On an Exponential Functional Inequality and its Distributional Version

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Abstract. Let $G$ be a group and $K = \mathbb{C}$ or $\mathbb{R}$. In this article, as a generalization of the result of Albert and Baker, we investigate the behavior of bounded and unbounded functions $f : G \to K$ satisfying the inequality

$$\left| f\left( \sum_{k=1}^{n} x_k \right) - \prod_{k=1}^{n} f(x_k) \right| \leq \phi(x_2, \ldots, x_n), \quad \forall x_1, \ldots, x_n \in G,$$

where $\phi : G^{n-1} \to [0, \infty)$. Also, as a distributional version of the above inequality we consider the stability of the functional equation

$$u \circ S - \underbrace{u \otimes \cdots \otimes u}_{n\text{-times}} = 0,$$

where $u$ is a Schwartz distribution or Gelfand hyperfunction, $\circ$ and $\otimes$ are the pullback and tensor product of distributions, respectively, and $S(x_1, \ldots, x_n) = x_1 + \cdots + x_n$.

1 Introduction

Throughout this paper, we denote by $G$ a group, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $K = \mathbb{C}$ or $\mathbb{R}$, $\phi : G^{n-1} \to [0, \infty)$, and $\epsilon \geq 0$. We call $m : G \to K$ an exponential function provided that

$$m(x+y) = m(x)m(y)$$

for all $x, y \in G$. Let $f : G \to K$ satisfy the exponential functional inequality

$$(1.1) \quad |f(x+y) - f(x)f(y)| \leq \epsilon$$

for all $x, y \in G$. Then $f$ is either an unbounded exponential function or a bounded function satisfying

$$(1.2) \quad |f(x)| \leq \frac{1}{2}(1 + \sqrt{1+4\epsilon})$$

for all $x \in G$ (see Baker [3]). In [2], Albert and Baker refined the inequality (1.2) when $G$ is a vector space over the field $\mathbb{Q}$ of rational numbers and proved that if $f : G \to \mathbb{R}$ is a bounded function satisfying (1.1) with $0 < \epsilon < \frac{1}{2}$, then $f$ satisfies either

$$(1.3) \quad -\epsilon \leq f(x) \leq \frac{1}{2}(1 - \sqrt{1-4\epsilon})$$
for all \( x \in G \), or
\[
\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \leq f(x) \leq \frac{1}{2}(1 + \sqrt{1 + 4\epsilon})
\]
for all \( x \in G \). The inequalities (1.3) and (1.4) imply that every bounded function satisfying the inequality (1.1) tends to 0 or 1 (the roots of the algebraic equation \( x^2 - x = 0 \)) as \( \epsilon \to 0 \).

In this paper, we investigate behaviors of bounded functions and unbounded functions \( f : G \to K \) satisfying the exponential functional inequality with \( n \)-variables \((n \geq 2)\)
\[
\left| f\left( \sum_{k=1}^{n} x_k \right) - \prod_{k=1}^{n} f(x_k) \right| \leq \phi(x_2, \ldots, x_n)
\]
for all \( x_1, \ldots, x_n \in G \). When we consider some exponential functional equations or unbounded solutions of exponential functional inequalities involving \( n \)-variables, we can follow the same approach as in the case of 2-variables. However, when we consider bounded solution of exponential functional inequality with \( n \)-variables, such as the inequality (1.5), the methods are quite different from that of 2-variables, such as those of Albert and Baker [2].

As a corollary of our main result we obtain that every bounded function \( f : G \to \mathbb{R} \) satisfying the inequality (1.5) with \( \phi(x_2, \ldots, x_n) = \epsilon \) for all \( x_2, \ldots, x_n \in G \) satisfies the following:

Let \( \alpha < \beta < \gamma \) be the positive real roots of the equation \( |t^n - t| = \epsilon \). If \( n \) is even, then \( f \) satisfies either \(-\epsilon \leq f(x) \leq \alpha \) for all \( x \in G \), or \( \beta \leq f(x) \leq \gamma \) for all \( x \in G \), and if \( n \) is odd, then \( f \) satisfies \( \beta \leq f(x) \leq \gamma \) for all \( x \in G \), \(-\alpha \leq f(x) \leq \alpha \) for all \( x \in G \), or \(-\beta \leq f(x) \leq \beta \) for all \( x \in G \).

As a direct consequence of this result, we also obtain that if \( n \) is even, then \( f \) satisfies either
\[
-\epsilon \leq f(x) \leq \frac{n}{n-1} \epsilon
\]
for all \( x \in G \), or
\[
-\sqrt[n]{n} \epsilon \leq f(x) - 1 \leq \frac{\epsilon}{n-1}
\]
for all \( x \in G \), and if \( n \) is odd, then \( f \) satisfies
\[
-\frac{n}{n-1} \epsilon \leq f(x) \leq \frac{n}{n-1} \epsilon
\]
for all \( x \in G \),
\[
-\sqrt[n]{n} \epsilon \leq f(x) - 1 \leq \frac{\epsilon}{n-1}
\]
for all \( x \in G \), or
\[
-\frac{\epsilon}{n-1} \leq f(x) + 1 \leq \sqrt[n]{n} \epsilon
\]
for all $x \in G$. We also consider the unbounded functions $f : G \to \mathbb{K}$ satisfying (1.5) and prove that if there exist $q_1, q_2, \ldots, q_n \in G$ such that

$$|f(q_1)(|f(q_2) \cdots f(q_n)| - 1)| > \phi(q_2, \ldots, q_n),$$

then the function $f$ satisfying (1.5) is unbounded and has the form $f(x) = Cm(x)$, where $C \in \mathbb{K}$ with $C^{n-1} = 1$ and $m$ is an exponential function. In the last section of the paper, as a distributional version of the inequality (1.5), we consider the inequality

