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Nagoya Math. J.
Vol. 49 (1973), 77–89

ON THE DISTRIBUTION OF VALUES OF FUNCTIONS IN THE UNIT DISK

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1. Introduction.

Let $f(z)$ be a function analytic and bounded, $|f(z)| < 1$, in $|z| < 1$. Then, by Fatou's theorem the radial limit $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere on $|z| = 1$. Seidel [8, p. 208] and Calderón, González-Domínguez, and Zygmund [1] (see also [9, pp. 281–282]) proved the following: if $f^*(e^{i\theta})$ is of modulus 1 almost everywhere on an arc $a < \theta < b$ of $|z| = 1$, then either $f(z)$ is analytically continuable across this arc or the values $f^*(e^{i\theta})$, $a < \theta < b$, cover the circumference $|w| = 1$ infinitely many times. In this paper we shall be primarily concerned with the behavior of $f^*(e^{i\theta})$ on each side of a singular point $P = e^{i\theta_0}$, $a < \theta_0 < b$, for $f(z)$.

2. One-side Limits.

We shall say that $f(z)$ has a right-sided (left-sided) limit at $e^{i\theta_0}$ if there is a positive number δ such that $f^*(e^{i\theta})$ exists and is continuous for all θ, $\theta_0 - \delta \leq \theta \leq \theta_0$ ($\theta_0 \leq \theta \leq \theta_0 + \delta$). We, now, state the first result of this paper which extends the theorem of Seidel and Calderón, González-Domínguez, and Zygmund.

THEOREM 1. *Let $f(z)$ be analytic and bounded, $|f(z)| < 1$, in $|z| < 1$. If $f^*(e^{i\theta})$ is of modulus 1 almost everywhere on an arc $a < \theta < b$ of $|z| = 1$ and if $P = e^{i\theta_0}$, $a < \theta_0 < b$, is a singular point for $f(z)$, then either*

i) *the values of $f^*(e^{i\theta})$, $a < \theta < \theta_0$, cover $|w| = 1$ infinitely many times and $f(z)$ has a left-sided limit at $e^{i\theta_0}$ of modulus 1, or*

ii) *the values of $f^*(e^{i\theta})$, $\theta_0 < \theta < b$, cover $|w| = 1$ infinitely many times*

Received April 10, 1972.

* This research was supported in part by a grant from the Oklahoma State University College of Arts and Sciences Office of Research and Graduate Studies.

and $f(z)$ has a right-sided limit at $e^{i\theta_0}$ of modulus 1, *or*

iii) *the values of $f^*(e^{i\theta})$ for both arcs $a<\theta<\theta_0$ and $\theta_0<\theta<b$, respectively, cover $|w|=1$ infinitely many times.*

Proof. Without loss of generality, we may assume $a=2\pi-\gamma$, $b=\gamma$, $\theta_0=0$ where $0<\gamma<\pi$.

Suppose $f^*(e^{i\theta})$ assumes α, $|\alpha|=1$, only finitely many times on the arc $2\pi-\gamma<\theta<2\pi$. Then, we may also assume, without loss of generality, that $f^*(e^{i\theta})$ omits α on the arc $2\pi-\gamma<\theta<2\pi$.

Let $\zeta=L(w)$ be a bilinear transformation mapping $|w|\leq 1$ onto $\operatorname{Re}(\zeta)\geq 0$ such that $L(\alpha)=\infty$. The function $L(f(z))$ is analytic in $|z|<1$. The harmonic function $\operatorname{Re}(L(f(z)))$ is positive in $|z|<1$ with boundary values 0 almost everywhere on the arc $2\pi-\gamma<\theta<2\pi$. Thus,

$$L(f(z))=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)+i\operatorname{Im}(L(f(0)))$$

where $\mu(t)$ is a bounded non-decreasing function $[0,2\pi]$ [9, p. 152]. Let

$$\begin{aligned}u(r,\theta)&=\operatorname{Re}(L(f(z)))\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}d\mu(t)\end{aligned}$$

and

$$\begin{aligned}v(r,\theta)&=\operatorname{Im}(L(f(z)))\\&=\frac{1}{\pi}\int_0^{2\pi}\frac{r\sin(\theta-t)}{1+r^2-2r\cos(\theta-t)}d\mu(t)+\operatorname{Im}(L(f(0)))\,.\end{aligned}$$

We wish, now, to examine the function $\mu(\theta)$. Since $\mu(\theta)$ is non-decreasing on $[0,2\pi]$, its derivative $\mu'(\theta)$ exists almost everywhere on $[0,2\pi]$. The harmonic function $u(r,\theta)$ tends radially to $\mu'(\theta)$ at every point of differentiability of $\mu(\theta)$. Since $u(r,\theta)$ has boundary values 0 almost everywhere for $2\pi-\gamma<\theta<2\pi$, $\mu'(\theta)=0$ almost everywhere on $2\pi-\gamma<\theta<2\pi$.

