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A NOTE ON THE DISTRIBUTION FUNCTION OF ADDITIVE ARITHMETICAL FUNCTIONS IN SHORT INTERVALS

BY

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ABSTRACT. Let f be an additive arithmetical function having a distribution F. For any sequence $1 \leqq b(n) \leqq n$, $b(n) \longrightarrow \infty$, let

$$Q_n(b, f)(x) = \text{card}\{n \leqq m \leqq n + b(n) : f(m) \leqq x\}/b(n).$$

In this note, we determine the slowest growing function b so that $Q_n(b, f)$ tends weakly to F, for various f.

Introduction. Let f be an additive function. The well known theorem of Erdös and Wintner gives the necessary and sufficient conditions for the existence of the distribution function of f; that is, there exists a distribution function F such that the density of integer m satisfying $f(m) < x$ exists and equals $F(x)$. These conditions are that the two series

$$\sum_p \frac{f'(p)}{p} \quad \text{and} \quad \sum_p \frac{(f'(p))}{p} \tag{1}$$

converge, where $f'(p) = f(p)$ if $|f(p)| \leqq 1$ and $f'(p) = 1$ otherwise. Here and in what follows p stands for prime numbers.

For any sequence $1 \leqq b(n) \leqq n$, of integers, let

$$Q_n(b, f)(x) = (b(n))^{-1} \text{card}\{n < m \leqq n + b(n) : f(m) < x\}.$$

Babu (1981) obtains conditions for the existence of mean values of complex-valued multiplicative functions in short intervals, when $b(n) = n^{\alpha}$; where $0 < \alpha < 1$. Applying these results to $g(n) = \exp(\text{itf}(n))$, f additive, it is not difficult to obtain results on the existence of limits for $Q_n(b, f)(x)$, when $b(n) = n^{\alpha}$. Babu (1982) showed the existence of distribution in short interval for ω, when $n \geqq b(n) \geqq \exp(a(n) \log n/(\log\log n)^{1/2})$ with $1 \leqq a(n) \leqq (\log\log n)^{1/2}$ and $a(n) \longrightarrow \infty$. K.-H. Indlekofer (1987) generalizes this result to strongly additive functions belonging to the Kubilius class H. In a recent article Hildebrand (1987) proves short interval version of Halàsz's Theorem (see Halàsz (1968)) on mean-values for multiplicative functions.

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There he assumes that the interval length $b(n)$ satisfies $\log b(n)/\log n \to 1$ as $n \to \infty$. Without any further restriction on f, this result cannot be improved. In this note we investigate the following problem: To determine as accurately as possible the slowest growing function b so that $Q_n(b, f)$ converges weakly to F. For related work see Babu (1981 and 1982) and Erdös (1935).

The proofs of Theorems 1–3 below, are easy and given in the next section. Throughout the paper we assume that the two series in (1) converge.

THEOREM 1. *If f is bounded and if $b(n) \to \infty$, then $Q_n(b, f)$ converges weakly to F.*

If f is not bounded then this result no longer holds. To simplify the presentation for unbounded f, *the remaining results are stated for strongly additive functions f satisfying* $0 \leqq f(p) \leqq 1$. Let

$$A(y) = \sum_{p \leqq y} f(p) \quad \text{and} \quad f_y(m) = \sum_{p|m, p \leqq y} f(p). \tag{1}$$

It follows easily from the Chinese remainder theorem that unless $A(\log n)/b(n) \to 0$, the sequence $Q_n(b, f)$ cannot converge weakly to F. On the other hand, by using elementary methods we could prove the following.

THEOREM 2. *If $A(\log n)/b(n) \to 0$, then $Q_n(b, f_{\log n})$ converges weakly to F.*

We shall show that if $f(p) = O(p^{\epsilon-1})$ for every $\epsilon > 0$ and further satisfies some very mild regularity conditions, then $Q_n(b, f)$ converges weakly to F. The conditions are given in the next theorem.

THEOREM 3. *Suppose for some $d_n \geqq 1$, the sequence $\{A((\log 2n)^{d_n})/A(\log n)\}$ is bounded and*

$$(g_n \log 2n)/\operatorname{loglog} n \to 0 \text{ as } n \to \infty, \tag{2}$$

where $g_n = \max\{f(p) : (\log n)^{d_n} \leqq p \leqq 2n\}$. If $A(\log n)/b(n) \to 0$, then $Q_n(b, f)$ converges weakly to F.

Thus in this case we have a very satisfactory solution. We shall illustrate Theorem 3, with some examples.

EXAMPLE 1. If $f(p) = p^{-1}$, then $A(x) = \operatorname{loglog} x + O(1)$. The conditions of Theorem 3 hold with $d_n \equiv 1$. So $Q_n(b, f)$ converges weakly to F if and only if $(\operatorname{loglog} n)/b(n) \to 0$. This result is essentially contained in Erdös (1935).

EXAMPLE 2. If $f(p) = (\log p)/p$, then $A(x) = \log x + O(1)$. If we take $d_n \equiv 2$ in Theorem 3, we get that $Q_n(b, f)$ converges weakly to F if and only if $(\operatorname{loglog} n)/b(n) \to 0$.

