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ON PRODUCTS OF SOBOLEV–ORLICZ SPACES

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We give conditions under which pointwise multiplication is a continuous bounded operation on kth order Sobolev-Orlicz spaces. This result is used to derive a sufficient condition under which the superposition operator is a continuous bounded operator on these spaces.

The Sobolev-Orlicz spaces W^kL_M generalise the classical Sobolev spaces W_p^k in rather the same way as the Orlicz spaces L_M generalise the classical Lebesgue spaces L_p. In applications they are useful, roughly speaking, whenever one has to deal with differential equations involving *strong nonlinearities*; for a typical application see [**7**]. To apply the standard principles of nonlinear analysis to such equations, one has to carry out a systematic study of various properties (like acting conditions, boundedness conditions, or continuity conditions) of the nonlinear *superposition operator*

$$Fu(x) = f(x, u(x)) \tag{1}$$

generated by some real function f on $\Omega \times \mathbb{R}$, with Ω being say, a bounded domain in the Euclidean space $\mathbb{R}^d$.

In the case $k = 1$, some results of this type for *first order Sobolev-Orlicz spaces* have been obtained by Hardy [**8-10**] which in turn generalise corresponding results for *first order Sobolev spaces* by Marcus and Mizel [**14, 15**]; see also [**16-21**]. In the case $k > 1$, however, nothing is known about the properties of the operator (1) in the space W^kL_M. On the other hand, [**25**] and [**26**] contain (sufficient) conditions under which the operator (1) acts in a *higher order Sobolev space* W_p^k and is continuous and bounded. These results build essentially on certain algebraic operations with Lebesgue spaces which were introduced and studied in detail in [**28, 29**] in the setting of so-called *multiplicator spaces of ideal spaces* of measurable functions, covering not only Lebesgue and Orlicz spaces, but also many classes of spaces arising in the interpolation theory of linear operators.

The purpose of this paper is to give results for the superposition operator (1) in *higher order Sobolev-Orlicz spaces* W^kL_M (respectively W^kE_M) which are parallel to

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those in [**25**]. The main tools are the general theory of multiplicators of Orlicz spaces, see [**2, 27, 28**], imbedding and density theorems for Sobolev-Orlicz spaces, see [**6**], and some general facts about superposition operators in function spaces, see [**3, 5**].

In the first section, we discuss some properties of *Sobolev conjugates and anti-conjugates* of Young functions which may be of independent interest for the general theory of ideal spaces. Afterwards, we give a description of the multiplicator space of two Sobolev-Orlicz spaces which may be regarded as a generalisation of both [**2**] and [**25**]. Finally, in the last section we apply this to obtain an acting condition for the operator (1) in the space $W^k E_M$; this condition ensures the boundedness and continuity of the operator (1) as well.

1. Sobolev-Orlicz spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which has the cone property, see [**13**], and let $M = M(u)$ be a real *Young function*, that is, M is even and convex on $\mathbb{R}$, $M(0) = 0$, and $M(\infty) = \infty$. The *Orlicz space* $L_M = L_M(\Omega)$ consists of all measurable functions $u = u(x)$ on Ω for which the (Luxemburg) norm

$$\|u\|_M = \inf\{\lambda : \lambda > 0, \int_\Omega M[u(x)/\lambda]dx \leqslant 1\} \tag{2}$$

is finite. Apart from the space L_M, we consider the (separable) subspace $E_M = E_M(\Omega)$ of all functions $u \in L_M$ with absolutely continuous norms; the space E_M is just the closure of $L_\infty = L_\infty(\Omega)$ with respect to the norm (2), and coincides with L_M if and only if the Young function M satisfies a Δ_2 condition (see for example [**11**]).

Since the Orlicz space $X = L_M$ is a *symmetric* (or *rearrangement-invariant*) *space*, one may define its *fundamental function* (see [**12**])

$$\phi(X;\lambda) = \|\chi_D\|_X \quad (\text{mes } D = \lambda); \tag{3}$$

it is easy to see that

$$\phi(L_M;\lambda) = \frac{1}{M^{-1}(1/\lambda)}. \tag{4}$$

Given a Young function M, the function

$$J_M(s,t) = \int_s^t \frac{M^{-1}(\tau)}{\tau^{1+1/d}}d\tau \quad (0 \leqslant s < t \leqslant \infty) \tag{5}$$

will be of fundamental importance in what follows (d is the dimension of Ω). We will always suppose that $J_M(0,1) < \infty$, replacing M, if necessary, by an equivalent Young function with the same asymptotic growth at infinity.