Suppose $\mu(\theta)$ is not absolutely continuous on $2\pi-\gamma<\theta<2\pi$. Notice that $\mu(\theta)$ is of bounded variation on $[0,2\pi]$. Then one of the following is true: i) $\mu(\theta)$ is continuous and not identically constant on $(2\pi-\gamma,2\pi)$, ii) there exists θ^*, $2\pi-\gamma<\theta^*<2\pi$, such that $\mu(\theta)$ is discontinuous at θ^*. If i) is the case, then from a theorem in Saks [7, p. 128] it follows that there exists θ_1, $2\pi-\gamma<\theta_1<2\pi$, such that $\mu'(\theta)$ exists and is infinite

https://doi.org/10.1017/S0027763000015294 Published online by Cambridge University Press

at $\theta = \theta_1$. Thus, $\lim_{r\to 1} u(r, \theta_1) = \mu'(\theta_1) = +\infty$. This implies that $f^*(e^{i\theta_1}) = \alpha$, which is a contradiction. If ii) is the case, then by a lemma of Lohwater [4, p. 244] $\lim_{r\to 1} u(r, \theta^*) = +\infty$. Again, we have a contradiction, namely, $f^*(e^{i\theta^*}) = \alpha$. Thus, it follows that $\mu(\theta)$ is absolutely continuous on $2\pi - \gamma < \theta < 2\pi$.

Since $\mu'(\theta) = 0$ almost everywhere on $2\pi - \gamma < \theta < 2\pi$, $\mu(\theta)$ is constant on $2\pi - \gamma < \theta < 2\pi$. Therefore, $L(f(z))$ is analytic at each point $e^{i\theta}$, $2\pi - \gamma < \theta < 2\pi$, and, in particular, we have

$$L(f(e^{i\theta})) = \frac{1}{2\pi}\int_0^{2\pi-\gamma} \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} d\mu(t) + i \operatorname{Im}(L(f(0)))$$

on $2\pi - \gamma < \theta < 2\pi$. Hence, $f(z)$ is analytic at each point $e^{i\theta}$, $2\pi - \gamma < \theta < 2\pi$, and $|f(e^{i\theta})| = 1$ at each $e^{i\theta}$, $2\pi - \gamma < \theta < 2\pi$.

Let ε be a sufficiently small positive number. Since

$$\begin{aligned} v(1, \theta) &= \operatorname{Im}(L(f(0))) + \frac{1}{2\pi}\int_0^{2\pi-\gamma} \frac{\sin(\theta - t)}{1 - \cos(\theta - t)} d\mu(t) \\ &= \operatorname{Im}(L(f(0))) + \frac{1}{2\pi}\int_0^{2\pi-\gamma} \cot\frac{1}{2}(\theta - t) d\mu(t) \\ &> \operatorname{Im}(L(f(0))) + \frac{1}{2\pi}\int_0^{2\pi-\gamma} \cot\frac{1}{2}(\theta + \varepsilon - t) d\mu(t) \\ &= v(1, \theta + \varepsilon) , \end{aligned}$$

it follows that as θ approaches 2π through increasing values in $(2\pi - \gamma, 2\pi)$, $f(e^{i\theta})$ moves along $|w| = 1$ in a counterclockwise direction. Since $f(e^{i\theta})$ omits α, $|\alpha| = 1$, on $(2\pi - \gamma, 2\pi)$, $f(e^{i\theta})$ cannot wind indefinitely around $|w| = 1$ as θ approaches 2π, $2\pi - \gamma < \theta < 2\pi$. Hence, $f(z)$ has a right-sided limit w_1 of modulus 1 at $\theta_0 = 0$.

Suppose, next, that there exists a complex value β, $|\beta| = 1$, such that $f^*(e^{i\theta})$ assumes β only finitely many times on the arc $0 < \theta < \gamma$. Then, by the above argument, it follows that $f(z)$ has a left-sided limit w_2 of modulus 1 at $\theta_0 = 0$. By a well-known theorem of Lindelöf, $w_1 = w_2$ [2, p. 43]. Another well-known theorem of Lindelöf [6, p. 75], then, implies that the cluster set of $f(z)$ at $P = 1$ is $C(f, 1) = \{w_1\}$. But, since $P = 1$ is a singular point for $f(z)$, a theorem of Seidel [2, p. 95] states $C(f, 1) = \{|w| \leq 1\}$. We have a contradiction. Thus, $f(z)$ cannot have right-sided and left-sided limit at $P = 1$ simultaneously. This completes our proof.

