On Hardy's Theory of m-Functions.

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§ 1. The Cardinal Function of Interpolation Theory[1] is the function

$$C(x) = \sum_{-\infty}^{\infty} a_n \frac{\sin \pi (x-n)}{\pi (x-n)}$$

which takes the values a_n at the points $x = n$. Ferrar[2] has recently proved

Theorem 1. *If* $\sum_{1}^{\infty} | a_n \log n | / n$ *and* $\sum_{1}^{\infty} | a_{-n} \log n | / n$ *are convergent, $C(x)$ is an m-function*[3] *for* $m \geqslant \pi$.

This means that $C(x)$ is a solution of the integral equation

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin m(x-t)}{x-t} f(t)\, dt \quad \ldots\ldots\ldots\ldots (1)$$

Ferrar's proof deals with functions of a real variable and involves some rather difficult double limit considerations. In § 2 of the present paper is given a complex variable treatment, which provides a much more direct proof of the property in question.

In the concluding sections,[4] we show that this m-function property of the Cardinal Function is closely allied to the fact that it can be represented, under certain circumstances, by an integral of the form

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt.$$

[1] This function was introduced by Prof. Whittaker, *Proc. Roy. Soc. Edin.*, **35** (1915), 181–194.

[2] *ibid.*, **46** (1926), 323–333; in particular 330–333.

[3] The theory of m-functions is due to Prof. Hardy, *Proc. Lond. Math. Soc.* (2), **7** (1909), 445–472.

[4] §§ 3, 4 have been rewritten in accordance with the valuable suggestions of Mr W. L. Ferrar, who kindly read the paper in manuscript.

§ 2. *Proof of Theorem* 1.

Under the conditions of Theorem 1, the Cardinal Function Series is uniformly and absolutely convergent in any finite part of the z-plane, and represents an integral function. Consider now

$$\int_{\Gamma} \frac{e^{im(z-x)}}{z-x} C(z)\, dz$$

where Γ is the contour formed of the segment of the real axis from $-R$ to R, indented at x, and the semicircle in the upper half plane on this segment as diameter; we suppose that $R = N + \delta$, where N is an integer, and $0 < \delta < 1$. This contour integral vanishes, since the integrand is analytic inside and on the contour. The evaluation of the contour integral gives at once

$$P \int_{-R}^{R} \frac{e^{im(t-x)}}{t-x} C(t)\, dt = \pi i\, C(x) - I(R)$$

where the integral on the left-hand-side is a Cauchy principal value, and where $I(R)$ is the integral round the semicircle.

Now we easily see that, if $0 \leqslant \theta \leqslant \pi$, $|C(Re^{\theta i})|$ is less than

$$\frac{1}{\pi} e^{\pi R \sin\theta} \left[\frac{|a_0|}{R} + \sum_1^{\infty} \frac{|a_n|}{\{R^2 + n^2 - 2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^{\infty} \frac{|a_{-n}|}{\{R^2 + n^2 + 2Rn\cos\theta\}^{\frac{1}{2}}} \right]$$

the series on the right-hand-side being uniformly convergent with respect to θ. Since

$$I(R) = \int_0^{\pi} \frac{e^{im(R\cos\theta - x) - mR\sin\theta}}{Re^{\theta i} - x} C(Re^{\theta i})\, iRe^{\theta i}\, d\theta,$$

$\pi(R - |x|)\,|I(R)|\,/\,R$ is less than

$$\int_0^{\pi} e^{-(m-\pi)R\sin\theta} \left[\frac{|a_0|}{R} + \sum_1^{\infty} \frac{|a_n|}{\{R^2 + n^2 - 2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^{\infty} \frac{|a_{-n}|}{\{R^2 + n^2 + 2Rn\cos\theta\}} \right] d\theta$$

$$\leqslant \int_0^{\pi} \left[\frac{|a_0|}{R} + \sum_1^{\infty} \frac{|a_n|}{\{R^2 + n^2 - 2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^{\infty} \frac{|a_{-n}|}{\{R^2 + n^2 + 2Rn\cos\theta\}^{\frac{1}{2}}} \right] d\theta$$

$$\leqslant \pi \left\{ \frac{|a_0|}{R} + \frac{2}{\pi} \sum_1^{\infty} \frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n} \right] \right\}$$

where $K(k)$ denotes the complete elliptic integral of the first kind, modulus k. We have here used the condition that $m \geqslant \pi$, and have integrated term-by-term, which is obviously valid in this case.

