We have that $AR \in \mathfrak{J}_a$ and so $APR \in \mathfrak{J}_a$. We may find an element $A_1 \in \mathfrak{A}$ such that $A_1(1 - P_{i(1)+1}) = 0$ and a projection Q_1' in $\mathfrak{P}$ majorized by $RP_{i(1)}'$ with $||(Q_1' - P_{i(1)}')x|| \leqq \epsilon/3n$ for every $x \in \mathfrak{X}$ such that $APRQ_1' = A_1Q_1'$ [**6**, Lemma 4]. Setting $Q_1 = RQ_1'$, we obtain a central projection Q_1 majorized by $P_{i(1)}'$ with

$$||(P_{i(1)}' - Q_1)x|| \leqq ||(P_{i(1)}' - Q_1')x|| + ||Q_1'|| \, ||(1 - R)x|| \leqq 2\epsilon/3n$$

for every $x \in \mathfrak{X}$ such that $APQ_1 = A_1Q_1$. Continuing by recursion, we obtain central projections Q_j majorized by $P_{i(j)}'$ with

$$||(P_{i(j)}' - Q_j)x|| \leqq 2\epsilon/3n$$

for every $x \in \mathfrak{X}$ and elements A_j in $\mathfrak{A}$ with $A_j(1 - P_{i(j)+1}) = 0$ such that

$$(AP - (A_1 + \ldots + A_{j-1}))Q_j = A_jQ_j$$

for $2 \leqq j \leqq n$. Setting $Q = \sum Q_j$ and $A' = \sum A_j$, we obtain a central projection Q with

$$||x - Qx|| \leqq ||(1 - \sum {}_j P_{i(j)}')x|| + \sum ||(P_{i(j)}' - Q_j)x|| \leqq \epsilon$$

for every $x \in \mathfrak{X}$ such that $APQ = A'Q$.

We can now obtain the following theorem.

THEOREM 2. *A C^*-algebra $\mathfrak{A}$ is a CCR algebra if and only if $\mathfrak{P}_0$ is strongly dense in $\mathfrak{P}$.*

Proof. Let $\mathfrak{A}$ be a CCR algebra. Since $\mathfrak{P}_0^-$ is clearly closed under taking least upper bounds of monotonely increasing nets, it is sufficient to show that every nonzero $P \in \mathfrak{P}$ majorizes a nonzero Q in $\mathfrak{P}_0^-$. We proceed to do this. Since $\mathfrak{A}$ is strongly dense in $\mathfrak{B}$, there is an $A \in \mathfrak{A}$ such that $AP \neq 0$. There is a net $\{Q_n\}$ of projections in $\mathfrak{P}$ which converges strongly to 1 and a corresponding net $\{A_n\}$ of elements of $\mathfrak{A}$ such that $APQ_n = A_nQ_n$ (Lemma 1). But the nonzero projection C_AP is in $\mathfrak{P}_0^-$ since $C_APQ_n = C_{A_n}Q_n$ and thus

$$\lim||(C_AP - C_{A_n})x|| \leqq \lim\sup||C_AP - C_{A_n}||\ ||(1 - Q_n)x|| = 0$$

for every x in the Hilbert space of $\mathfrak{B}$. This completes the proof of this part of the theorem.

Conversely, let $\mathfrak{P}_0^- = \mathfrak{P}$. Let $\mathfrak{B}$ be considered as a module over its centre $\mathfrak{Z}$. Let ϕ be a bounded module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$ and let ψ be the unique extension of the restriction of ϕ to $\mathfrak{A}$ to a σ-weakly continuous linear function of $\mathfrak{B}$ into $\mathfrak{Z}$ [**6**, Proposition 1]. It is sufficient to show that ψ is a module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$ [**6**, Theorem 6]. If $P \in \mathfrak{P}_0$, then there is a monotonely increasing net $\{A_n\}$ in $\mathfrak{A}^+$ which converges σ-strongly to P [**1**, III.4]. Then for every $A \in \mathfrak{A}$ we have that

$$(1 - P)\psi(PA) = \lim(1 - P)\psi(A_nA) = \lim(1 - P)\phi(A_nA) = \lim \phi((1 - P)A_nA) = 0$$

