J. Austral. Math. Soc. **20** (Series A) (1975), 142–145.

LEFT INVARIANT MEASURE IN TOPOLOGICAL SEMIGROUPS

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(Received 27 October 1972)

Communicated by G. B. Preston

Abstract

We consider the problem of the existence of a left invariant measure in a class of topological semigroups. Several authors have considered this and related problems on semigroups satisfying similar conditions, but the invariance they considered is right invariance. This paper is different in that it deals with left invariance.

0. Preliminaries

Any undefined terms from Measure Theory will be as in Halmos (1950); undefined terms from Topology will be understood to be as in Kelley (1955); undefined terms from the Algebraic Theory of Semigroups will be as in Clifford and Preston (1961).

A *topological semigroup* is a Hausdorff space together with a continuous associative multiplication.

Throughout this paper $(S,\cdot,\mathcal{T})$ denotes a locally compact topological semigroup such that $(S,\cdot)$ is a left group; i.e., $(S,\cdot)$ is left simple and right cancellative.

Recall that a left group $(S,\cdot)$ is the union of pairwise disjoint groups (called its group components). See Clifford and Preston (1961; page 39) for a characterization of left groups. The following list of properties shows that $(S,\cdot,\mathcal{T})$ shares many of the pleasant properties of topological groups.

1. $S = \bigcup\{G(e): e \in E\}$ where $G(e) = \{x \in S: x = ex\}$ is the group component of S with identity e and E is the set of all idempotents of S.
2. E is a left zero subsemigroup of S; i.e., $e \in E, f \in E \Rightarrow ef = e$.
3. Each $e \in E$ is a right identity for all of S; i.e., $e \in E,\ x \in S \Rightarrow xe = x$.
4. $G(e)\cdot S = G(e)$.
5. $a \in G(e) \Rightarrow aS \subset G(e)$.
6. $A \subset S \Rightarrow A \cap G(e) \subset eA$.

7. E is (topologically) closed. (Doyle and Warne (1963; Theorem 2.4.))

8. Each $G(e)$ is (topologically) closed. This follows from the fact that $G(e)$ is the set of all fixed points of the map $\phi\colon S \to S$ given by $\phi(x) = ex$. (Problem (1966), see comments at the end of the solution of the problem).

9. Each $G(e)$ (with the relative topology) is a locally compact Hausdorff topological group (Ellis (1957)).

10. The mapping $\pi\colon G(f) \to G(e)$, given by $\pi(x) = ex$, is both an isomorphism and a homeomorphism of $G(f)$ onto $G(e)$.

11. Let $a \in G(e)$. The mapping $\phi\colon G(f) \to G(e)$, given by $\phi(x) = ax$, may be factored as $\phi = \alpha \circ \pi$ where π and α are mappings $G(f) \xrightarrow{\pi} G(e) \xrightarrow{\alpha} G(e)$ given by $\pi(x) = ex$ and $\alpha(y) = ay$ respectively.

Here is an *example* of a topological semigroup satisfying our conditions. Let G_i = the line on the xy-plane parallel to the x-axis such that the y-coordinate of each point of G_i is i. Let $S = \bigcup_{i=1}^{\infty} G_i$ and let S be given the topology it inherits by virtue of its being a subset of the xy-plane endowed with the usual (euclidean) topology. For $s = (a, i) \in S$ and $t = (b, j) \in S$, define $s.t = (a + b, i)$ where $a + b$ denotes the usual sum of the real numbers a and b.

1. Lemmas

Each $G(e)$ is a locally compact Hausdorff topological group and as such, possesses a left Haar measure. We *single out* a particular group $G(f)$. Let μ_f be a left Haar measure on $G(f)$. That is, $\mu_f(xB) = \mu_f(B)$ for each Borel set B in $G(f)$ and for each $x \in G(f)$; and, $\mu_f(U) > 0$ for each non-void open Borel set U in $G(f)$. We define μ_e on $G(e)$, $e \neq f$, as follows. We have noted that the mapping $\pi\colon G(f) \to G(e)$, given by $\pi(x) = ex$, is both an isomorphism and a homeomorphism of $G(f)$ onto $G(e)$. If B is a Borel set in $G(e)$, then $A = \pi^{-1}(B)$ is a Borel set in $G(f)$. Define $\mu_e(B) = \mu_f(A)$. Then μ_e is a left Haar measure on $G(e)$. We have thus set up a left Haar measure on each of the group components of S. Moreover, we have

LEMMA 1. *The measures on the groups agree under the mappings* $\pi\colon G(g) \to G(e)$ *given by* $\pi(x) = ex$. *That is, if* B *is a Borel set in* $G(g)$, *then* $\mu_g(B) = \mu_e(\pi(B))$.

