ANZIAM J. 52(2011), 250–262
doi:10.1017/S144618111100068X

OPTIMAL INVESTMENT AND REINSURANCE IN A JUMP DIFFUSION RISK MODEL

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(Received 12 January, 2010; revised 10 March, 2011)

Abstract

We consider an insurance company whose surplus is governed by a jump diffusion risk process. The insurance company can purchase proportional reinsurance for claims and invest its surplus in a risk-free asset and a risky asset whose return follows a jump diffusion process. Our main goal is to find an optimal investment and proportional reinsurance policy which maximizes the expected exponential utility of the terminal wealth. By solving the corresponding Hamilton–Jacobi–Bellman equation, closed-form solutions for the value function as well as the optimal investment and proportional reinsurance policy are obtained. We also discuss the effects of parameters on the optimal investment and proportional reinsurance policy by numerical calculations.

2010 *Mathematics subject classification*: primary 60G65; secondary 91B30.

Keywords and phrases: jump diffusion risk model, proportional reinsurance, investment, compound Poisson process, exponential utility, Hamilton–Jacobi–Bellman equation.

1. Introduction

In recent years, optimization problems with various objectives in the control of investment and reinsurance have become a topic of much interest in actuarial science. In particular, maximizing the expected utility, maximizing the expected discounted dividend payment and minimizing the probability of ruin are three important objective functions which have attracted a great deal of interest. These problems have been dealt with through optimization and stochastic control techniques. There have been three major types of mathematical model for the insurance company surplus. In the first type of model, the surplus process is represented by the classical risk process, that is, the Cramer–Lundberg model [6]. In the second type, the surplus process is represented by a diffusion process, namely a diffusion approximation to the classical

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Cramer–Lundberg risk model. In the third, the surplus process is represented by a jump diffusion process, namely the perturbed compound Poisson process [7].

For the diffusion risk model, Jeanblanc-Picque and Shiryaev [18], Asmussen and Taksar [1], Højgaard and Taksar [15, 16] and others considered the optimal policy to maximize the expected value of discounted dividends paid until time of ruin. For the diffusion surplus process, Browne [3–5], Taksar and Markussen [29], Promislow and Young [25], Bai and Guo [2] and Luo et al. [24] studied the optimal policy to minimize the ruin probability. For the classical risk process, Hipp and Plum [13, 14], Schmidli [27, 28] and Gaier et al. [9] studied the optimal policy to minimize the ruin probability. For the jump diffusion risk process, Liu and Yang [23], Yang and Zhang [30], Liang and Guo [20, 21], Lin [22] and Qian and Lin [26] considered the optimal policy to minimize the ruin probability.

Browne [3] first considered the case where the surplus process is modelled by a Brownian motion with drift and the risky asset follows a geometric Brownian motion, and found the optimal investment strategy to maximize the expected exponential utility of the terminal wealth. Bai and Guo [2] found the optimal investment and proportional reinsurance policy to maximize the expected exponential utility of the terminal wealth for the diffusion risk model. For the jump diffusion risk model, Irgens and Paulsen [17] obtained the optimal investment and proportional reinsurance in a layered policy to maximize the expected exponential utility of the terminal wealth. Yang and Zhang [30] obtained the optimal investment policy to maximize the expected exponential utility of the terminal wealth for the jump diffusion risk model, but without reinsurance.

In the aforementioned works, the price of the risky asset was assumed to follow a geometric Brownian motion. It is well known that pure diffusion processes are not an adequate representation of the characteristics of investment returns from risky assets. Therefore, such an assumption is unrealistic. Jump diffusion processes have traditionally been favoured as a better representation of investment returns from risky assets. The relationships between the optimal policy and the parameters, and the effects of a jump on the optimal policy were also not considered in the aforementioned works. We note also that in most of the literature, the premium of reinsurance is usually obtained via the expected value principle. Hald and Schmidli [12] studied the optimal reinsurance policy to maximize the adjustment coefficient of the ruin probability with the variance premium principle. So far, the application of the variance premium principle to optimal investment and reinsurance for the jump diffusion risk process has not been reported in academic articles.

