A NOTE ON MULTIVARIATE POISSON FLOWS ON STOCHASTIC PROCESSES

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Abstract

In [1], a deterministic counting rate condition is shown to be necessary and sufficient for a counting process induced on a Markov step process Z to be multivariate Poisson. We show here that the result continues to hold without Z being a Markov step process.

MARKOV STEP PROCESS

It was assumed in [1] that a Markov step process Z induces a multivariate counting process \( N = (N_1, N_2, \cdots, N_c) \). The infinitesimal generator \( A \) of \( Z \) was used there to characterize a vector process whose respective components \( r_i(Z(t)) \) can be heuristically interpreted as the counting rates for the corresponding \( N_i \) at time \( t \). It is shown in [1] that if the components of \( N \) do not have simultaneous jumps, a determinacy condition based on the sigma algebras \( N_t = \sigma\{N(u), u \leq t\} \) is necessary and sufficient for \( N \) to consist of mutually independent Poisson processes. This condition is that for each \( t \) we have almost surely

\[
E[r(Z(t)) | N_t] = E[r(Z(t))].
\]

The above result is extended in the present letter to processes \( Z \) that need not be Markov. To this end, let \( Z \) be measurable with respect to an increasing family of sigma algebras \( \{F_t\} \), and suppose further that \( Z \) induces the counting process \( N \) (as defined in [2], Chapter 2) in the sense that \( N_t \subseteq F_t \) for each \( t \). Let \( E[N_i(t)] < \infty \) for each \( t, i = 1, 2, \cdots, c \), with the \( N_i \) having the respective \( F_t \)-intensities (see [2], II.D7) \( \lambda_i \). It is also presumed that the conditional expectations \( E[\lambda_i(\cdot) | N] \) have an \( N_t \)-progressive version, which we can (and shall) assume to be \( N_t \)-predictable ([2], Theorem II.T13) without loss of generality.

Now if \( I \) stands for the indicator function, it is tautologically true that

\[
I[N_i(t) - N_i(s) > n_i] = \int_s^t I[N_i(u) - N_i(s) = n_i] dN_i(u)
\]

for any \( 0 \leq s \leq t \). (Equation (2), together with its possible implications, were called to the author’s attention by Dr B. Melamed.) Moreover, \( N_i(t) - \int_0^t \lambda_i(s) ds \) is not only an \( F_t \)-martingale, but also a fortiori an \( N_t \)-martingale. It then follows from the definition of

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intensity that on the right side of (2)

\[
(3) \quad E \left[ \int \left[ N_t(u) - N_t(s) = n_i \right] dN_t(u) \right| N_t] = E \left[ \int \left[ N_t(u) - N_t(s) = n_i \right] \lambda_t(u) \, du \right| N_t].
\]

Equations (2) and (3) may be combined by taking the conditional expectation in (2) respective to \( N_t \), and substituting. If we then also add over \( n_i = 0, 1, 2, \cdots \) and apply Fubini’s theorem, we obtain

\[
(4) \quad E[N_t(t) - N_t(s) \mid N_t] = \int_s^t E[\lambda_t(u) \mid N_t] \, du.
\]

This equation effectively generalizes (1.18) of [1]; our \( \lambda_t \) plays the role of the \( \rho_i \) of [1], which in [1] is generated by a Markov step process \( Z \). Indeed, under the assumptions of [1], our (4) specializes precisely to Equation (1.18) in [1].

Condition (3.2) in [1] may be replaced by

\[
(5) \quad E[\lambda_t(t) \mid N_t] = E[\lambda_t(t)]
\]

almost surely with respect to \( dt \, dP \) measure. As in [1], this condition (in the presence of the preceding hypotheses on \( N, N_t, F, \) and \( E[\lambda(\cdot) \mid N] \) above) is necessary and sufficient for \( N \) to be a multivariate Poisson process respective to \( N_t \). The proofs are easy exercises in the martingale theory of multivariate counting processes.

If (5) is met, we have in (4)

\[
(6) \quad E[\lambda_t(u) \mid N_t] = E\{E[\lambda_t(t) \mid N_t] \mid N_t\} = E[\lambda_t(t)].
\]

Thus \( N \) is a multivariate Poisson process according to the multichannel Watanabe theorem (see [2], Theorem II.T6). Conversely, let \( N \) be multivariate Poisson. From (4) and the \( N_t \)-independent increment property it follows that \( N_t \) has the predictable \( N_t \)-intensity \( E[\lambda(\cdot)] \). But also, a version of \( E[\lambda(\cdot) \mid N] \) is such an intensity (see [2], Theorem II.T14). The uniqueness of predictable intensities ([2], Theorem II.T12) then yields (5), as was desired.

References