A natural question which theorem 1 raises is this question: can functions $f(z)$ analytic and bounded, $|f(z)|<1$, in $|z|<1$ be found which exhibit each type of behavior as described in theorem 1? With regard to this question, we shall show by means of Blaschke products that theorem 1 is sharp in this sense. In fact, we shall give a necessary and sufficient condition for a Blaschke product to have a right-sided limit at $e^{i\theta_0}$.

3. Blaschke Products.

Let $\{a_k\}$ be a sequence of points in $|z|<1$ such that

$$\sum_{k=1}^{\infty}(1-|a_k|)<+\infty .$$

Then, the infinite product

$$B(z)=\prod_{k=1}^{\infty}\frac{|a_k|}{a_k}\frac{a_k-z}{1-\bar{a}_k z}$$

is a bounded, non-constant, holomorphic function in $|z|<1$. The function $B(z)$ is called a Blaschke product with zeros $\{a_k\}$. By Fatou's theorem the radial limit $B^*(e^{i\theta})$ exists almost everywhere on $|z|=1$. It is also known that the modulus of $B^*(e^{i\theta})$ is 1 almost everywhere on $|z|=1$. The following result of Frostman [3] (see also [2, p. 33–35]) gives a necessary and sufficient condition for $B^*(e^{i\theta_0})$ to be of modulus 1.

THEOREM A. *Let $B(z)$ be a Blaschke product with zeros $\{a_k\}$. Then, a necessary and sufficient condition that $B(z)$ and all its partial products have radial limit of modulus* 1 *at $e^{i\theta_0}$ is the convergence of*

$$\sum_{k=1}^{\infty}\frac{1-|a_k|}{|e^{i\theta_0}-a_k|} .$$

Remark. Geometrically, Frostman's condition implies that at most a finite number of zeros $\{a_k\}$ of $B(z)$ lie in any Stolz angle at $e^{i\theta_0}$.

For further properties of Blaschke products we refer the reader to [2, p. 28–38] or [9, p. 271–285].

THEOREM 2. *Let $B(z)$ be a Blaschke product with zeros $\{a_k\}$ which have $e^{i\theta_0}$ as a limit point and which lie in a Stolz angle at $e^{i\theta_0}$. Then, for each δ, $0<\delta<\pi/2$, the values of $B^*(e^{i\theta})$ for the arcs $\theta_0-\delta<\theta<\theta_0$ and $\theta_0<\theta<\theta_0+\delta$, respectively, cover $|w|=1$ infinitely many times.*

Proof. Suppose $B(z)$ had either a right-sided or a left-sided limit at $e^{i\theta_0}$. This limit would, of course, be of modulus 1. Then, by a theorem of Lindelöf, $B(z)$ would have angular limit at $e^{i\theta_0}$ of modulus 1. But, this cannot happen, since the sequence $\{a_k\}$, by assumption, lies in a Stolz angle at $e^{i\theta_0}$. Thus, theorem 2 follows from theorem 1.

We, now, state the main result of this section.

THEOREM 3. *Let $B(z)$ be a Blaschke product with zeros $\{a_k\}$. Then, $B(z)$ and all its partial products have a right-sided limit of modulus* 1 *at $e^{i\theta_0}$ if and only if*

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0}-a_k|} < +\infty ,$$

and there exist positive numbers δ and ε, $\varepsilon < 1$, such that there are no zeros $\{a_k\}$ in the region

$$\Delta = \{z \,|\, 1-\varepsilon < |z| < 1, \theta_0 - \delta < \arg(z) < \theta_0\} .$$

Proof. Utilizing the proper rotation of $|z|<1$, we may assume that $\theta_0 = 0$. Suppose the zeros $\{a_k\}$ of $B(z)$ satisfy:

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|1-a_k|} < +\infty , \tag{1}$$

and there exist positive numbers δ and ε, $\varepsilon < 1$, such that

$$\{a_1, a_2, a_3, \cdots\} \cap \{z \,|\, 1-\varepsilon < |z| < 1, -\delta < \arg(z) < 0\} = \emptyset . \tag{2}$$

