On Hardy's Theory of m-Functions.

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§ 1. The Cardinal Function of Interpolation Theory\(^1\) is the function

\[ C(x) = \sum_{-\infty}^{\infty} a_n \frac{\sin \pi(x-n)}{\pi(x-n)} \]

which takes the values \(a_n\) at the points \(x = n\). Ferrar\(^2\) has recently proved

**Theorem 1.** If \(\sum_{1}^{\infty} |a_n \log n| / n\) and \(\sum_{1}^{\infty} |a_{-n} \log n| / n\) are convergent, \(C(x)\) is an \(m\)-function\(^3\) for \(m \geq \pi\).

This means that \(C(x)\) is a solution of the integral equation

\[ f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin m(x-t)}{x-t} f(t) \, dt \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \]

Ferrar's proof deals with functions of a real variable and involves some rather difficult double limit considerations. In §2 of the present paper is given a complex variable treatment, which provides a much more direct proof of the property in question.

In the concluding sections,\(^4\) we show that this \(m\)-function property of the Cardinal Function is closely allied to the fact that it can be represented, under certain circumstances, by an integral of the form

\[ C(x) = \int_{0}^{1} [\phi(t) \cos \pi xt + \psi(t) \sin \pi xt] \, dt. \]

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\(^1\) This function was introduced by Prof. Whittaker, *Proc. Roy. Soc. Edin.*, 35 (1915), 181–194.

\(^2\) *ibid.*, 46 (1926), 323–333; in particular 330–333.

\(^3\) The theory of \(m\)-functions is due to Prof. Hardy, *Proc. Lond. Math. Soc. (2)*, 7 (1909), 445–472.

\(^4\) §§3, 4 have been rewritten in accordance with the valuable suggestions of Mr W. L. Ferrar, who kindly read the paper in manuscript.
§ 2. Proof of Theorem 1.

Under the conditions of Theorem 1, the Cardinal Function Series is uniformly and absolutely convergent in any finite part of the z-plane, and represents an integral function. Consider now

\[ \int_{\Gamma} \frac{e^{im(z-x)}}{z-x} C(z) \, dz \]

where \( \Gamma \) is the contour formed of the segment of the real axis from \(-R\) to \(R\), indented at \(x\), and the semicircle in the upper half plane on this segment as diameter; we suppose that \(R = N + \delta\), where \(N\) is an integer, and \(0 < \delta < 1\). This contour integral vanishes, since the integrand is analytic inside and on the contour. The evaluation of the contour integral gives at once

\[ \oint_{-R}^{R} \frac{e^{im(t-x)}}{t-x} C(t) \, dt = \pi i C(x) - I(R) \]

where the integral on the left-hand-side is a Cauchy principal value, and where \(I(R)\) is the integral round the semicircle.

Now we easily see that, if \(0 < \theta < \pi\), \(|C(Re^{i\theta})|\) is less than

\[ \frac{1}{\pi} e^{\pi R \sin \theta} \left[ \sum \frac{|a_n|}{R \left( R^2 + n^2 - 2Rn \cos \theta \right)^{\frac{3}{4}}} \right] \]

the series on the right-hand-side being uniformly convergent with respect to \(\theta\). Since

\[ I(R) = \int_{0}^{\pi} \frac{e^{im(Re^{i\theta} - x)} - mR \sin \theta}{Re^{i\theta} - x} C(Re^{i\theta}) \, d\theta \]

\[ |\pi(R - |x|)| \, |I(R)| \, |I(R)| / R \text{ is less than} \]

\[ \left\{ \sum \frac{|a_n|}{R \left( R^2 + n^2 - 2Rn \cos \theta \right)^{\frac{3}{4}}} + \sum \frac{|a_{-n}|}{R \left( R^2 + n^2 + 2Rn \cos \theta \right)^{\frac{3}{4}}} \right\} d\theta \]

where \(K(k)\) denotes the complete elliptic integral of the first kind, modulus \(k\). We have here used the condition that \(m > \pi\), and have integrated term-by-term, which is obviously valid in this case.
This inequality may be written in the form
\[
\frac{R - |x|}{R} |I(R)| \leq \left| \frac{a_n}{R} + \frac{2}{\pi} \left( \sum_{1}^{\infty} \frac{|a_n|}{R + n} \right) \right| K \left[ \frac{2\sqrt{Rn}}{R + n} \right]
\]
where \(R = N + \delta\). We shall shew from this that \(I(R)\) tends to zero as \(N\) tends to infinity, \(\delta\) being fixed.