We associate with M two sequences of Young functions $M_{(1)}$, $M_{(2)}, \cdots$ and $M_{(-1)}$, $M_{(-2)}, \cdots$ by putting $M_{(0)}(t) = M(t)$, $\left(M_{(1)}\right)^{-1}(t) = J_M(0, t), \cdots$, $\left(M_{(k+1)}\right)^{-1}(t) = J_{M_{(k)}}(0, t)$, and $\left(M_{(-1)}\right)^{-1}(t) = t^{1+1/d}(d/dt)\left(M^{-1}(t)\right), \cdots$, $\left(M_{(-k-1)}\right)^{-1}(t) = t^{1+1/d}(d/dt)\left(M_{(-k)}\right)^{-1}(t)$.

The functions $M_{(k)}$ and $M_{(-k)}$ $(k \geqslant 1)$ will be called the kth *Sobolev conjugate* and *anticonjugate*, respectively, of M. We denote the smallest integer $k \geqslant 0$ such that $J_{M_{(k)}}(1, \infty) < \infty$ by $\kappa = \kappa(M)$; obviously, $\kappa \leqslant d$. It is clear that $M_{(k)(l)} = M_{(k+l)}$ for all $k, l \in Z$. Observe that, by (4), the fundamental functions (3) of the space L_M, $L_{M_{(1)}}$, and $L_{M_{(-1)}}$ are related by the formulae

$$\frac{1}{\phi\left(L_{M_{(1)}};\lambda\right)} = \int_\lambda^\infty \frac{1}{\mu^{1-1/d}\phi(L_M;\mu)}d\mu,$$

$$\phi\left(L_{M_{(-1)}};\lambda\right) = \frac{\phi^2(L_M;\lambda)}{\phi'(L_M;\lambda)}\lambda^{-1+1/d}$$

respectively.

The *Sobolev-Orlicz space* $W^kL_M = W^kL_M(\Omega)$ (respectively $W^kE_M = W^kE_M(\Omega)$) consists, by definition, of all measurable functions $u = u(x)$ on Ω such that all (distributional) derivatives $D^\alpha u$ of u belong to L_M (respectively E_M) for $0 \leqslant |\alpha| \leqslant k$, equipped with the norm

$$\|u\|_{k,M} = \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_M. \tag{6}$$

The following *imbedding theorem* for Sobolev-Orlicz spaces is fundamental (see [6], and also Chapter 8 of [1] or Chapter 7 of [13]).

LEMMA 1. *Let M be a Young function. In case $J_M(1, \infty) = \infty$, the space W^1L_M is imbedded in the Orlicz space $L_{M_{(1)}}$, while in case $J_M(1, \infty) < \infty$ the space W^1L_M is imbedded in the (generalised) Hölder space $C^\mu = C^\mu\left(\overline{\Omega}\right)$ generated by the function $\mu(t) = J_M\left(t^{-d}, \infty\right)$. More generally, for $k \leqslant \kappa(M)$ the space W^kL_M is imbedded in the Orlicz space $L_{M_{(k)}}$, while for $k > \kappa(M)$ the space W^kL_M is imbedded in the Hölder space $C^{k-\kappa-1,\mu}$ with μ as above. All statements are true as well with L replaced by E.*

The simplest example of a Young function is of course

$$M(u) = M_p(u) = \frac{1}{p}|u|^p \quad (1 \leqslant p < \infty) \tag{7}$$

or

$$M(u) = M_\infty(u) = \begin{cases} 0 & |u| \leqslant 1, \\ \infty & |u| > 1, \end{cases} \tag{8}$$

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which gives the Lebesgue spaces L_p amd L_∞, respectively. In this case, the requirement $J_M(1,\infty)=\infty$ means that $p \leqslant d$, and $L_{M_{(1)}}$ is the Lebesgue space $L_{dp/(d-p)}$ (M as in (7)); on the other hand, in case $J_M(1,\infty)<\infty$ (that is, $p>d$) we get for C^μ the classical Hölder space generated by $\mu(t)=t^{1-d/p}$. More generally, the Young function (7) generates the sequence $M_{(k)}(u)=|u|^{dp/(d-kp)}$ ($k \in Z$); this shows that Lemma 1 generalises the classical Sobolev imbedding theorems for the spaces W_p^k (see for example [**1**, **13**]).

