A NOTE ON CONVERGENCE FIELDS

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The purpose of this note is to show that the (bounded) convergence field of a conservative matrix is closed under a certain diagonalization procedure.

As an application of the above result we establish a conjecture of Hill and Sledd in **(1)** and obtain a result of Lorentz originally proved in **(2)**.

First we introduce some notation and definitions, most of which are standard.

Let l_∞ denote the Banach space of bounded sequences with the supremum norm and let c denote the closed subspace of l_∞ consisting of convergent sequences.

If $A = (a_{ij})$ is a conservative matrix, that is $Ax \in c$ for $x \in c$, we define the norm of A by

$$||A|| = \sup_i \sum_j |a_{ij}| = \sup_{x \in l_\infty} \{||Ax||/||x||\}.$$

We assume all matrices to be conservative and of norm 1 unless otherwise stated.

We denote by $B(A)$ the (bounded) convergence field of A. That is,

$$B(A) = \{x | x \in l_\infty \quad \text{and} \quad Ax \in c\}.$$

If $x \in B(A)$, we define $\lim Ax$ by

$$\lim Ax = \lim_i \sum_{j=1}^{\infty} a_{ij}\, x(j).$$

Let Z^+ denote the positive integers. All indices will be taken from Z^+ unless otherwise noted.

If $x \in l_\infty$ and $n \in Z^+$, we define the sequence $T_n x$ by

$$T_n x(p) = \begin{cases} x(p) & \text{for } p \geqslant n, \\ 0 & \phantom{\text{for }} p < n. \end{cases}$$

We note that if $x \in B(A)$, then $\lim_n \lim AT_n x$ always exists. Indeed for $N > M$,

$$|\lim AT_M x - \lim AT_N x| \leqslant \left(\sum_{j=M}^{N-1} \left|\lim_i a_{ij}\right|\right) ||x||.$$

But

$$\sum_{j=1}^{\infty} \left|\lim_i a_{ij}\right| \leqslant \sup_i \sum_j |a_{ij}| = ||A||.$$

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We also note that if $x \in B(A)$, then for fixed p, $\lim_n [(AT_n x)(p)] = 0$.

Now we come to our principal result.

THEOREM 1. *Let A be a conservative matrix. Let x_n, $n = 1, 2, \ldots,$ be a set of sequences in $B(A)$ such that*

(1) $||x_n|| \leqslant 1$ *for each* n,

(2) $\lim_m \lim AT_m x_n = 0$ *for each* n.

For each n let $M_n \in Z^+$ be given. Then there exists $y \in B(A)$ and for each n there exists $k_n \in Z^+$ such that

$$y(p) = x_n(p) \qquad \text{for } k_n < p \leqslant k_n + M_n.$$

Proof. For each $q \in Z^+$ and $p \in Z^+$ such that $1 \leqslant p \leqslant 2^{q+1}$ choose $r(q, p) \in Z^+$ such that

(1) $r(q, p_1) < r(q, p_2)$ if $p_1 < p_2$,

(2) $r(q_1, p_1) < r(q_2, p_2)$ if $q_1 < q_2$,

(3) $r(q, 2^q + 1) - r(q, 2^q) > M_q$.

(4) If we define y_q by

$$y_q = \sum_{p=1}^{2^q} 2^{-q} T_{r(q,p)}\, x_q - \sum_{p=2^q+1}^{2^{q+1}} 2^{-q} T_{r(q,p)}\, x_q,$$

then $||Ay_q|| < 4 \cdot 2^{-q}$.

Note that

$$y_q(p) = \begin{cases} 0 & p < r(q, 1), \\ 0 & \text{for } p \geqslant r(q, 2^{q+1}), \\ x_q(p) & r(q, 2^q) \leqslant p < r(q, 2^q + 1). \end{cases}$$

The set y_q, $q = 1, 2, \ldots,$ forms a set of non-overlapping finite sequences. If we define $y \in l_\infty$ by

$$y(n) = \sum_{q=1}^{\infty} y_q(n),$$

we see that

$$(Ay)(n) = \left(\sum_{q=1}^{\infty} Ay_q\right)(n).$$

Since $Ay_q \in c$ and

$$\sum_{q=1}^{\infty} Ay_q$$

converges in l_∞, $Ay \in c$ and y is the desired sequence.

A few details regarding the existence of the set $r(q, p)$ may be in order. For a given $x_q \in B(A)$ and satisfying the hypotheses of our theorem, given $N \in Z^+$ and given $\epsilon > 0$, there exists J such that for $j > J$,

(1) $|(AT_j x_q)(n)| \leqslant 1$ for all n,

(2) $|\lim (AT_j x_q)| < \epsilon$,

and, in accordance with our remark preceding Theorem 1,

(3) $|(AT_j x_q)(n)| < \epsilon$ for $n < N$.