Choose δ so that $0 < \delta < \pi/2$. Let $\{a_{m_j}\}$ be the set of zeros $\{a_k\}$ of $B(z)$ lying in $\{z \,|\, |z| < 1, 0 \leq \arg(z) \leq \pi/2\}$. Let $\{a_{n_j}\}$ be the remaining set of zeros $\{a_k\}$ of $B(z)$. From theorem A, (1) and (2), it follows that the radial limit $B^*(e^{i\theta})$ exists and is of modulus 1 for all θ, $-\delta/2 \leq \theta \leq 0$. In order to prove that conditions (1) and (2) are sufficient for $B(z)$ to have a right-sided limit at $z=1$ it suffices to show that $\arg(B^*(e^{i\theta}))$ is continuous for θ, $-\delta/2 \leq \theta \leq 0$. To do this we shall prove that for k sufficiently large

$$\left|\arg\left(\frac{|a_k|}{a_k}\frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right)\right|$$

is dominated by positive numbers M_k whose sum forms a convergent series. By virtue of this, we assume that $\{a_{m_j}\}$ and $\{a_{n_j}\}$ are both subsequences of $\{a_k\}$.

It is clear that

$$\begin{aligned}\left|\arg\left(\frac{|a_k|}{a_k}\frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right)\right| &= \left|\arg\left(\frac{\bar{a}_k(a_k - re^{i\theta})}{1-\bar{a}_k re^{i\theta}}\right)\right| \\ &= \left|\arg\left(1 - \frac{1-|a_k|^2}{1-\bar{a}_k re^{i\theta}}\right)\right| \\ &= \left|\arcsin\frac{(1-|a_k|^2)r(\beta_k\cos\theta - \alpha_k\sin\theta)}{|\bar{a}_k|\,|a_k - re^{i\theta}|\,|1-\bar{a}_k re^{i\theta}|}\right| \\ &= \arcsin\frac{(1-|a_k|^2)r\,|\beta_k\cos\theta - \alpha_k\sin\theta|}{|\bar{a}_k|\,|a_k - re^{i\theta}|\,|1-\bar{a}_k re^{i\theta}|}\end{aligned}$$

where $a_k = \alpha_k + i\beta_k$. Since $\arcsin x \leq \pi x/2$ for $0 \leq x \leq 1$, it suffices to show that, for k sufficiently large, the argument of the arc sin is dominated by positive numbers whose sum is a convergent series.

We, first, consider the zeros $\{a_{n_j}\}$. Let

$$d_1 = \inf\left|\frac{1}{\bar{a}_{n_j}} - re^{i\theta}\right|, \quad j = 1, 2, 3, \cdots, \quad 1 - \frac{\varepsilon}{2} < r < 1, \quad \frac{-\delta}{2} \leq \theta \leq 0,$$

and

$$d_2 = \inf|a_{n_j} - re^{i\theta}|, \quad j = 1, 2, 3, \cdots, \quad 1 - \frac{\varepsilon}{2} < r < 1, \quad -\frac{\delta}{2} \leq \theta \leq 0.$$

By (2), we have $d_1 > 0$ and $d_2 > 0$. Let $d = \min(d_1, d_2)$. If $k = n_j$, then, from the way in which d was chosen,

$$\begin{aligned}\frac{(1-|a_k|^2)r\,|\beta_k\cos\theta - \alpha_k\sin\theta|}{|\bar{a}_k|\,|a_k - re^{i\theta}|\,|1-\bar{a}_k re^{i\theta}|} &\leq \frac{4(1-|a_k|)}{|\bar{a}_k|^2\,|a_k - re^{i\theta}|\left|\dfrac{1}{\bar{a}_k} - re^{i\theta}\right|} \\ &\leq \frac{4}{|\bar{a}_k|^2\,d^2}(1-|a_k|)\end{aligned}$$

for $1 - \varepsilon/2 < r < 1$ and $-\delta/2 \leq \theta \leq 0$.

Next, we consider the zeros $\{a_{m_j}\}$. Let L_1 and L_2 be chords of $|z| < 1$ drawn from $z = 1$ inclined from the radius to $z = 1$ by an angle of $\delta/2$ and $\delta/4$, respectively. Let Δ_1 be the triangle with sides L_1 and the radii to the endpoints of the chord L_1. Let Δ_2 be the triangle formed in the same way as Δ_1, but, instead of L_1, we use the chord L_2. For θ, $-\delta/2 < \theta < 0$, let $\Delta_2(\theta)$ be the triangle obtained by rotating Δ_2

through an angle of θ about its vertex $z=0$ (see figure 1). Let $L_2(\theta)$ be the side of $\Delta_2(\theta)$ which is a chord of $|z|<1$. From the construction, it is clear that