In this paper, the surplus process is modelled by a jump diffusion process. The insurance company is allowed to take proportional reinsurance with the variance premium principle and invest its surplus into a financial market consisting of a risk-free asset and a risky asset whose return follows a jump diffusion process. We consider the optimal investment and proportional reinsurance policy to maximize the expected exponential utility of the terminal wealth. We study the relationships between the optimal policy and the parameters, and also investigate the effects of the jump on the optimal policy by numerical calculations.

The rest of the paper is organized as follows. The risk model and market model are given in Section 2. We state the optimization problem and give the corresponding Hamilton–Jacobi–Bellman equation for the value function and the optimal investment and proportional reinsurance policy in Section 3. The closed-form solutions for the maximal expected exponential utility of the terminal wealth and the optimal investment and proportional reinsurance policy are studied in Section 4. Numerical calculations showing the relationships between the optimal policy and the parameters, and an economic analysis, are given in Section 5.

2. The risk and market models

In this paper, we use the standard assumptions of continuous-time financial models: continuous trading is allowed, no transaction cost or tax is involved in trading, and all assets are infinitely divisible. We also assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, F, P)$ satisfying the usual conditions, that is, $F = \{\mathcal{F}_t, t \geq 0\}$ is right continuous and P-complete.

We consider the classical compound Poisson risk model perturbed by a diffusion, or a jump diffusion risk model, in which the surplus $R(t)$ of the insurance company at time t is

$$R(t) = x + ct - \sum_{k=1}^{N_1(t)} Y_k + \beta W_t^1 = x + ct - S(t) + \beta W_t^1,$$

where $x \geq 0$ denotes the initial capital; $c > 0$ is the premium rate per unit of time; $\{Y_k, k = 1, 2, \ldots\}$ is a sequence of independent and identically distributed nonnegative random variables with a common distribution function F with $F(0) = 0$ and density function f with finite mean μ_1 and second moment μ_2, where Y_k denotes the amount of the kth claim; $\{N_1(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 > 0$, representing the number of claims up to time t; $\{W_t^1, t \geq 0\}$ is a standard Brownian motion; and β is a constant, representing the diffusion volatility parameter. In addition, we assume that $\{Y_k, k = 1, 2, \ldots\}$, $\{N_1(t), t \geq 0\}$ and $\{W_t^1, t \geq 0\}$ are mutually independent.

The proportional reinsurance level is associated with the value $1 - a$, where $0 \leq a \leq 1$ is called the risk exposure. Suppose that the reinsurer uses the variance premium principle, that is, the premium becomes $(1 - a)\lambda_1\mu_1 + \alpha(1 - a)^2\lambda_1\mu_2$ with $c \leq \lambda_1(\mu_1 + \alpha\mu_2)$, where $\alpha > 0$ is a constant. The corresponding surplus process of the insurance company after proportional reinsurance becomes

$$R(t, a) = x + [c - (1 - a)\lambda_1\mu_1 - \alpha(1 - a)^2\lambda_1\mu_2]t - aS(t) + \beta W_t^1.$$

We assume that there are two assets available for investment in the financial market: a risk-free asset, whose price at time t is denoted by B_t, and a risky asset, whose price at time t is denoted by P_t. The price B_t of the risk-free asset is assumed to follow

$$dB_t = rB_t\, dt,$$

where $r > 0$ is the risk-free interest rate. The price P_t of the risky asset is assumed to follow the geometric Levy process

$$dP_t = P_t\Big[\mu\, dt + \sigma\, dW_t^2 + \int_R zN\,(dt, dz)\Big], \tag{2.1}$$

where $\mu \geq r$, σ are positive constants, $\{W_t^2, t \geq 0\}$ is a standard Brownian motion and $\int_0^t \int_R zN\,(ds, dz)$ is a compound Poisson process, that is,