and hence $\psi(PA) = P\psi(PA)$. Here the limits are taken in the σ-weak topology. Since $\mathfrak{A}$ is σ-weakly dense in $\mathfrak{B}$, $\mathfrak{P}_0$ is a bounded and σ-strongly dense in $\mathfrak{P}$, and ψ is σ-weakly continuous, we see that $\psi(PA) = P\psi(PA)$ for every $A \in \mathfrak{B}$ and $P \in \mathfrak{P}$. But this means that

$$\psi(PA) = P\psi(PA) + P(1 - P)\psi((1 - P)A) = P\psi(PA) + P\psi((1 - P)A) = P\psi(A)$$

for every $A \in \mathfrak{B}$ and $P \in \mathfrak{P}$. Now because ψ is a uniformly continuous linear function into $\mathfrak{Z}$ and because elements of $\mathfrak{Z}$ are uniform limits of linear combinations of projections of $\mathfrak{Z}$, we obtain that $\psi(CA) = C\psi(A)$ for every $A \in \mathfrak{B}$ and $C \in \mathfrak{Z}$. Hence ψ is a module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$.

We now consider GCR algebras.

THEOREM 3. *If $\mathfrak{A}$ is a GCR algebra, then $\langle\mathfrak{P}_0\rangle = \mathfrak{P}$.*

Proof. It is sufficient to show that $\langle\mathfrak{P}_0\rangle^- = \mathfrak{P}$ since the complete Boolean algebra of projections $\langle\mathfrak{P}_0\rangle$ contains every projection in the weakly closed algebra which it generates [**2**, Theorem 2.8]. Given $P \in \mathfrak{P}$, a finite subset $\mathfrak{X}$ of the Hilbert space H of $\mathfrak{B}$, and $\epsilon > 0$, it is sufficient to find a Q in $\langle\mathfrak{P}_0\rangle^-$ such that $||(P - Q)x|| \leqq \epsilon$ for every $x \in \mathfrak{X}$. Let $\{P_i|0 \leqq i \leqq i_0\}$ be a maximal set of central projections indexed by the ordinals such that

(i) $P_0 = 0$;

(ii) $P_i < P_{i+1}$ $(i < i_0)$;

(iii) $\text{lub}\{P_i|i < j\} = P_j$ if j is a limit ordinal; and

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(iv) $\mathfrak{B}(P_{i+1} - P_i)$ is the strong closure of a CCR ideal $\mathfrak{J}_i$ contained in $\mathfrak{A}(1 - P_i)$ (cf. proof of Lemma 1).

Then clearly we must have $P_{i_0} = 1$.

We notice that each P_i is in $\mathfrak{P}_0$ since P_i supports the ideal

$$\{A \in \mathfrak{A} | A(1 - P_i) = 0\}.$$

Thus, every projection $P_i' = P_{i+1} - P_i$ is in $\langle \mathfrak{P}_0 \rangle$. Since $\sum P_i' = 1$, we may find ordinals $i(1), \ldots, i(n)$ such that $||(1 - \sum_j P_{i(j)}')x|| \leqq \epsilon/3$ for every $x \in \mathfrak{X}$. Now $\mathfrak{J}_{i(j)}$ is strongly dense in $\mathfrak{B}P_{i(j)}'$, and thus there is positive element A_j in the unit sphere of $\mathfrak{J}_{i(j)}$ with

$$||(A_j - P_{i(j)})x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. Therefore, we have

$$||(C_{A_j} - P_{i(j)}')x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. We may find a central projection Q_j majorized by $P_{i(j)}'$ with

$$||(P_{i(j)}' - Q_j)x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$ such that

$$A_j P Q_j = B_j Q_j$$

for some $B_j \in \mathfrak{A}$ (Lemma 1). Since all $C_{B_j} P_{i(j)}'$ are in $\langle \mathfrak{P}_0 \rangle$, their least upper bound $Q = \sum C_{B_j} P_{i(j)}'$ is also in $\langle \mathfrak{P}_0 \rangle$. We then have that

$$||(P - Q)x|| \leqq ||P(1 - \sum_j P_{i(j)}')x|| + \sum_j ||P(P_{i(j)}' - C_{A_j})x|| + \sum_j ||(C_{A_j}P - C_{B_j})P_{i(j)}'||\,||(P_{i(j)}' - Q_j)x|| \leqq \epsilon$$

for every $x \in \mathfrak{X}$.