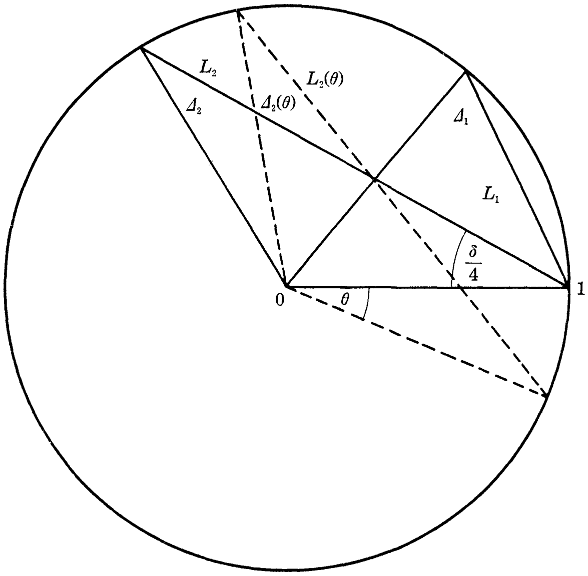

Fig. 1

$$L_2(\theta) \cap L_1 = \phi \quad \text{for} \quad -\frac{\delta}{2} < \theta < 0\,. \tag{3}$$

By (1) and the remark following theorem A, there exists a positive integer J such that a_{m_j} lies outside Δ_1 for $j>J$. Let $s(a,b)$ denote line segment joining the complex numbers a and b. Then, we denote by $\phi_j(\theta)$ the angle formed at $e^{i\theta}$ by $s(0,e^{i\theta})$ and $s(a_{m_j}, e^{i\theta})$, $-\delta/2 \le \theta \le 0$. We lengthen the segment $s(a_{m_j}, e^{i\theta})$ so that it is a chord $L'_j(\theta)$ of $|z|<1$ (see figure 2). Then,

$$L'_j(\theta) \cap L_1 \neq \emptyset \tag{4}$$

for $j<J$ and $-\delta/2 \le \theta < 0$. Thus, from (3) and (4) we have that the arc of $|z|=1$ cut off by $L_2(\theta)$ is greater than the arc of $|z|=1$ cut off by $L'_j(\theta)$ for $j>J$ and $-\delta/2 \le \theta < 0$. Thus,

$$\pi - 2\phi_j(\theta) < \pi - \frac{\delta}{2}$$

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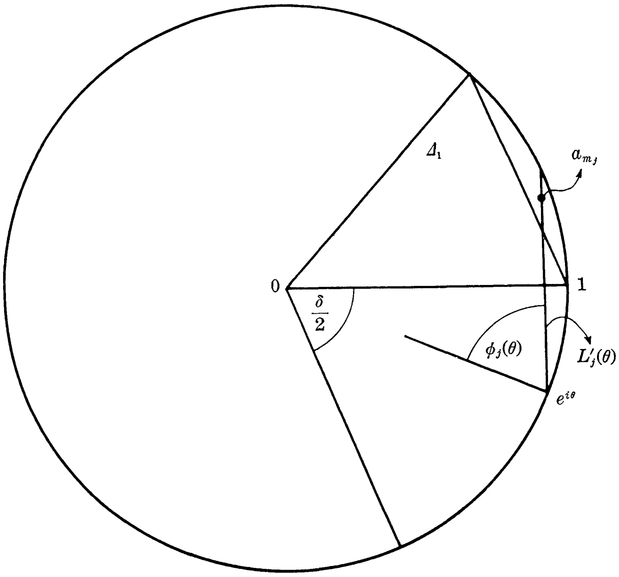

Fig. 2

for $j > J$ and $-\delta/2 \leq \theta < 0$. Also, since a_{m_j} lies outside Δ_1 for $j > J$, $\phi_j(0) > \delta/2 > \delta/4$ for $j > J$. Thus, $\phi_j(\theta) > \delta/4$ for $j > J$ and $-\delta/2 \leq \theta \leq 0$. This implies

$$\begin{aligned} |a_{m_j} - re^{i\theta}| &\geq |a_{m_j} - e^{i\theta}| \sin \phi_j(\theta) \\ &\geq |a_{m_j} - e^{i\theta}| \sin \frac{\delta}{4} \end{aligned}$$

for $j > J$ and $-\delta/2 \leq \theta \leq 0$. But, for all j and $-\delta/2 \leq \theta \leq 0$, $-\sin \theta \leq |a_{m_j} - e^{i\theta}|$. Thus,