$$\int_0^t \int_R zN\,(ds, dz) = \sum_{i=1}^{N_2(t)} Z_i,$$

where $\{N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_2 > 0$ and $\{Z_i, i = 1, 2, \ldots\}$ are independent identically distributed random variables with a common distribution G. We assume that $\{Y_k, k = 1, 2, \ldots\}$, $\{N_1(t), t \geq 0\}$, $\{Z_i, i = 1, 2, \ldots\}$ and $\{N_2(t), t \geq 0\}$ are mutually independent. Let ρ denote the correlation coefficient of the two standard Brownian motions, that is, $\langle W^1, W^2\rangle_t = \rho t$, $\rho \in [-1, 1]$.

Let $b(t)$ be the amount of money being invested in the risky asset at time t. The remaining portion of the surplus is invested in the risk-free asset. At any time $t \geq 0$, $a = a(t)$ and $b = b(t)$ are chosen by the insurance company. We denote $\pi(\cdot) = (a(\cdot), b(\cdot))$. Once the policy $\pi(\cdot)$ is chosen, the dynamics of the wealth process of the insurance company become

$$dX(t, \pi) = b(t)\frac{dP_t}{P_t} + (X(t, \pi) - b(t))\frac{dB_t}{B_t} + dR(t, a),$$

or, more explicitly,

$$\begin{aligned} dX(t, \pi) = {} & [(\mu - r)b(t) + rX(t, \pi) + c - (1 - a(t))\lambda_1\mu_1 - \alpha(1 - a(t))^2\lambda_1\mu_2]\, dt \\ & + \beta dW_t^1 - a(t)\, dS(t) + b(t)\sigma\, dW_t^2 + b(t)\int_R zN\,(dt, dz), \\ X(0) = {} & x. \end{aligned} \tag{2.2}$$

A control policy $\pi(\cdot) = (a(\cdot), b(\cdot))$ is said to be admissible if $a(\cdot)$ and $b(\cdot)$ are predictable with respect to F and for each $t \geq 0$.

(1) $0 \leq a(t) \leq 1$, and
(2) $P\{\int_0^\infty b^2(t)\, dt < \infty\} = 1$.

The set of all admissible policies is denoted by Π.

Let

$$h(r) = E[e^{rY}] = \int_0^\infty e^{ry}\, dF(y)$$

be the moment generating function of the claim size Y, and assume that there exists $r_\infty > 0$ such that $h(r) \uparrow \infty$ when $r \uparrow r_\infty$ (we allow the possibility $r_\infty = \infty$). Then $h(0) = 1$ and h is increasing, convex and continuous on $[0, r_\infty)$.

3. Maximizing the expected exponential utility

Suppose now that the insurance company is interested in maximizing the utility function for its terminal wealth, that is, the wealth at time T. The utility function is denoted by $u(x)$, and we assume that $u' > 0$ and $u'' < 0$. For a policy π, we define the utility attained by the insurance company from state x at time t as

$$V^{\pi}(t, x) = E[u(X(T, \pi)) \mid X(t) = x],$$

where $\{X(t, \pi), t \geq 0\}$ is the wealth process under the policy π. Our objective is to find the optimal value function

$$V(t, x) = \sup_{\pi \in \Pi} V^{\pi}(t, x) \tag{3.1}$$

and the optimal policy $\pi^* = (a^*, b^*)$ such that

$$V^{\pi^*}(t, x) = V(t, x).$$

Since u is an increasing concave function, there exists a unique optimal policy π^* such that the expected utility reaches its maximum.

In this paper, we assume that the insurance company has an exponential utility function

$$u(x) = m - \frac{\delta}{\gamma} e^{-\gamma x},$$

where $\delta > 0$ and $\gamma > 0$ are constants. The utility function has constant absolute risk aversion parameter γ. Such utility functions play a prominent role in insurance mathematics and actuarial practice, since they are the only utility functions under which the principle of "zero utility" gives a fair premium that is independent of the level of reserves of an insurance company [10, 11].

We start with the associated Hamilton–Jacobi–Bellman (HJB) equation for the value function $V(t, x)$. The proof is standard [8, Lemma 4.2].