2. Multiplicator spaces

Given two function spaces X and Y over Ω, we denote by $Y:X$ the *multiplicator space* (see [**28**, **29**]) of all functions $v=v(x)$ on Ω such that $uv \in Y$ for all $u \in X$, equipped with the natural norm

$$\|v\|_{Y:X}=\sup\{\|uv\|_Y : \|u\|_X \leqslant 1\}. \tag{9}$$

It is evident that the "size" of the multiplicator space $Y:X$ depends on the relation between X and Y. If $X=L_M$ and $Y=L_N$ are Orlicz spaces, this can be made precise.

We write $N \preccurlyeq M$ if

$$\overline{\lim_{u\to\infty}} \frac{N(\lambda u)}{M(u)}<\infty \tag{10}$$

for some $\lambda>0$, and $N \prec M$ if

$$\lim_{u\to\infty} \frac{N(\lambda u)}{M(u)}=0 \tag{11}$$

for all $\lambda>0$. In the first case we have $L_M \subseteq L_N$ (continuous imbedding); in the second case this imbedding is absolutely continuous (which means that the elements of the unit ball in L_M have uniformly absolutely continuous norms in L_N). From general results (see [**4**]) about the fundamental function (3) of a symmetric space it follows that (10) is equivalent to

$$\overline{\lim_{t\to\infty}} \frac{M^{-1}(t)}{N^{-1}(t)}<\infty, \tag{12}$$

and (11) is equivalent to

$$\lim_{t\to\infty} \frac{M^{-1}(t)}{N^{-1}(t)}=0. \tag{13}$$

Given two Young functions M and N, we denote by $N : M$ the Young function (8) if $N \preccurlyeq M$, but $N \not\prec M$, and the Young function

$$(N : M)(u) = \sup_{v \geqslant 0}\{N(uv) - M(v)\} \tag{14}$$

if $N \prec M$. With this terminology, the formula

$$L_N : L_M = L_{N:M} \tag{15}$$

holds; see [**2**, **27**, **28**]. Observe that the case $N \not\preccurlyeq M$ is not interesting, since the multiplicator space $L_N : L_M$ then contains only the zero function. In the particular case $M(u) = |u|^p/p$ and $N(u) = |u|^q/q$ $(1 \leqslant p, q < \infty)$ we have $N \preccurlyeq M$ if and only if $q \leqslant p$, and $N \prec M$ if and only if $q < p$; in the last case, the Young function (14) is just $(N : M)(u) = |u|^r/r$ with $1/r = 1/q - 1/p$.

We are now going to study the Sobolev conjugates and anti-conjugates of a Young function from the viewpoint of multiplicators. To begin with, we remark that, if M is a Young function satisfying

$$M^{-1}(at) < a^{1/d}M^{-1}(t) \quad (t > 0) \tag{16}$$

for some $a < 1$, then $M \prec M_{(1)}$. In fact from (16) it follows that

$$\int_0^t \frac{M^{-1}(\tau)}{\tau^{1+1/d}}d\tau \sim \frac{M^{-1}(t)}{t^{1/d}} \quad (t > 0)$$

(see [**12**], Lemma 1.4); consequently,

$$\frac{\left(M_{(1)}\right)^{-1}(t)}{M^{-1}(t)} = \frac{1}{M^{-1}(t)}\int_0^t \frac{M^{-1}(\tau)}{\tau^{1+1/d}}d\tau \leqslant Ct^{-1/d} \to 0$$

as $t \to \infty$, that is (13) holds. Observe that (16) is satisfied for $M(u) = |u|^p/p$ if and only if $p < d$.