Now when \(n > N + 1\), we have
\[
\frac{2\sqrt{Rn}}{R + n} \leq \frac{2\sqrt{(N + \delta)(N + 1)}}{2N + 1 + \delta} < 1 - \frac{C_1}{N^2}
\]
where \(C_1\) is a positive constant depending only on \(\delta\); since \(K(k)\) is a monotone increasing function of \(K\), if \(0 < k < 1\), we see that
\[
\sum_{N+1}^{\infty} \frac{|a_n| + |a_{-n}|}{R + \delta} K \left[ \frac{2\sqrt{Rn}}{R + n} \right] \leq \sum_{N+1}^{\infty} \frac{|a_n| + |a_{-n}|}{N + \delta + n} K \left[ 1 - \frac{C_1}{N^2} \right]
\]
Now it can be easily shewn\(^1\) that
\[
K \left[ 1 - \frac{C_1}{N^2} \right] / \log N
\]
is positive and finite for all \(N( > 1)\) and tends to unity as \(N \to \infty\). Consequently
\[
\sum_{N+1}^{\infty} \frac{|a_n| + |a_{-n}|}{R + n} K \left[ \frac{2\sqrt{Rn}}{R + n} \right] \leq C_2 \sum_{N+1}^{\infty} \frac{|a_n| + |a_{-n}|}{N + \delta + n} \log N
\]
\[
\leq C_2 \sum_{N+1}^{\infty} \frac{|a_n| + |a_{-n}|}{n} \log n
\]
\[
\to 0 \text{ as } N \to \infty,
\]
since the two series \(\sum |a_n| \log n|/n\) and \(\sum |a_{-n}| \log n|/n\) are convergent.

It is a consequence of Tannery’s Theorem\(^2\) that
\[
\left| \frac{a_0}{R} + \frac{2}{\pi} \sum_{1}^{\infty} \frac{|a_n| + |a_{-n}|}{R + n} K \left[ \frac{2\sqrt{Rn}}{R + n} \right] \right| \to 0 \text{ as } N \to \infty,
\]

\(^1\) It is an elementary consequence of the result (given in Whittaker and Watson, Modern Analysis (1920), § 22. 737), \(\lim_{k \to 0} \left( K' - \log \frac{4}{k} \right) = 0\).

\(^2\) See Bromwich, Infinite Series (1926). § 49.
if we can shew that

$$|a_n| + |a_{-n}| \frac{2 \sqrt{Rn}}{R + n} \leq M_n$$

where $M_n$ is independent of $R$, and $\sum M_n$ is convergent. But, as above, we may shew that

$$\frac{|a_n| + |a_{-n}|}{R + n} K \left( \frac{2 \sqrt{Rn}}{R + n} \right) \leq C_3 \frac{|a_n| + |a_{-n}|}{n} \log n,$$

which is sufficient for our purpose.

We have thus shewn that, under the conditions of Theorem 1, $I(R) \to 0$ as $N \to \infty$, and hence that

$$P \int_{-\infty}^{\infty} \frac{e^{im(t-x)}}{t-x} C(t) \, dt = \pi i \, C(x)$$

where the integral on the left-hand-side is a principal value, both at $t = x$, and $t = \infty$. Equating imaginary parts, we have at once, if $m > \pi$,

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x} C(t) \, dt = \pi C(x);$$

the principal value sign has been omitted because $t = x$ is a removable singularity, and because $C(t)$ is an integral function finite on the real axis and

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x} \, dt$$

exists. This completes the proof of Theorem 1.

It may be pointed out that the proof that $I(R) \to 0$ may be considerably shortened in the case $m > \pi$, by the use of the inequality

$$|C(Re^{\theta i})| \ll Ke^{\pi R \sin \theta} R / \log R$$

if $0 \leq \theta \leq \pi$. But the proof by the use of this inequality fails in the case $m = \pi$.