THEOREM 3.1. *Assume that $V(t, x)$ defined by (3.1) is continuously differentiable in t on $[0, T]$ and twice continuously differentiable in x on $\mathbb{R}$, that is, $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$. Then $V(t, x)$ satisfies the Hamilton–Jacobi–Bellman equation*

$$\begin{aligned} 0 = \max_{\pi \in \Pi} \Big\{ & V_t + [(\mu - r)b + rx + c - (1 - a)\lambda_1\mu_1 - \alpha(1 - a)^2\lambda_1\mu_2]V_x \\ & + \frac{1}{2}[\beta^2 + 2b\beta\sigma\rho + b^2\sigma^2]V_{xx} + \lambda_1 \int_0^{\infty} [V(t, x - ay) - V(t, x)]\, F(dy) \\ & + \lambda_2 \int_{-\infty}^{\infty} [V(t, x + bz) - V(t, x)]\, G(dz) \Big\} \end{aligned} \tag{3.2}$$

with boundary condition

$$V(T, x) = m - \frac{\delta}{\gamma} e^{-\gamma x}. \tag{3.3}$$

The following verification theorem is essential in solving the associated stochastic control problem. The proof is similar to that of Theorem 5.1 of Fleming and Soner [8].

THEOREM 3.2. *If* $W(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ *satisfies the Hamilton–Jacobi–Bellman equation* (3.2) *subject to the boundary conditions* (3.3)*, then the value function* $V(t, x)$ *given by* (3.1) *coincides with* $W(t, x)$*, that is,*

$$V(t, x) = W(t, x).$$

Furthermore, if $\pi^* = (a^*, b^*)$ *satisfies*

$$\begin{aligned}0 = W_t &+ [(\mu - r)b^* + rx + c - (1 - a^*)\lambda_1\mu_1 - \alpha(1 - a^*)^2\lambda_1\mu_2]W_x \\ &+ \frac{1}{2}[\beta^2 + 2b^*\beta\sigma\rho + (b^*)^2\sigma^2]W_{xx} + \lambda_1 \int_0^\infty [W(t, x - a^*y) - W(t, x)]\, F(dy) \\ &+ \lambda_2 \int_{-\infty}^\infty [W(t, x + b^*z) - W(t, x)]\, G(dz)\end{aligned}$$

for all $(t, x) \in [0, T] \times \mathbb{R}$*, then the policy* $\pi^* = (a^*, b^*)$ *is an optimal policy. That is,*

$$W(t, x) = V(t, x) = V^{\pi^*}(t, x).$$

4. Optimal investment and reinsurance

According to the boundary condition (3.3), we try a solution to (3.2) with the parametric form

$$V(t, x) = m - \frac{\delta}{\gamma} \exp(-\gamma x e^{r(T-t)} + h(T - t)), \tag{4.1}$$

where $h(\cdot)$ is a suitable function such that (4.1) is a solution of (3.2), and (3.3) implies that $h(0) = 0$.

By (4.1),

$$V_t = [V(t, x) - m][\gamma x r e^{r(T-t)} - h'(T - t)], \tag{4.2}$$

$$V_x = -\gamma e^{r(T-t)}[V(t, x) - m], \quad V_{xx} = \gamma^2 e^{2r(T-t)}[V(t, x) - m], \tag{4.3}$$

$$\begin{aligned}&\int_0^\infty [V(t, x - ay) - V(t, x)]\, F(dy) \\ &\quad = [V(t, x) - m] \int_0^\infty [\exp(ay\gamma e^{r(T-t)}) - 1]\, F(dy),\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\int_{-\infty}^\infty [V(t, x + b^*z) - V(t, x)]\, G(dz) \\ &\quad = [V(t, x) - m] \int_{-\infty}^\infty [\exp(-bz\gamma e^{r(T-t)}) - 1]\, G(dz).\end{aligned} \tag{4.5}$$