Another relation of Sobolev conjugates with the orderings (10) and (11) are given in the following

Lemma 2. *Let M and N be two Young functions such that $N \preccurlyeq M$ (respectively $N \prec M$). Then the following holds.*

(a) $N_{(1)} \preccurlyeq M_{(1)}$ *(respectively* $N_{(1)} \prec M_{(1)}$*),*
(b) $N_{(-1)} \preccurlyeq M_{(-1)}$ *(respectively* $N_{(-1)} \prec M_{(-1)}$*).*

PROOF: The proof of (a) is trivial, therefore we drop it. The proof of (b) follows from the following general fact on real functions: if $\mu = \mu(t)$ and $\nu = \nu(t)$ are increasing C^1 functions such that

$$\mu(t) \to \infty,\ v(t) \to \infty,\ \frac{\mu(t)}{\nu(t)} \to \lambda \quad (t \to \infty)$$

from some $\lambda \in [0, \infty)$, then $$\overline{\lim_{t\to\infty}}\, \frac{\mu'(t)}{\nu'(t)} < \infty.$$

To see this, suppose that $$\overline{\lim_{t\to\infty}}\, \frac{\mu'(t)}{\nu'(t)} = \infty,$$

and choose $T > 0$ such that $\mu(t) \leqslant (\lambda+1)\nu(t)$ and $\mu'(\tau) \geqslant (\lambda+2)\nu'(\tau)$ for $t, \tau \geqslant T$. Given $t > T$, we find $\tau \in (T, t)$ such that $[\mu(t) - \mu(T)]\nu'(\tau) = [\nu(t) - \nu(T)]\mu'(\tau)$. This implies that $(\lambda+2)[\nu(t) - \nu(T)] \leqslant \mu(t) - \mu(T) \leqslant (\lambda+1)\nu(t) - \mu(T)$, hence $\nu(t) \leqslant (\lambda+2)\nu(T) - \mu(T)$, a contradiction for t large enough.

The assertion with $\preccurlyeq$ replaced by $\prec$ is proved similarly. ∎

We shall call two Young functions M and N *compatible* if $N \preccurlyeq M$ implies that $N : M_{(1)} \preccurlyeq (N : M)_{(-1)}$. Observe that the Young functions $M(u) = |u|^p/p$ and $N(u) = |u|^q/q$ are always compatible, as a simple computation shows.

We are now ready to prove our main results on multiplicators of Sobolev-Orlicz spaces.

THEOREM 1. *Let M, N and R be Young functions such that $R \preccurlyeq M$, $R \preccurlyeq N$, R and M are compatible, R and N are compatible, and*

$$R: M \prec N_{(k)},\ R: N \prec M_{(k)} \tag{17}$$

for some $k \in \mathbb{N}$. Then $u \in W^k E_M$ and $v \in W^k E_N$ implies that $uv \in W^k E_R$ and

$$\|uv\|_{k,R} \leqslant C\,\|u\|_{k,M}\,\|v\|_{k,N}. \tag{18}$$

PROOF: We prove the assertion by induction on k. Suppose first that (17) holds for $k = 1$, and let $u \in W^1 E_M$ and $v \in W^1 E_N$. We set $D_i u = \partial u/\partial x_i$ $(i = 1, \ldots, d)$, and distinguish four cases.

(i) $J_M(1, \infty) = J_N(1, \infty) = \infty$. Lemma 1 implies in this case that $u \in E_{M_{(1)}}$, $v \in E_{N_{(1)}}$, $D_i u \in E_M$, and $D_i v \in E_N$ $(i = 1, \ldots, d)$, hence $vD_i u \in E_R$ and $uD_i v \in E_R$, by (17).

(ii) $J_M(1, \infty) = \infty$, $J_N(1, \infty) < \infty$. In this case, we conclude from Lemma 1 that $u \in E_{M_{(1)}}$, $v \in C^\mu \subset L_\infty$, $D_i u \in E_M$, and $D_i v \in E_N$ $(i = 1, \ldots, d)$. From (17) and the hypothesis $R \preccurlyeq M$ we get again that $vD_i u \in E_R$ and $uD_i v \in E_R$.