§ 3. We have just seen that the fact that the Cardinal Function is a solution of the equation (1) depends chiefly upon the result that $I(R) \to 0$ as $R \to \infty$, the other parts of the proof being straightforward deductions from Cauchy's Integral Theorem.
Ferrar\textsuperscript{1} has recently shown that, if \(\sum_{n=-\infty}^{\infty} |a_n|^{\frac{1}{p}}\) is convergent \((p \geq 1)\), then the Cardinal Function has the definite integral representation

\[
C(x) = \int_{0}^{1} [\phi(t) \cos \pi xt + \psi(t) \sin \pi xt] \, dt,
\]

where \(\phi\) and \(\psi\) are each \(L^{1+p}\) over \((0, 1)\). From this result we are able to prove, with very little trouble, that \(I(R) \to 0\) as \(R \to \infty\), and thus to bring out the connection between the two properties of the Cardinal Function which we have noted in § 1.

For, considered as a function of the complex variable \(z\), \(C(z)\) is an integral function which remains finite as \(z\) tends to infinity in either direction along the real axis. In the upper half plane, we have

\[
z = r e^{i\theta} \quad (0 \leq \theta \leq \pi)
\]

Hence, by the use of H\ölder's inequality for integrals, we have

\[
|C(re^{i\theta})| \leq \int_{0}^{1} |\phi| |\cos \pi xt| \, dt + \int_{0}^{1} |\psi| |\sin \pi xt| \, dt
\]

\[
\leq \left\{ \left( \int_{0}^{1} |\phi|^{1+p} \, dt \right)^{\frac{1}{1+p}} + \left( \int_{0}^{1} |\psi|^{1+p} \, dt \right)^{\frac{1}{1+p}} \right\} \left\{ \int_{0}^{1} e^{\pi \left( 1 + \frac{1}{p} \right) t} \, dt \right\}^{\frac{p}{1+p}}
\]

\[
< Ke^{\pi r \sin \theta} (r \sin \theta)^{-\frac{p}{1+p}}
\]

where \(K\) is a finite constant, since \(\phi\) and \(\psi\) are each \(L^{1+p}\) over \((0, 1)\).

We now have

\[
|I(R)| \leq \frac{RK}{R - |x|} \int_{0}^{\pi} e^{-2(m-\pi)R \sin \theta} R^{\frac{p}{1+p}} \sin^{-\frac{p}{1+p}} \theta \, d\theta
\]

\[
\leq \frac{2RK}{R - |x|} \int_{0}^{\pi} e^{-2(m-\pi)R \theta / \pi} (\frac{2R \theta}{\pi})^{-\frac{p}{1+p}} \, d\theta
\]

\(\to 0\), as \(R \to \infty\), if \(m \geq \pi\).

We can now easily complete the proof, exactly as in § 2, of the following theorem:—

**Theorem 1.** If $\sum_{-\infty}^{\infty} |a_n|^{1 + \frac{1}{p}} (p > 1)$ is convergent, then $C(x)$ possesses the definite integral representation

$$C(x) = \int_{0}^{1} [\phi(t) \cos \pi xt + \psi(t) \sin \pi xt] \, dt$$

where $\phi$ and $\psi$ are each $L^{1+p}$ over $(0, 1)$, and is an $m$-function for $m \geq \pi$.

Theorem 1* is, of course, included in Theorem 1; for, by Holder’s inequality, the convergence of $\sum_{-\infty}^{\infty} |a_n|^{1 + \frac{1}{p}} (p > 1)$ implies the convergence of $\sum_{1}^{\infty} |a_n \log n|/n$ and $\sum_{1}^{\infty} |a_{-n} \log n|/n$, but not conversely.

§ 4. Finally, the use of functions of class $L^{p}$ enables us to prove, by the same direct method, Theorems 2 and 3 below.

**Theorem 2.** The integrals

$$f(x) = \int_{a}^{A} \frac{\phi(w)}{\sin wx} \, dw$$

represent $m$-functions, if $-m < a < A < m$, provided only that $\phi(w)$ is $L^{p}$ ($p > 1$) over $(a, A)$.

**Theorem 3.** The integral

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin \mu (w - x)}{w - x} \phi(w) \, dw$$

represents an $m$-function, if $m \geq \mu > 0$, provided only that $\phi(w)$ is $L^{p}$ ($p > 1$) over $(-\infty, \infty)$.

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1 Compare the rather similar theorems given by Hardy, *loc. cit.*, 457, 459.