On substitution of (4.2)–(4.5) into (3.3), we have after simplification

$$0 = \inf_{\pi\in\Pi} g(a, b), \tag{4.6}$$

where

$$g(a,b) = -h'(T-t) - \gamma[(\mu - r)b + c - (1-a)\lambda_1\mu_1 - \alpha(1-a)^2\lambda_1\mu_2]e^{r(T-t)} + \frac{\gamma^2}{2}[\beta^2 + 2b\beta\sigma\rho + b^2\sigma^2]e^{2r(T-t)} + \lambda_1 \int_0^\infty [\exp(ay\gamma e^{r(T-t)}) - 1]\, F(dy) + \lambda_2 \int_{-\infty}^\infty [\exp(-bz\gamma e^{r(T-t)}) - 1]\, G(dz).$$

Setting $\partial g(a,b)/\partial a = 0$, we obtain

$$\mu_1 + 2\alpha\mu_2(1-a) - \int_0^\infty y \exp(ay\gamma e^{r(T-t)})\, F(dy) = 0. \tag{4.7}$$

Setting $\partial g(a,b)/\partial b = 0$, we obtain

$$\mu - r + \lambda_2 \int_{-\infty}^\infty z \exp(-bz\gamma e^{r(T-t)})\, G(dz) - b\gamma\sigma^2 e^{r(T-t)} - \beta\gamma\sigma\rho e^{r(T-t)} = 0. \tag{4.8}$$

Since the proportional reinsurance policy is assumed to satisfy $0 \le a \le 1$, we need the following lemma.

Lemma 4.1. *Equation* (4.7) *has a unique positive root* $\hat{a}$ *with* $0 < \hat{a} < 1$. *Equation* (4.8) *has a finite root* $\hat{b}$.

Proof. Let

$$h(a) = \mu_1 + 2\alpha\mu_2(1-a) - \int_0^\infty y \exp(ay\gamma e^{r(T-t)})\, F(dy).$$

Then

$$h'(a) = -2\alpha\mu_2 - \int_0^\infty y^2\gamma e^{r(T-t)} \exp(ay\gamma e^{r(T-t)})\, F(dy) < 0$$

and

$$h''(a) = -\int_0^\infty y^3\gamma^2 e^{2r(T-t)} \exp(ay\gamma e^{r(T-t)})\, F(dy) < 0,$$

so $h(a)$ is a monotone decreasing concave function. As

$$h(0) = 2\alpha\mu_2 > 0 \quad \text{and} \quad h(1) = \mu_1 - \int_0^\infty y \exp(y\gamma e^{r(T-t)})\, F(dy) < 0,$$

(4.7) has a unique positive root $\hat{a}$ with $0 < \hat{a} < 1$.

Let

$$g(b) = \mu - r + \lambda_2 \int_{-\infty}^\infty z \exp(-bz\gamma e^{r(T-t)})\, G(dz) - b\gamma\sigma^2 e^{r(T-t)} - \beta\gamma\sigma\rho e^{r(T-t)}.$$

Then

$$g'(b) = -\lambda_2\gamma e^{r(T-t)} \int_{-\infty}^\infty z^2 \exp(-bz\gamma e^{r(T-t)})\, G(dz) - \gamma\sigma^2 e^{r(T-t)} < 0,$$

so $g(b)$ is a monotone decreasing function. Furthermore, $\lim_{b\to-\infty} g(b) > 0$ and $\lim_{b\to\infty} g(b) < 0$, and hence equation (4.8) has a finite root $\hat{b}$. □

Substituting $\hat{a}$ and $\hat{b}$ into (4.6), we obtain

$$\begin{aligned} h'(T-t) = &-\gamma[(\mu - r)\hat{b} + c - (1-\hat{a})\lambda_1\mu_1 - \alpha(1-\hat{a})^2\lambda_1\mu_2]e^{r(T-t)} \\ &+ \frac{\gamma^2}{2}[\beta^2 + 2\hat{b}\beta\sigma\rho + \hat{b}^2\sigma^2]e^{2r(T-t)} + \lambda_1\int_0^\infty [\exp(\hat{a}y\gamma e^{r(T-t)}) - 1]\, F(dy) \\ &+ \lambda_2\int_{-\infty}^\infty [\exp(-\hat{b}z\gamma e^{r(T-t)}) - 1]\, G(dz), \end{aligned}$$

together with the initial condition

$$h(0) = 0.$$

If we know the distributions of the claim size Y and the jump size Z of the risky asset then we can obtain the closed-form expression for $h(T-t)$.