(iii) $J_M(1, \infty) < \infty$, $J_N(1, \infty) = \infty$. This is of course analogous to (ii).

(iv) $J_M(1, \infty) < \infty$, $J_N(1, \infty) < \infty$. In this case, the relations $R \preccurlyeq M$ and $R \preccurlyeq N$ imply directly that $vD_iu \in E_R$ and $uD_iv \in E_R$. Now since the space $C^\infty(\overline{\Omega}) \cap W^kE_M(\Omega)$ is dense in $W^kE_M(\Omega)$ (see [6]), we may find sequences $u_m \in C^\infty \cap W^1E_M$ and $v_m \in C^\infty \cap W^1E_N$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{1,M} = \lim_{m\to\infty} \|v_m - v\|_{1,N} = 0.$$

An easy calculation shows then that

$$\begin{aligned}\lim_{m\to\infty} &\|(v_mD_iu_m + u_mD_iv_m) - (vD_iu + uD_iv)\|_R \\ &\leqslant \lim_{m\to\infty} \|u_m - u\|_{1,M}\left(\|v_m\|_{1,N} + \|v\|_{1,N}\right) \\ &\quad + \lim_{m\to\infty} \|v_m - v\|_{1,N}\left(\|u_m\|_{1,M} + \|u\|_{1,M}\right) = 0.\end{aligned}$$

From this it follows that $D_i(uv) = vD_iu + uD_iv \in E_R$ $(i = 1, \dots, d)$, hence $uv \in W^1E_R$ as claimed. The estimate (18) is proved by a straightforward computation.

Assume now that the statement holds for $k-1$, and let $u \in W^kE_M$ and $v \in W^kE_N$, with k satisfying (17). As in the first step, the case when $k > \kappa(M)$ or $k > \kappa(N)$ is easy; therefore we shall discuss only the case when $k \leqslant \kappa(M)$ and $k \leqslant \kappa(N)$. Lemma 1 implies then that $u \in W^{k-1}E_{M_{(1)}}$, $v \in W^{k-1}E_{N_{(1)}}$, $D_iu \in W^{k-1}E_M$, and $D_iv \in W^{k-1}E_N$ $(i = 1, \dots, d)$. Now, applying Lemma 2 (b) to (17) and using the compatibility of R and M and of R and N yields

$$R\colon M_{(1)} \prec N_{(k-1)}, \quad R : N_{(1)} \prec M_{(k-1)}. \tag{19}$$

By the induction hypothesis, we get $vD_iu \in W^{k-1}E_R$ and $uD_iv \in W^{k-1}E_R$, hence $uv \in W^kE_R$ as above. The estimate (18) is proved as before using a density argument. □

The formula (18) shows that, under the hypothesis (17), the space W^kE_M (respectively W^kE_N) is continuously imbedded in the multiplicator space $W^kE_R : W^kE_N$ (respectively $W^kE_R : W^kE_M$). The special case when $M = N = R$ will be of particular interest in the next section. Indeed, it follows from Theorem 1 that the space W^kE_M is an *algebra* if $L_{M_{(k)}}$ is absolutely continuously imbedded in L_∞; this is possible only if $L_{M_{(k)}} = \{0\}$ or, equivalently, $k > \kappa(M)$.

In the case $M(u) = |u|^p/p$, $N(u) = |u|^q/q$, and $R(u) = |u|^r/r$ $(1 \leqslant r < p,\ q < \infty)$, condition (17) reads

$$\frac{pr}{p-r} < \frac{dq}{d-kq}, \quad \frac{qr}{q-r} < \frac{dp}{d-kp}.$$

From Theorem 1 we conclude that the space W_p^k (respectively W_q^k) is then continuously imbedded in the multiplicator space $W_r^k : W_q^k$ (respectively $W_r^k : W_p^k$). This result is essentially due to Valent [25]. However, it seems that the author of [25] was unaware of the survey article [22] and the book [23] where a complete characterisation of the multiplicator space $W_r^k : W_p^k$ is given in terms of capacities.