This inequality may be written in the form

$$\frac{R-|x|}{R}|I(R)| \leqslant \frac{|a_0|}{R}+\frac{2}{\pi}\left(\sum_1^N+\sum_{N+1}^\infty\right)\frac{|a_n|+|a_{-n}|}{R+n}\ K\left[\frac{2\sqrt{Rn}}{R+n}\right]$$

where $R=N+\delta$. We shall shew from this that $I(R)$ tends to zero as N tends to infinity, δ being fixed.

Now when $n \geqslant N+1$, we have

$$\frac{2\sqrt{Rn}}{R+n} \leqslant \frac{2\sqrt{(N+\delta)(N+1)}}{2N+1+\delta} < 1-\frac{C_1}{N^2}$$

where C_1 is a positive constant depending only on δ; since $K(k)$ is a monotone increasing function of K, if $0 \leqslant k \leqslant 1$, we see that

$$\sum_{N+1}^\infty \frac{|a_n|+|a_{-n}|}{R+\delta}\ K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant \sum_{N+1}^\infty \frac{|a_n|+|a_{-n}|}{N+\delta+n}\ K\left[1-\frac{C_1}{N^2}\right]$$

Now it can be easily shewn[1] that

$$K\left[1-\frac{C_1}{N^2}\right]/\log N$$

is positive and finite for all $N\,(>1)$ and tends to unity as $N \to \infty$. Consequently

$$\begin{aligned}\sum_{N+1}^\infty \frac{|a_n|+|a_{-n}|}{R+n}\ K\left[\frac{2\sqrt{Rn}}{R+n}\right] &\leqslant C_2\sum_{N+1}^\infty \frac{|a_n|+|a_{-n}|}{N+\delta+n}\ \log N \\ &\leqslant C_2\sum_{N+1}^\infty \frac{|a_n|+|a_{-n}|}{n}\ \log n \\ &\to 0 \text{ as } N\to\infty,\end{aligned}$$

since the two series $\overset{\infty}{\Sigma}\,|a_n \log n|/n$ and $\overset{\infty}{\Sigma}\,|a_{-n}\log n|/n$ are convergent.

It is a consequence of Tannery's Theorem[2] that

$$\frac{|a_0|}{R}+\frac{2}{\pi}\sum_1^N \frac{|a_n|+|a_{-n}|}{R+n}\ K\left[\frac{2\sqrt{Rn}}{R+n}\right] \to 0 \qquad \text{as } N\to\infty,$$

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[1] It is an elementary consequence of the result (given in Whittaker and Watson, *Modern Analysis* (1920), § 22.737), $\lim_{k\to 0}\{K'-\log(4/k)\}=0$.

[2] See Bromwich, *Infinite Series* (1926). § 49.

if we can shew that

$$\frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant M_n$$

where M_n is independent of R, and ΣM_n is convergent. But, as above, we may shew that

$$\frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant C_3 \frac{|a_n| + |a_{-n}|}{n} \log n,$$

which is sufficient for our purpose.

We have thus shewn that, under the conditions of Theorem 1, $I(R) \to 0$ as $N \to \infty$, and hence that

$$P\int_{-\infty}^{\infty} \frac{e^{im(t-x)}}{t-x} C(t)\, dt = \pi i\, C(x)$$

where the integral on the left-hand-side is a principal value, both at $t = x$, and $t = \infty$. Equating imaginary parts, we have at once, if $m \geqslant \pi$,

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x} C(t)\, dt = \pi\, C(x);$$

the principal value sign has been omitted because $t = x$ is a removable singularity, and because $C(t)$ is an integral function finite on the real axis and

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x}\, dt$$

exists. This completes the proof of Theorem 1.

It may be pointed out that the proof that $I(R) \to 0$ may be considerably shortened in the case $m > \pi$, by the use of the inequality

$$|C(Re^{\theta i})| \leqslant Ke^{\pi R \sin\theta} R / \log R$$

if $0 \leqslant \theta \leqslant \pi$. But the proof by the use of this inequality fails in the case $m = \pi$.