Now we consider the corollary mentioned in the introduction. We remind the reader that an automorphism α of $\mathfrak{A}$ is said to be ideal-preserving if $\alpha(\mathfrak{J}) \subset \mathfrak{J}$ for each ideal $\mathfrak{J}$ of $\mathfrak{A}$. For GCR algebras this is equivalent to the requirement that $\alpha(\mathfrak{J}) = \mathfrak{J}$ [**7**]. Further, the automorphism α is said to be universally weakly inner if, for every representation π of $\mathfrak{A}$, the automorphism α_π of $\pi(\mathfrak{A})$ defined by $\alpha_\pi(\pi(A)) = \pi(\alpha(A))$ is implemented by a unitary element of $\pi(\mathfrak{A})^-$. (Note that α_π is well-defined whenever α is ideal-preserving.) Lance [**7**] showed that ideal-preserving automorphism of a separable GCR algebra is universally weakly inner. In a recent paper Elliott [**5**] proved that same result for an arbitrary GCR algebra by using a decomposition into "champs continus". At the end of the paper he asked if the fact that the enveloping von Neumann algebra of a GCR algebra is type I could be used to prove the same result. The following corollary answers this question in the affirmative.

COROLLARY 4. *An ideal preserving automorphism α of a GCR algebra $\mathfrak{A}$ is universally weakly inner.*

Proof. We identify $\mathfrak{A}$ with a weakly dense *-subalgebra of the enveloping von Neumann algebra $\mathfrak{B}$ of $\mathfrak{A}$ and prove that α is inner in $\mathfrak{B}$. The automor-

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phism α has an extension to an automorphism of $\mathfrak{B}$. We also denote this extension by α. Since $\mathfrak{A}$ is GCR, the algebra $\mathfrak{B}$ is of type I and so it suffices to show that $\alpha(P) = P$ for every $P \in \mathfrak{P}$ [**4**, III, § 3, Corollary 2 of Proposition 3]. Clearly, if P is open, we have that $\alpha(P) = P$ by the strong continuity and ideal preserving property of α and α^{-1}. Thus the same relation holds for finite Boolean polynomials in open projections and hence for strong limits of such polynomials. By Theorem 2, every central projection of $\mathfrak{B}$ is obtained in this way and so there is a unitary $U \in$ such that $\alpha(A) = UAU^{-1}$ for every A in $\mathfrak{B}$.

Now let π be an arbitrary nondegenerate representation of $\mathfrak{A}$. There is a normal homomorphism ϕ of $\mathfrak{B}$ onto the von Neumann algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$ such that $\phi(A) = \pi(A)$ for every $A \in \mathfrak{A}$ [**3**, 12.1.5]. Then we have that

$$\begin{aligned}\alpha_\pi(\pi(A)) &= \pi(\alpha(A)) = \phi(\alpha(A))\\ &= \phi(UAU^*) = \phi(U)\phi(A)\phi(U)^*\\ &= \phi(U)\pi(A)\phi(U)^*\end{aligned}$$

for every $A \in \mathfrak{A}$. Clearly, the operator $\phi(U)$ is unitary in $\pi(\mathfrak{A})''$. Hence the automorphism α_π is implemented by the unitary operator $\phi(U)$ in

$$\pi(\mathfrak{A})^- = \pi(\mathfrak{A})''.$$

Addendum (July 13, 1972). Trond Digernes has shown that a separable GCR algebra is completely characterized by the fact that $\mathfrak{P}_0 = \mathfrak{P}$. This will appear in an article, *A new characterization of separable GCR-algebras* in Proc. Amer. Math. Soc. Herbert Halpern has recently shown that a C^*-algebra (perhaps non-separable) is GCR if and only if the Boolean σ-algebra generated by $\mathfrak{P}_0$ contains all minimal projections of $\mathfrak{P}$.

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