3. The superposition operator

In this section we shall apply Theorem 1 to give an acting condition for the superposition operator (1) in the space W^kE_M which also implies the boundedness and continuity of this operator.

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which has the cone property, let M be a real Young function on $\mathbb{R}$, and let f be a real C^k function on $\overline{\Omega} \times \mathbb{R}$ such that $k > \kappa(M) + 1$. Then the superposition operator (1) generated by f maps the space $W^kE_M(\Omega)$ into itself and is bounded and continuous.*

Proof: We prove the assertion again by induction on k. Suppose first that f is C^1 on $\overline{\Omega} \times \mathbb{R}$. Since $J_M(1, \infty) < \infty$, W^1E_M is imbedded in the Hölder space C^μ with $\mu(t) = J_M\big(t^{-d}, \infty\big)$ (see Lemma 1). Let

$$f_0(x, u) = \frac{\partial}{\partial u} f(x, u), \quad f_i(x, u) = \frac{\partial}{\partial x_i} f(x, u) \quad (i = 1, \dots, d), \tag{20}$$

and denote the superposition operator generated by the function f_i by F_i $(i = 0, 1, \dots, d)$. By assumption, the functions (20) are continuous, and hence the operators F_i $(i = 0, 1, \dots, d)$ act in the space $L_\infty = L_\infty(\Omega)$ and are continuous and bounded (see for example [30]). Given $u \in W^1E_M$, choose a sequence $u_m \in C^\infty \cap W^1E_M$ such that $\|u_m - u\|_{1,M} \to 0$; hence $\|D_iu_m - D_iu\|_M \to 0$ and $\|u_m - u\|_\infty \to 0$ $(m \to \infty)$. Putting $v = Fu$, $v_m = Fu_m$, and passing in the equality

$$D_iv_m(x) = F_iu_m(x) + F_0u_m(x)D_iu_m(x)$$

to the (distributional) limit as $m \to \infty$, we get

$$D_iv(x) = F_iu(x) + F_0u(x)D_iu(x), \tag{21}$$

and thus $v \in W^1E_M$ as claimed. The proof shows also that the operator F is bounded and continuous in W^1E_M.

Assume now that the statement holds for $k-1$, suppose that f is of class C^k with $k > \kappa(M) + 1$, and let $u \in W^kE_M$. From Lemma 1 it follows that $u \in C^{k-\kappa-1,\mu}$ with $\kappa = \kappa(M)$ and $\mu(t) = J_M\big(t^{-d}, \infty\big)$. Moreover, all the operators F_i $(i = 0, 1, \dots, d)$

are continuous and bounded in $W^{k-1}E_M$, by the induction hypothesis. Since $D_i u \in W^{k-1}E_M$, (21) shows that $D_i v \in W^{k-1}E_M$, because $W^{k-1}E_M$ is an algebra for $k-1 > \kappa(M)$. This proves the assertion for k. □

Theorem 2 shows that a sufficient condition for the operator F to act in $W^k E_M$ is that the corresponding function f is of class C^k. This requirement may probably be considerably weakened; some results of this type for the Sobolev space W_p^k may be found, for example, in [**24**].

References

[1] R.A. Adams, *Sobolev spaces* (Academic Press, New York, 1976).

[2] T. Andô, 'On products of Orlicz spaces', *Math. Ann.* **140** (1960), 174–186.

[3] J. Appell, *Untersuchungen zur Theorie nichtlinearer Operatoren und Operatorgleichungen* (Habilitationsschrift, Univ. Augsburg, 1985).

[4] J. Appell and P.P. Zabrejko, 'On the degeneration of the class of differentiable superposition operators in function spaces', *Analysis* **7** (1987), 305–312.

[5] J. Appell and P.P. Zabrejko, *Nonlinear superposition operators* (Cambridge University Press, Cambridge, 1989).

[6] T.K. Donaldson and N.S. Trudinger, 'Orlicz-Sobolev spaces and imbedding theorems', *J. Funct. Anal.* **8** (1971), 52–75.

[7] J-P. Gossez, 'Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients', *Trans. Amer. Math. Soc.* **190** (1974), 163–205.