$$\begin{aligned} \frac{-\sin \theta}{|a_{m_j} - re^{i\theta}|} &\leq \frac{-\sin \theta}{|a_{m_j} - e^{i\theta}|} \frac{1}{\sin (\delta/4)} \\ &\leq \frac{1}{\sin (\delta/4)} \end{aligned}$$

for $j > J$ and $-\delta/2 \leq \theta \leq 0$. Also,

$$\operatorname{Im}(a_{m_j}) = \beta_{m_j} \leq |a_{m_j} - re^{i\theta}|$$

for all j and $-\delta/2 \leq \theta \leq 0$. Therefore,

$$(5) \qquad \frac{\beta_{m_j} - \sin \theta}{|a_{m_j} - re^{i\theta}|} \leq 1 + \frac{1}{\sin (\delta/4)} = C < +\infty$$

for $j > J$ and $-\delta/2 \leq \theta \leq 0$.

Let γ_j be the angle between $s(a_{m_j}, 1)$ and $s(0, 1)$. Recalling that a_{m_j} lies outside Δ_1 for $j > J$, we have $\pi/2 > \gamma_j > \delta/2$ for $j > J$ and $-\delta/2 \leq \theta \leq 0$. Hence,

$$\begin{aligned}\left|\frac{1}{\bar{a}_{m_j}} - re^{i\theta}\right| &\geq \operatorname{Im}\left(\frac{1}{\bar{a}_{m_j}}\right) \\ &\geq \operatorname{Im}(a_{m_j}) \\ &= |1 - a_{m_j}| \sin \gamma_j \\ &\geq |1 - a_{m_j}| \sin \frac{\delta}{2}\end{aligned}$$

for $j > J$ and $-\delta/2 \leq \theta \leq 0$. Thus,

$$(6) \qquad \frac{1}{|1/\bar{a}_{m_j} - re^{i\theta}|} \leq \frac{1}{|1 - a_{m_j}| \sin(\delta/2)}$$

for $j > J$ and $-\delta/2 \leq \theta \leq 0$.

Using the estimates (5) and (6), we have, for $j > J$, $-\delta/2 \leq \theta \leq 0$, and $0 < r < 1$,

$$\begin{aligned}\frac{(1 - |a_{m_j}|^2) r\, |\beta_{m_j} \cos\theta - \alpha_{m_j} \sin\theta|}{|\bar{a}_{m_j}|^2\, |a_{m_j} - re^{i\theta}|\, |1/\bar{a}_{m_j} - re^{i\theta}|} &\leq \frac{2(1 - |a_{m_j}|)(\beta_{m_j} - \sin\theta)}{|\bar{a}_{m_j}|^2\, |a_{m_j} - re^{i\theta}|\, |1/\bar{a}_{m_j} - re^{i\theta}|} \\ &\leq \frac{2C}{|a_{m_j}|^2 \sin(\delta/2)} \frac{1 - |a_{m_j}|}{|1 - a_{m_j}|} .\end{aligned}$$

Thus, for a positive integer K chosen sufficiently large, we have

$$\left|\arg\left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1 - \bar{a}_k re^{i\theta}}\right)\right| = \begin{cases} C_1(1 - |a_k|) = M_k\,, & \text{if } k = n_j\,, \\ C_2 \dfrac{1 - |a_k|}{|1 - a_k|} = M_k\,, & \text{if } k = m_j\,, \end{cases}$$

for $k > K$ and $1 - \varepsilon/2 < r < 1$, $-\delta/2 \leq \theta \leq 0$, where C_1 and C_2 are constants. Note that

$$\sum_{k=K}^{\infty} M_k \leq C_1 \sum_{k=1}^{\infty} (1 - |a_k|) + C_2 \sum_{k=1}^{\infty} \frac{1 - |a_k|}{|1 - a_k|} < +\infty .$$

Note, also, that for any integer K', $K' > K$,

$$\begin{aligned}\sum_{k=1}^{\infty} \left|\arg\left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1 - \bar{a}_k r^{i\theta}}\right)\right| &\leq \sum_{k=1}^{K'} \left|\arg\left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1 - \bar{a}_k re^{i\theta}}\right)\right| + \sum_{k=K'+1}^{\infty} M_k \\ &\leq \sum_{k=1}^{K} \left|\arg\left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1 - \bar{a}_k re^{i\theta}}\right)\right| + \sum_{k=K+1}^{\infty} M_k\end{aligned}$$

for $1-\varepsilon/2<r<1$ and $-\delta/2 \leq \theta \leq 0$.