EXAMPLE 3. If $f(p) = p^{-1} \exp((\log p)^a)$, $0 < a < 1/2$, then

$$(\log x)^{-a} \exp((\log x)^a) \ll A(x) \ll (\log x)^{-a} \exp((\log x)^a).$$

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The assumptions of Theorem 3 hold with

$$d_n = (1 + (\log\log n)^{-a})^{1/a} \geqq 1 + a^{-1}(\log\log n)^{-a}.$$

If follows that $Q_n(b, f)$ converges weakly to F if and only if $A(\log n)/b(n) \to 0$.

We could not get such a complete answer, when $f(p) \geqq p^{-a}$ for some $a > 0$. For example if $f(p) = p^{-a}$, $0 < a < 1$, then $(\log n)^{1-a}/b(n)\log\log n \to 0$ is necessary for $Q_n(b, f)$ to converge weakly to F, but we could prove the weak convergence only if $(\log n)^{-1+1/a}/b(n)(\log\log n)^{1/a} \to 0$. We expect that the lower bound is the correct one. If $f(p) = (\log p)^{-1}$, then $(\log n)/b(n)(\log\log n)^2 \to 0$ is necessary. But we could prove the convergence of $Q_n(b, f)$ to F only when $\sqrt{\log n}/\log b(n) \to 0$.

Consider now $f(p) = 1$, that is $f(n) = \omega(n)$. It was shown in Babu (1982) that if $b(n) = n^{a(n)(\log\log n)^{-1/2}}, a(n) \to \infty$, then the Erdös-Kac Theorem holds. But it is very likely that much more is true. The Chinese remainder theorem only gives that $(\log n)/b(n)(\log\log n)^2$ must tend to zero, but the fact in this case may very well be between these extremes. Perhaps it will not be very easy to improve these conditions.

Proofs of the Theorems

Proof of Theorem 1. Without loss of generality we can assume that $|f(m)| \leqq 1$. Clearly, $\sup \sum_p |f(p^{m(p)})| \leqq 2$, where the supremum is taken over all sequences $\{m(p)\}$ of non-negative integers.

Suppose $\sum_{p>k} |f(p^{m(p)})| \not\to 0$ uniformly for all sequences $\{m(p)\}$ of non-negative integers. Then there exists $\epsilon > 0$ such that for all integers $k \geqq 1$, $\sum_{p>k} |f(p^{m(p,k)})| > \epsilon$ for some sequence $\{m(p, k)\}$ of non-negative integers.

It follows that there exist $1 = n_0 < n_1 < n_2 < \ldots$, integers such that, for $i \geqq 1$

$$\sum_{n_{i-1}<p\leqq n_i} |f(p^{m(p,n_i)})| > \epsilon.$$

So if $r = [2/\epsilon] + 2$, $m(p) = m(p, n_i)$ for $n_{i-1} < p \leqq n_i$, $1 \leqq i \leqq r$ and $m(p) = 1$ for $p > n_r$, then

$$\sum_p |f(p^{m(p)})| > (2 + [2/\epsilon])\epsilon \geqq 2 + \epsilon,$$

which is impossible. So $\sum_{p>k} |f(p^{m(p)})| \to 0$ uniformly for all sequences $\{m(p)\}$ of non-negative integers. As a consequence we have

$$f(m) - f_k(m) \to 0 \tag{3}$$

uniformly in m, as $k \to \infty$.

It is well known that there exists a sequence $\{X_p\}$ of independent random variables (see Babu (1978)) such that $\sum X_p$ converges a.e. and both f and $\sum X_p$ have the same distribution. Further, if $b(n) \to \infty$, then for all integers $k \geqq 1$, $Q_n(b, f_k)$ converges

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weakly to the distribution of $\sum_{p\leqq k} X_p$. The result now follows from (3) and the fact that $\sum X_p$ converges a.e.

Proof of Theorem 2. In view of the proof of Theorem 1, it is enough to show that for every $\delta > 0$, that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \delta(b(n))^{-1} \operatorname{card}\{n < m \leqq n + b(n) : f_{\log n}(m) - f_k(m) > \delta\} = 0. \tag{4}$$

The expression in (4) is not more than

$$\begin{aligned}
& b(n)^{-1} \sum_{n<m\leqq n+b(n)} (f_{\log n}(m) - f_k(m)) \\
& \leqq (b(n))^{-1} \sum_{k<p\leqq \log n} f(p) \left(\left[\frac{n+b(n)}{p} \right] - \left[\frac{n}{p} \right] \right) \\
& \leqq \sum_{k<p\leqq \log n} \frac{f(p)}{p} + \frac{1}{b(n)} \sum_{p\leqq \log n} f(p) \\
& \leqq \sum_{p>k} \frac{f(p)}{p} + A(\log n)(b(n))^{-1} \\
& \to \sum_{p>k} \frac{f(p)}{p} \quad \text{as } n \to \infty \\
& \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

This completes the proof.

Proof of Theorem 3. By (2),

$$0 \leqq f(m) - f_{((\log n)^{d_n})}(m) \leqq (g_n \log 2n)/\operatorname{loglog} n$$

tends to zero uniformly in $m \leqq 2n$, as $n \to \infty$. As in the proof of Theorem 2, we have for any $\delta > 0$,

$$\begin{aligned}
& \frac{\delta}{b(n)} \operatorname{card}\{n < m \leqq n + b(n) : f_{((\log n)^{d_n})}(m) - f_{\log n}(m) > \delta\} \\
& \leqq \sum_{p>\log n} \frac{f(p)}{p} + \frac{A((\log n)^{d_n})}{b(n)} \to 0
\end{aligned}$$

as $n \to \infty$. The result now follows from Theorem 2.

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