[8] G. Hardy, 'Extensions of theorems of Gagliardo and Marcus and Mizel to Orlicz spaces', *Bull. Austral. Math. Soc.* **23** (1981), 121–138.

[9] G. Hardy, 'Nemitsky operators between Orlicz-Sobolev spaces', *Bull. Austral. Math. Soc.* **30** (1984), 251–269.

[10] G. Hardy, 'Demicontinuity of Nemitsky operators on Orlicz-Sobolev spaces', *Bull. Austral. Math. Soc.* **37** (1988), 29–42.

[11] M.A. Krasnosel'skiĭ and Ya.B. Rutickiĭ, *Convex functions and Orlicz spaces*, (Russian) (Fizmatgiz, Moscow, 1958). English translation (Noordhoff, Gronigen, 1961).

[12] S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of linear operators*, (Russian) (Nauka, Moscow, 1978). English translation, *Math. Monogr. Amer. Math. Soc.* **54**. Providence, 1982 .

[13] A. Kufner, O. John and S. Fučik, *Function spaces* (Noordhoff Leyden, 1977).

[14] M. Marcus and V.J. Mizel, 'Absolute continuity on tracks and mappings of Sobolev spaces', *Arch. Rational Mech. Anal.* **45** (1972), 294–320.

[15] M. Marcus and V.J. Mizel, 'Nemitsky operators on Sobolev spaces', *Arch. Rational Mech. Anal.* **51** (1973), 347–370.

[16] M. Marcus and V.J. Mizel, 'Continuity of certain Nemitsky operators on Sobolev spaces and the chain rule', *J. Analyse Math* **28** (1975), 303–334.

[17] M. Marcus and V.J. Mizel, 'Representation theorems for nonlinear disjointly additive

functionals and operators on Sobolev spaces', *Trans. Amer. Math. Soc.* **228** (1977), 1–45.

[18] M. Marcus and V.J. Mizel, 'Superposition mappings which operate on Sobolev spaces', *Nonlinear Anal. TMA* **2** (1978), 257–258.

[19] M. Marcus and V.J. Mizel, 'Every superposition operator mapping one Sobolev space into another is continuous', *J. Funct. Anal.* **33** (1979), 217–229.

[20] M. Marcus and V.J. Mizel, 'Complete characterization of functions which act, via superposition, on Sobolev spaces', *Trans. Amer. Math. Soc.* **251** (1979), 187–218.

[21] M. Marcus and V.J. Mizel, 'A characterization of first order non-linear partial differential operators on Sobolev spaces', *J. Funct. Anal.* **38** (1980), 118–138.

[22] V.G. Maz'ya and T.O. Shaposhnikova, 'Multipliers of S.L. Sobolev spaces in a domain', (Russian), *Math. Nachr.* **99** (1980), 165–183.

[23] V.G. Maz'ya and T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions* (Pitman, London, 1985).

[24] V.B. Moseenkov, 'Composition of functions in Sobolev spaces', (Russian), *Ukrain. Mat. Zh.* **34** (1982), 384–388.

[25] T. Valent, 'A property of multiplication in Sobolev spaces: some applications', *Rend. Sem. Mat. Univ. Padova* **74** (1985), 63–73.

[26] T. Valent, *Boundary value problems of finite elasticity - local theorems on existence, uniqueness and analytic dependence on data* (Springer-Verlag, Berlin, Heidelberg, New York, 1987).

[27] S.W. Wang, 'On the products of Orlicz spaces', *Bull. Acad. Polon. Sci.* **11** (1963), 19–22.

[28] P.P. Zabrejko, 'Nonlinear integral operators', (Russian), *Voron. Gos. Univ. Trudy Sem. Funk. Anal.* **8** (1966), 1–148.

[29] P.P. Zabrejko, *On the theory of integral operators in ideal function spaces*, (Russian) (Doct. Dissertation, Univ. Voronezh, 1968).

[30] P.P. Zabrejko, A.I. Koshelev, M.A. Krasnosel'skiĭ, S.G. Mikhlin, L.S. Rakovshchik and V.Ya Stetsenko, *Integral equations*, (Russian) (Nauka, Moscow, 1968). English translation (Noordhoff, Leyden, 1975).

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