Choose an arbitrary, but fixed, point $re^{i\theta}$, $1-\varepsilon/2<r<1$ and $-\delta/2 \leq \theta \leq 0$. Let ε_0 be an arbitrary positive number. We can choose an integer $K'>K$ sufficiently large that

$$\left|\arg B(re^{i\theta}) - \arg \prod_{k=1}^{K'} \frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right| < \frac{\varepsilon_0}{2}$$

and

$$\sum_{k=K'+1}^{\infty} M_k < \frac{\varepsilon_0}{2} .$$

Then,

$$\begin{aligned}
&\left|\arg B(re^{i\theta}) - \sum_{k=1}^{\infty} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right)\right| \\
&\qquad \leq \left|\arg B(re^{i\theta}) - \sum_{k=1}^{K'} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right)\right| + \sum_{k=K'+1}^{\infty} M_k \\
&\qquad \leq \left|\arg B(re^{i\theta}) - \arg \prod_{k=1}^{K'} \frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right| + \sum_{k=K'+1}^{\infty} M_k < \varepsilon_0
\end{aligned}$$

for $1-\varepsilon/2<r<1$ and $-\delta/2 \leq \theta \leq 0$. Since ε_0 is arbitrary, we have

$$\arg B(re^{i\theta}) = \sum_{k=1}^{\infty} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right)$$

for $1-\varepsilon/2<r<1$ and $-\delta/2 \leq \theta \leq 0$. Also, this series converges uniformly, with θ fixed, in the region $1-\varepsilon/2<r<1$ and $-\delta/2 \leq \theta \leq 0$, and the uniform convergence implies

$$\begin{aligned}
\arg B^*(e^{i\theta}) &= \lim_{r\to 1} \sum_{k=1}^{\infty} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - re^{i\theta}}{1-\bar{a}_k re^{i\theta}}\right) \\
&= \sum_{k=1}^{\infty} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - e^{i\theta}}{1-\bar{a}_k e^{i\theta}}\right)
\end{aligned}$$

for $-\delta/2 \leq \theta \leq 0$. Finally, we remark that

$$\left|\arg \left(\frac{|a_k|}{a_k} \frac{a_k - e^{i\theta}}{1-\bar{a}_k e^{i\theta}}\right)\right| \leq M_k$$

for $k>K$ and $-\delta/2 \leq \theta \leq 0$. This implies that the series

$$\sum_{k=1}^{\infty} \arg \left(\frac{|a_k|}{a_k} \frac{a_k - e^{i\theta}}{1-\bar{a}_k e^{i\theta}}\right)$$

converges uniformly to $\arg B^*(e^{i\theta})$ for $-\delta/2 \le \theta \le 0$. Thus, $\arg B^*(e^{i\theta})$ is continuous for $-\delta/2 \le \theta \le 0$. This completes the proof of theorem 3 in one. direction.

It is clear that the same conclusion holds for all partial products of $B(z)$.

Conversely, let us assume that $B(z)$ has right-sided limit of modulus 1 at $e^{i\theta_0}$. Then, by a theorem of Lindelöf, $B(z)$ has angular limit of modulus 1 at $e^{i\theta_0}$, and, hence, radial limit of modulus 1 at $e^{i\theta_0}$. Thus, by theorem A, the zeros $\{a_k\}$ of $B(z)$ satisfy the condition

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0}-a_k|} < +\infty \ .$$

Since $B(z)$ has right-sided limit of modulus 1 at $e^{i\theta_0}$, there exists $\delta > 0$ such that $B^*(e^{i\theta})$ exists and is continuous for all θ, $\theta_0 - \delta \le \theta \le \theta_0$. Let $R = \{z \mid |z| < 1, \ \theta_0 - \delta < \arg(z) < \theta_0\}$. Now, in the sector R, we have that $B(z)$ is analytic and bounded. Moreover, $B(z)$ has radial limit of modulus 1 at $e^{i\theta_0}$ and right-sided limit of modulus 1 at $e^{i\theta_0}$. Thus, by another theorem of Lindelöf, $B(z)$ converges to a value of modulus 1 as z tends to $e^{i\theta_0}$, $z \in R$. It follows that R contains at most a finite number of zeros of $B(z)$. Thus, there exists ε, $0 < \varepsilon < 1$, such that there are no zeros $\{a_k\}$ in the region

$$\Delta = \{z \mid 1-\varepsilon < |z| < 1, \theta_0 - \delta < \arg(z) < \theta_0\} \ .$$

This completes the proof of theorem 3.

A direct consequence of theorem 3 is the following theorem.