From the above discussion, we obtain the following theorem.

Theorem 4.1. *For the wealth process (2.2), the optimal proportional reinsurance policy $a^*(t)$ to maximize the expected exponential utility is the unique positive root of the equation*

$$\mu_1 + 2\alpha\mu_2(1 - a(t)) - \int_0^\infty y\exp(a(t)y\gamma e^{r(T-t)})\, F(dy) = 0. \tag{4.9}$$

The optimal investment policy $b^*(t)$ to maximize the expected exponential utility is the finite root of the equation

$$\mu - r + \lambda_2\int_{-\infty}^\infty z\exp(-b(t)z\gamma e^{r(T-t)})\, G(dz) - b(t)\gamma\sigma^2 e^{r(T-t)} - \gamma\beta\sigma\rho e^{r(T-t)} = 0. \tag{4.10}$$

The maximal expected exponential utility is

$$V(t, x) = m - \frac{\delta}{\gamma}\exp(-\gamma x e^{r(T-t)} + h(T-t)),$$

where $h(T-t)$ satisfies

$$\begin{aligned} h'(T-t) = &-\gamma[(\mu - r)b^*(t) + c - (1-a^*(t))\lambda_1\mu_1 - \alpha(1-a^*(t))^2\lambda_1\mu_2]e^{r(T-t)} \\ &+ \frac{\gamma^2}{2}[\beta^2 + 2b^*(t)\beta\sigma\rho + (b^*(t))^2\sigma^2]e^{2r(T-t)} \\ &+ \lambda_1\int_0^\infty [\exp(a^*(t)y\gamma e^{r(T-t)}) - 1]\, F(dy) \\ &+ \lambda_2\int_{-\infty}^\infty [\exp(-b^*(t)z\gamma e^{r(T-t)}) - 1]\, G(dz), \end{aligned}$$

with the initial condition $h(0) = 0$.

Remark 4.1. If $a^* = 0$ and $\lambda_2 = 0$ then the result is consistent with that of Yang and Zhang [30].

5. Numerical examples and economic analysis

In this section, we present some illustrative numerical examples. We study the relationships between the optimal investment and proportional reinsurance policy and the parameters, and also investigate the effects of the jump on optimal investment.

Suppose that the claim sizes are exponentially distributed with parameter 1, that is, the density function is

$$f(y) = e^{-y}, \quad y \geq 0.$$

Then equation (4.9) becomes

$$1 + 4\alpha[1 - a(t)] - \frac{1}{[1 - a(t)\gamma e^{r(T-t)}]^2} = 0. \tag{5.1}$$

Assume that the jump size Z of the risky asset has a double exponential distribution with density

$$g(z) = \begin{cases} p\eta_1 e^{-\eta_1 z} & \text{if } z \geq 0 \\ q\eta_2 e^{\eta_2 z} & \text{if } z < 0, \end{cases}$$