§ 3. We have just seen that the fact that the Cardinal Function is a solution of the equation (1) depends chiefly upon the result that $I(R) \to 0$ as $R \to \infty$, the other parts of the proof being straightforward deductions from Cauchy's Integral Theorem.

Ferrar[1] has recently shown that, if $\sum_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ is convergent $(p \geqslant 1)$, then the Cardinal Function has the definite integral representation

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt,$$

where ϕ and ψ are each L^{1+p} over $(0, 1)$. From this result we are able to prove, with very little trouble, that $I(R) \to 0$ as $R \to \infty$, and thus to bring out the connection between the two properties of the Cardinal Function which we have noted in § 1.

For, considered as a function of the complex variable z, $C(z)$ is an integral function which remains finite as z tends to infinity in either direction along the real axis. In the upper half plane, we have

$$z = re^{\theta i} \qquad (0 \leqslant \theta \leqslant \pi)$$

$$\left| \begin{matrix} \cos \\ \sin \end{matrix} \pi z t \right| \leqslant e^{\pi r \sin \theta . t} \quad (t \geqslant 0).$$

Hence, by the use of Hölder's inequality for integrals, we have

$$|C(re^{\theta i})| \leqslant \int_0^1 |\phi| \; |\cos \pi r e^{\theta i} t|\, dt + \int_0^1 |\psi| \; |\sin \pi r e^{\theta i} t|\, dt$$

$$\leqslant \left\{ \left(\int_0^1 |\phi|^{1+p} dt \right)^{\frac{1}{1+p}} + \left(\int_0^1 |\psi|^{1+p} dt \right)^{\frac{1}{1+p}} \right\} \left\{ \int_0^1 e^{\pi \left(1 + \frac{1}{p}\right) r \sin \theta . t} dt \right\}^{\frac{p}{1+p}}$$

$$< K e^{\pi r \sin \theta} (r \sin \theta)^{-\frac{p}{1+p}}$$

where K is a finite constant, since ϕ and ψ are each L^{1+p} over $(0, 1)$.

We now have

$$|I(R)| \leqslant \frac{RK}{R - |x|} \int_0^{\pi} e^{-(m-\pi) R \sin \theta} R^{-\frac{p}{1+p}} \sin^{-\frac{p}{1+p}} \theta \, d\theta$$

$$\leqslant \frac{2RK}{R - |x|} \int_0^{\frac{1}{2}\pi} e^{-2(m-\pi) R\theta / \pi} \left(\frac{2R\theta}{\pi} \right)^{-\frac{p}{1+p}} d\theta$$

$$\to 0, \text{ as } R \to \infty, \text{ if } m \geqslant \pi.$$

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[1] *Proc. Roy. Soc. Edin.*, **47** (1927), 230–242. The particular case $p=1$ was previously discussed by J. M. Whittaker, *Proc. Edin. Math. Soc.* (2), **1** (1927), 41–46.

We can now easily complete the proof, exactly as in § 2, of the following theorem:—

Theorem 1.* *If* $\sum_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ $(p \geqslant 1)$ *is convergent, then* $C(x)$ *possesses the definite integral representation*

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt$$

where ϕ *and* ψ *are each* L^{1+p} *over* $(0, 1)$, *and is an m-function for* $m \geqslant \pi$.

Theorem 1* is, of course, included in Theorem 1; for, by Hölder's inequality, the convergence of $\sum_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ $(p \geqslant 1)$ implies the convergence of $\sum_1^{\infty} |a_n \log n| / n$ and $\sum_1^{\infty} |a_{-n} \log n| / n$, but not conversely.

§ 4. Finally, the use of functions of class L^p enables us to prove, by the same direct method, Theorems[1] 2 and 3 below.

Theorem 2. *The integrals*

$$f(x) = \int_a^A \phi(w) \begin{matrix}\cos \\ \sin\end{matrix}\, wx\, dw$$

represent m-functions, if $-m \leqslant a < A \leqslant m$, *provided only that* $\phi(w)$ *is* L^p $(p > 1)$ *over* (a, A).

Theorem 3. *The integral*

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin \mu (w - x)}{w - x} \phi(w)\, dw$$

represents an m-function, if $m \geqslant \mu > 0$, *provided only that* $\phi(w)$ *is* L^p $(p > 1)$ *over* $(-\infty, \infty)$.

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[1] Compare the rather similar theorems given by Hardy, *loc. cit.*, 457, 459.