THEOREM 4. *A necessary and sufficient condition for a Blaschke product $B(z)$ with zeros $\{a_k\}$ to have a right-sided limit of modulus 1 at $e^{i\theta_0}$ but not a left-sided limit at $e^{i\theta_0}$ is that the zeros $\{a_k\}$ satisfy the following properties:*

i) *$e^{i\theta_0}$ is a limit point of $\{a_k\}$,*

ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0}-a_k|} < +\infty$, *and*

iii) *there exist positive numbers δ and ε, $\varepsilon < 1$, such that there are no zeros $\{a_k\}$ in the region*

$$\Delta = \{z \mid 1-\varepsilon < |z| < 1, \theta_0 - \delta < \arg(z) < \theta_0\} \ .$$

Proof. Theorem 1 and theorem 3 imply that properties i), ii), and

https://doi.org/10.1017/S0027763000015294 Published online by Cambridge University Press

iii) are sufficient for $B(z)$ to have right-sided limit at $e^{i\theta_0}$ but not left-sided limit at $e^{i\theta_0}$. This follows easily once we notice that property i) implies that $e^{i\theta_0}$ is a singular point for $B(z)$ and properties ii) and iii) imply that $B(z)$ has right-sided limit at $e^{i\theta_0}$. The Blaschke product $B(z)$ cannot have left-sided limit at $e^{i\theta_0}$, otherwise we contradict theorem 1.

To prove that properties i), ii), and iii) are necessary, let us suppose that $B(z)$ has right-sided limit of modulus 1 at $e^{i\theta_0}$, but not left-sided limit at $e^{i\theta_0}$. Thus, by theorem 3, we have that

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0}-a_k|} < +\infty$$

and that there exist positive numbers δ and ε, $\varepsilon < 1$, such that there are no zeros $\{a_k\}$ in the region

$$\Delta = \{z \,|\, 1-\varepsilon < |z| < 1, \theta_0 - \delta < \arg(z) < \theta_0\} .$$

Since $B(z)$ does not have left-sided limit at $e^{i\theta_0}$, $P = e^{i\theta_0}$ is a singular point for $B(z)$. Suppose $e^{i\theta_0}$ is not a limit point of the zeros $\{a_k\}$. Then, there exists a positive number p such that there are no zeros $\{a_k\}$ in $\Delta' = \{z \,|\, |z| < 1, |z - e^{i\theta_0}| < p\}$. Thus, by theorem A, $B^*(e^{i\theta})$ exists and is of modulus 1 for each $e^{i\theta}$ on the boundary of Δ'. But, by a theorem of Lohwater [5, p. 153], since $P = e^{i\theta_0}$ is a singular point for $B(z)$, there exists a point $e^{i\theta^*}$ on the boundary of Δ' such that $B^*(e^{i\theta^*}) = 0$. This is a contradiction. Therefore, $e^{i\theta_0}$ is a limit point of $\{a_k\}$. This completes the proof of theorem 4.

Remark. We point out that theorem 3 and theorem 4 can be modified in the obvious way to give necessary and sufficient conditions for $B(z)$ to have left-sided limit of modulus 1 at $e^{i\theta_0}$.

Acknowledgment. The author is grateful to Professor W. Seidel for several valuable conversations concerning theorem 3.

References

[1] A. P. Calderón, A. González-Domínguez, and A. Zygmund, Nota sobre los valores limites de funciones analiticas, Revista de la Unión Matemática Argentina, **14** (1949), pp. 16–19.

[2] E. F. Collingwood and A. J. Lohwater, The Theory of Cluster Sets, Cambridge University Press, New York, 1966.

[3] O. Frostman, Sur les produits de Blaschke, Kungl. Fysiogr. Sällsk. i Lund Förh., Bd. 12, Nr. **15** (1942), pp. 169–182.

[4] A. J. Lohwater, The boundary values of a class of meromorphic functions, Duke Math. J. **19** (1952), pp. 243–252.
[5] A. J. Lohwater, On the Schwarz reflection principle, Mich. Math. J. **2** (1953-54), pp. 151–156.
[6] R. Nevanlinna, Analytic Functions, Springer-Verlag, New York, 1970.
[7] S. Saks, Theory of the Integral, 2nd Ed., Dover Publications, Inc., New York, 1964.
[8] W. Seidel, On the distribution of values of bounded analytic functions, Trans. Amer. Math. Soc. **36** (1934), pp. 201–226.
[9] A. Zygmund, Trigonometric Series, Vol. **1**, 2nd Ed., Cambridge University Press, New York, 1959.

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