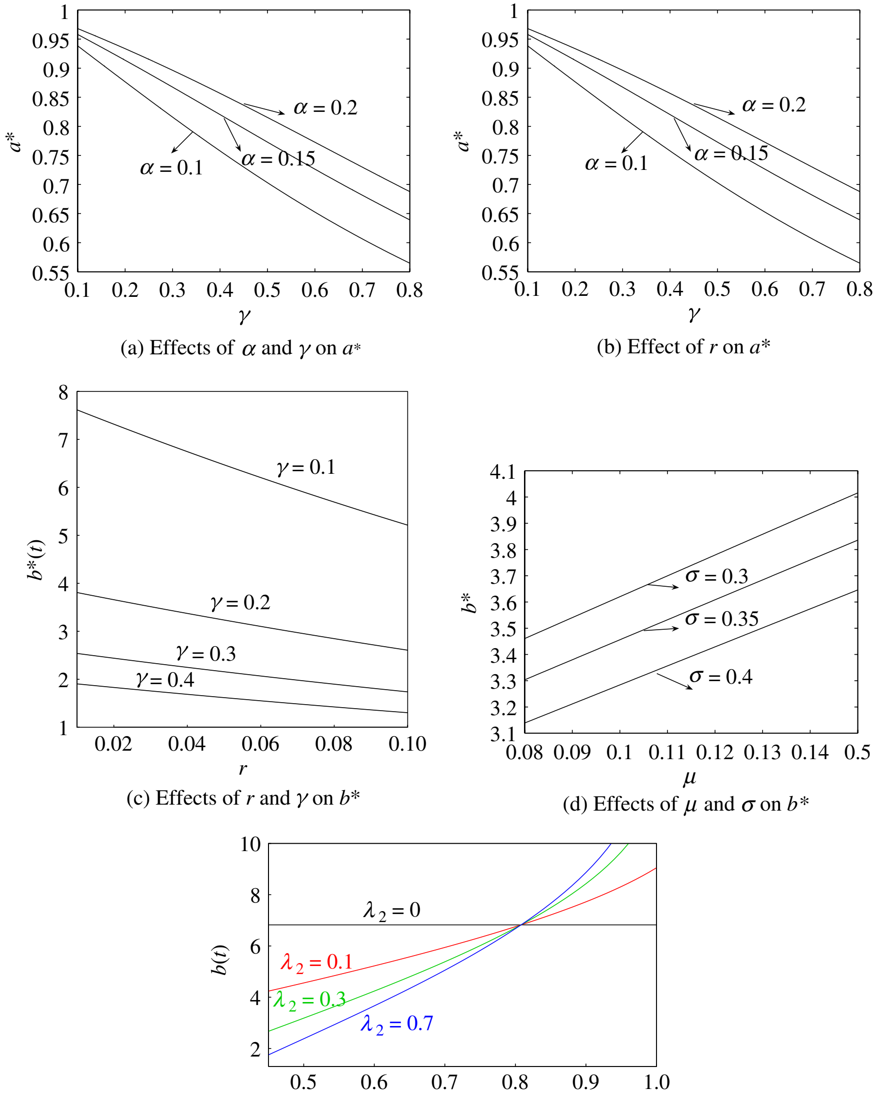
where $p, q \geq 0$, $p + q = 1$ represent the probabilities of upward and downward jumps [19]. Then equation (4.10) becomes

$$\mu - r + \frac{\lambda_2 p\eta_1}{[\eta_1 + b(t)\gamma e^{r(T-t)}]^2} - \frac{\lambda_2 q\eta_2}{[\eta_2 - b(t)\gamma e^{r(T-t)}]^2} - b(t)\sigma^2\gamma e^{r(T-t)} - \gamma\beta\sigma\rho e^{r(T-t)} = 0. \tag{5.2}$$

5.1. Optimal proportional reinsurance policy

Example 5.1. Assume that $r = 0.05$, $T = 4$, $t = 2$, $\rho = 0$, $\alpha = 0.1, 0.15, 0.2$ and $\gamma \in [0.1, 0.8]$. We calculate the optimal proportional reinsurance policy $a^*(t)$ by (5.1). The results are presented in Figure 1(a). We see that $a^*(t)$ is a decreasing function of γ. This is intuitive, as γ is the absolute risk aversion (ARA) parameter. The larger γ is, the less aggressive the insurer will be, and hence the less retention level the insurer will hold. We also see that $a^*(t)$ is decreasing in α. A large α yields a high retention level of proportional reinsurance. This simply states that as the premium of reinsurance increases, the insurer should retain a greater share of each claim.