For each Borel set B in S, we now *define* $\mu(B) = \sum_{e \in E} \mu_e(B \cap G(e))$ where the summation is to be understood as follows:

(a) when $\mu_e(B \cap G(e)) = 0$ for all except a countable number of e, the summation is taken over those e for which $\mu_e(B \cap G(e)) \neq 0$;

(b) when $\mu_e(B \cap G(e)) \neq 0$ for an uncountable number of e, the sum is to be regarded as ∞.

LEMMA 2. *μ is a Borel measure on S.*

LEMMA 3. *$\mu(B) \geqq \mu(aB)$ for each Borel set B and for each $a \in S$.*

PROOF. We may assume that there is a countable subfamily $E_0 = \{e_1, e_2, \cdots\}$ of E such that

$$\mu_e(B \cap G(e)) = 0 \quad \text{for } e \in E - E_0$$

and that $\mu(B) = \sum_{i=i}^{\infty} \mu_{e_i}(B \cap G(e_i))$. Here each $B_i = B \cap G(e_i)$ is a Borel set in $G(e_i)$. Let $a \in G(e)$. For each i, the mapping from $G(e_i)$ to $G(e)$ given by $x \to ax$ may be factored:

$$G(e_i) \xrightarrow{\pi} G(e) \xrightarrow{\alpha} G(e) \quad \text{where } \pi(x) = ex \text{ and } \alpha(y) = ay.$$

Then $B_i \xrightarrow{\pi} eB_i \xrightarrow{\alpha} aB_i$ and so

$$\mu_{e_i}(B_i) = \mu_e(eB_i) = \mu_e(aB_i).$$

Hence

$$\mu(B) = \sum_{i=1}^{\infty} \mu_{e_i}(B_i) = \sum_{i=1}^{\infty} \mu_e(aB_i) \geqq \mu_e\left(\bigcup_{i=1}^{\infty} aB_i\right) = \mu_e\left(a \bigcup_{i=1}^{\infty} B_i\right)$$
$$= \mu_e(aB) = \mu(aB).$$

2. Main Theorem

THEOREM. *Let $(S, \cdot\ \mathscr{T})$ be a locally compact toplogical semigroup such that $(S, \cdot)$ is a left group. Then there exists a Borel measure μ on $(S, \cdot, \mathscr{T})$ such that the restriction μ_e of μ to $G(e)$ is a left Haar measure for each $e \in E$. Here E is the set of all idempotents of S and $G(e)$ is the group component of S containing the idempotent e. Moreover, μ agrees under the topological isomorphisms of $G(f)$ onto $G(e)$ given by $x \to ex$, in the sense that $\mu_f(B) = \mu_e(eB)$ for each Borel set B in $G(f)$. We also have $\mu(aB) \leqq \mu(B)$ for each Borel set B in S and for each $a \in S$. To within a multiplicative constant, the measure μ is unique. Finally, μ is left invariant if and only if $(S, \cdot, \mathscr{T})$ is a topological group.*

PROOF. We prove the essential uniqueness of μ. Let ν be another such measure on $(S, \cdot, \mathscr{T})$. For each $e \in E$, the restrictions μ_e and ν_e of μ and ν to $G(e)$ are left Haar measures on the locally compact group $G(e)$ and so there exists a positive constant c such that $\nu_e = c\mu_e$. We have to show that the constant c does not depend on the cho ce of $e \in E$. Let B a Borel set in $G(e)$ such that $e \in B$. Now take $f \in E - \{e\}$. Then $A = fB$ is a Borel set in $G(f)$ such that $f \in A$. We have $\nu_f(A) = \nu_e(B)$ and $\mu_f(A) = \mu_e(B)$. Hence $\nu_e(B) = c\mu_e(B)$ implies $\nu_f(A) = c\mu_f(A)$. Hence $\nu_e = c\mu_c$ where the constant c is independent of $e \in E$. This means $\nu = c\mu$.

Suppose now that $(S, \cdot, \mathscr{T})$ admits a left Haar measure. We prove $(S, \cdot)$ is a group. Suppose not. Let e and f be two distinct idempotents of S. Let A be an open (relative topology) set in $G(e)$ such that $e \in A$ and $\bar{A}$ is compact. Let $B = fA$. Then B is a non-void open set in $G(f)$. Let $C = A \cup B$. Then $eC = A$,

https://doi.org/10.1017/S1446788700020437 Published online by Cambridge University Press

and so from the invariance $\mu(eC) = \mu(C)$ we have $\mu(A) = \mu(A) + \mu(B)$ or $\mu(B) = 0$ contradicting the fact that nonvoid open sets have positive measure.

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https://doi.org/10.1017/S1446788700020437 Published online by Cambridge University Press