Now let $\epsilon = 4^{-q}$. By the above observations we may choose $r(q, 1)$ so large and $r(q, p)$, $p = 2, \ldots, 2^{q+1}$, so rapidly increasing that

(1) $|\lim AT_{r(q,p)} x_q| < 4^{-q}$ for $p = 1, \ldots, 2^{q+1}$, and

(2) if $|(AT_{r(q,p)} x_q)(n)| > 4^{-q}$, then

$$\max_{j \leqslant p-1} \sup_{m \geqslant n} |(AT_{r(q,j)} x_q)(m)| < 2^{-q} \qquad \text{for } p = 2, \ldots, 2^{q+1}.$$

For such a set of $r(q, p)$ we see that

$$\left\| A\left(\sum_{p=1}^{2^q} 2^{-q} T_{r(q,p)} x_q\right)\right\| < 2 \cdot 2^{-q},$$

$$\left\| A\left(\sum_{p=2^q+1}^{2^{q+1}} 2^{-q} T_{r(q,p)} x_q\right)\right\| < 2 \cdot 2^{-q}.$$

If, in addition, we choose $r(q, 2^q + 1)$ so that $r(q, 2^q + 1) - r(q, 2^q) > M_q$, and if we choose inductively $r(q, 1)$ for each $q > 1$ so that $r(q, 1) > r(q - 1, 2^q)$, we have the desired set $r(q, p)$.

This completes the proof.

The conditions on the x_n hypothesized are easy to satisfy.

We define $\delta = (1, 1, 1, \ldots)$.

If A is co-regular, that is, $\lim_n \lim AT_n \delta = \rho \neq 0$, and if $x \in B(A)$, $||x|| \leqslant 1$, then $\lim_n \lim AT_n(x - \alpha\delta) = 0$ for properly chosen scalar α, $|\alpha| \leqslant 1/\rho$.

Observe that it is by no means necessary that the various x_n in the theorem be distinct. Hence if A is not co-regular, that is $\lim_n \lim AT_n \delta = 0$, then $x_n = \delta$ satisfies our hypotheses.

We now consider an application of our theorem.

Following Hill and Sledd in **(1)** we say that $x \in Z_p$ if $x \in l_\infty$ and

$$\lim_{r \to \infty} \sum_{n=r}^{n=r+p-1} x(n)$$

exists. We define $\bar{Z}$ by $\bar{Z} = \overline{\bigcup_p Z_p}$ where closure is in the supremum norm.

We let uap denote the ultimately almost periodic sequences; that is, $x \in$ uap if for $\epsilon > 0$ there exists $K \in Z^+$ such that

$$\sup_{p > K} \left| x(p) - \sum_{n=1}^{q} \alpha_n \exp(i\theta_n p)\right| < \epsilon$$

for suitable $q \in Z^+$, real θ_n, and complex α_n. Hill and Sledd call this space the almost periodic sequences and denote it by ap.

We let ac denote the space of almost convergent sequences, that is, the space of sequences such that each sequence possesses a unique Banach (translation-invariant) limit. Lorentz proved in **(2**, Theorem 1**)** that a necessary and

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sufficient condition that x be in ac with Banach limit s is that

$$\lim_{p} \frac{1}{p} \sum_{n=r}^{r+p-1} x(n) = s$$

hold uniformly in r.

It is known that uap $\subset$ ac **(2**, p. 173**)** and that $\bar{Z} \subset$ ac **(1**, p. 743**)**.

Hill and Sledd conjectured **(1**, p. 754**)** that neither uap nor $\bar{Z}$ were $B(A)$ for any regular A. (A regular matrix is a conservative matrix that preserves limits.) Lorentz proved in **(2**, §7**)** that ac is not $B(A)$ for any regular A.

We have the following corollary to our Theorem, which establishes slight generalizations of the above.

COROLLARY. *Let A be conservative. Neither $\bar{Z}$,* uap, *nor* ac *is $B(A)$.*

Proof. If A is co-regular, assume that $B(A)$ includes any one of the above three spaces. Then $B(A)$ includes the space of all periodic sequences. Let y_n consist of alternate blocks of zeros and ones, each block of length n. Suppose that $\lim_m \lim A T_m \delta = 1/\rho$. Choose scalars α_n, β_n so that if we define x_n by $x_n = \beta_n(y_n - \alpha_n \delta)$, then $||x_n|| = 1$ and $\lim_m \lim A T_m x_n = 0$. Since $|\alpha_n| \leqslant |\rho|$,

$$|\beta_n| \geqslant \frac{1}{1 + |\rho|}.$$

Hence

$$\lim_{p,q} \sup |x_n(p) - x_n(q)| \geqslant \frac{1}{1 + |\rho|}.$$

Now apply our theorem to this set of x_n with $M_n = 4n$ to obtain a sequence $y \in B(A)$. A glance at Lorentz's criterion makes it clear that $y \notin$ ac.

If A is not co-regular, we take $x_n = \delta$ for even n and $x_n = 0$ for odd n. Taking $M_n = n$ in our theorem, we construct $y \in B(A)$ such that $y \notin$ ac.

Since ac $\supset$ uap and ac $\supset \bar{Z}$, our desired result is obtained. It is also clear by considering y directly that $y \notin$ ac, $y \notin$ uap, and $y \notin \bar{Z}$.

This completes the proof.

REFERENCES

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