Example 5.2. Assume that $\gamma = 0.2$, $T = 4$, $t = 2$, $\rho = 0$, $\alpha = 0.15$ and $r \in [0.01, 0.08]$. We calculate the optimal proportional reinsurance policy $a^*(t)$ by (5.1) and present the results in Figure 1(b). We find that $a^*(t)$ is decreasing in r. As r is the risk-free interest rate, the larger r is, the greater the expected income of the risk-free asset, the larger the income the insurance company will obtain from investment, and hence the less risk the insurance company will wish to share in each claim.



(a) Effects of α and γ on $a*$

(b) Effect of r on $a*$

(c) Effects of r and γ on $b*$

(d) Effects of μ and σ on $b*$

(e) Effects of λ_2 and p on $b*$

FIGURE 1. Effects of parameters on the optimal proportional reinsurance policy a^* and optimal investment policy b^*.

5.2. Optimal investment policy

EXAMPLE 5.3. Assume that $\mu = 0.1$, $\sigma = 0.2$, $T = 4$, $t = 2$, $\lambda_2 = 2$, $p = 2/3$, $q = 1/3$, $\rho = 0$, $\eta_1 = 2$ and $\eta_2 = 3$. For $r \in [0.01, 0.10]$ and $\gamma = 0.1, 0.2, 0.3, 0.4$, we calculate the optimal investment policy $b^*(t)$ by (5.2). The results are presented in Figure 1(c). We see that $b^*(t)$ is decreasing in γ. As γ is the absolute risk aversion parameter, the larger γ is, the less aggressive the insurance company will be, and hence the less the insurance company will wish to invest in the risky asset. We also see that $b^*(t)$ is a decreasing function of r. As r is the risk-free interest rate, the larger r is, the greater the expected income of the risk-free asset, and hence the more the insurance company will wish to invest in the risk-free asset.

EXAMPLE 5.4. Assume that $r = 0.05$, $T = 4$, $t = 2$, $\lambda_2 = 2$, $p = 2/3$, $q = 1/3$, $\gamma = 0.15$, $\rho = 0$, $\eta_1 = 2$ and $\eta_2 = 3$. For $\mu \in [0.08, 0.15]$ and $\sigma = 0.30, 0.35, 0.40$, we calculate the optimal investment policy $b^*(t)$ by (5.2) and present the results in Figure 1(d). We find that $b^*(t)$ is decreasing in σ, which is the volatility of the risky asset (see (2.1)). The larger σ is, the riskier the risky asset will be, and hence the less the insurance company will wish to invest in the risky asset. We also see that $b^*(t)$ is an increasing function of μ, which describes the rate of the income of the risky asset (see (2.1)). The larger μ is, the greater the expected income of the risky asset will be, and hence the more the insurance company will wish to invest in the risky asset.

EXAMPLE 5.5. Assume that $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\gamma = 0.15$, $T = 4$, $t = 2$, $\rho = 0$, $\eta_1 = 2$ and $\eta_2 = 3$. For $\lambda_2 = 0, 0.5, 1.0, 2.0$ and $p \in [0.35, 0.80]$, we calculate the optimal investment policy $b^*(t)$ by (5.2) and present the results in Figure 1(e). We find that $b^*(t)$ is first decreasing and then increasing in λ_2, which is the jump intensity of the risky asset. If p is small then the larger λ_2 is, the greater the possibility of a downward jump of the risky asset will be, the riskier the risky asset will be, and hence the less the insurance company will wish to invest in the risky asset. If p is large then the larger λ_2 is, the greater the possibility of an upward jump of the risky asset will be, the greater the expected income of the risky asset will be, and hence the more the insurance company will wish to invest in the risky asset. We also see that $b^*(t)$ is an increasing function of p. As p represents the probability of an upward jump of the risky asset, the larger p is, the greater the expected income of the risky asset will be, and hence the more the insurance company will wish to invest in the risky asset.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (Project No. 10901164 and 90820302). We also thank the reviewers and the editor for many helpful and valuable comments and suggestions that improved the presentation.

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