$$\|u \circ S - \underbrace{u \otimes \cdots \otimes u}_{n \text{-times}}\| \leq \epsilon,$$

where $u$ is a Schwartz distribution[6] or Gelfand hyperfunction [4,5], $\circ$ and $\otimes$ denote the pullback and the tensor product of distributions, respectively, and $\| \cdot \| \leq \epsilon$ means that $|\langle \cdot, \varphi \rangle| \leq \epsilon \|\varphi\|_1$ for all test functions $\varphi$ (see Section 3). As a result, we prove that if $u$ satisfies (1.6), then either $u$ is a bounded measurable function satisfying

$$\|u\|_{L\infty} \leq \gamma,$$

where $\gamma > 1$ is the root of the algebraic equation $z^n - z = \epsilon$, or

$$u = e^{ikt} e^{cx}$$

for some $k \in \{0, 1, 2, \ldots, n - 2\}$, $c \in \mathbb{C}$. We refer the reader to [7–9, 11–14] for related results of Hyers–Ulam stability of functional equations.

## 2 Classical Solutions of (1.5)

In this section we investigate behaviors of bounded functions and unbounded functions $f : G \to \mathbb{K}$ satisfying the exponential functional inequality (1.5). We first investigate behaviors of bounded functions satisfying the inequality (1.5).

**Lemma 2.1** Let $f : G \to \mathbb{K}$ be a bounded function satisfying the inequality (1.5). Then $f$ satisfies

$$(2.1) \quad |f(x_1)(1 - |f(x_2) \cdots f(x_n)|)| \leq \phi(x_2, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in G$.

**Proof** Let $M = \sup_{x \in G}|f(x)|$. Using the triangle inequality with (1.5) we have

$$(2.2) \quad |f(x_1)f(x_2) \cdots f(x_n)| \leq |f(x_1 + \cdots + x_n)| + \phi(x_2, \ldots, x_n) \leq M + \phi(x_2, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in G$. From (2.2) we have

$$(2.3) \quad M|f(x_2) \cdots f(x_n)| = \sup_{x_1 \in G}|f(x_1)||f(x_2) \cdots f(x_n)| \leq M + \phi(x_2, \ldots, x_n)$$

for all $x_2, \ldots, x_n \in G$. Thus from (2.3), we get

$$(2.4) \quad M(|f(x_2) \cdots f(x_n)| - 1) \leq \phi(x_2, \ldots, x_n)$$
for all $x_2, \ldots, x_n \in G$. Replacing $x_1$ by $x_1 - x_2 - \cdots - x_n$ in (1.5) and using the triangle inequality with the result we have
\begin{equation}
|f(x_1)| \leq |f(x_1 - x_2 - \cdots - x_n)| |f(x_2) \cdots f(x_n)| + \phi(x_2, \ldots, x_n)
\end{equation}
\begin{equation}
\leq M|f(x_2) \cdots f(x_n)| + \phi(x_2, \ldots, x_n)
\end{equation}
for all $x_1, \ldots, x_n \in G$. From (2.5) we have
\[
M = \sup_{x_n \in G} |f(x_1)| \leq M|f(x_2) \cdots f(x_n)| + \phi(x_2, \ldots, x_n)
\]
for all $x_2, \ldots, x_n \in G$, which implies
\begin{equation}
M\left(1 - |f(x_2) \cdots f(x_n)|\right) \leq \phi(x_2, \ldots, x_n)
\end{equation}
for all $x_2, \ldots, x_n \in G$. Thus, from (2.4) and (2.6) we have
\[
M\left|1 - |f(x_2) \cdots f(x_n)|\right| \leq \phi(x_2, \ldots, x_n)
\]
for all $x_2, \ldots, x_n \in G$, which implies (2.1). This completes the proof. 

From now on, for each integer $n \geq 2$, we denote by $c_n := (n - 1)n^{-\frac{1}{n-1}}$ and $D := \{x \in G : \phi(x, \ldots, x) < c_n\}$. Note that $c_n$ is the (local) maximum of the polynomial $p(t) := t - t^n$. One can see that $\frac{1}{4} \leq c_n < c_{n+1} < 1$ for all $n = 2, 3, 4, \ldots$. It is easy to see that for each $x \in G$, the equation
\begin{equation}
|t^n - t| = \phi(x, \ldots, x)
\end{equation}
has only one real root $\gamma(x) > 1$, and for each $x \in D$, the equation (2.7) has the three positive real roots $\alpha(x) < \beta(x) < \gamma(x)$. Note that $0 < \alpha(x_1) < n^{-\frac{1}{n-1}} < \beta(x_2) < 1 < \gamma(x_3)$ for all $x_1, x_2, x_3 \in D$. In particular, we denote by $\alpha < \beta < \gamma$ the positive real roots of the equation $|t^n - t| = \epsilon$ when $\epsilon < c_n$.

As a main result of this section we have the following.

**Theorem 2.2** Let $f : G \to K$ be a bounded function satisfying the inequality (1.5). Then $f$ satisfies
\begin{equation}
|f(x)| \leq \gamma(x)
\end{equation}
for all $x \in G$. Furthermore, $f$ satisfies either
\begin{equation}
|f(x)| \leq \alpha(x)
\end{equation}
for all $x \in D$, or
\begin{equation}
\beta(x) \leq |f(x)| \leq \gamma(x)
\end{equation}
for all $x \in D$.

**Proof** Replacing $x_1, x_2, \ldots, x_n$ by $x$ in (2.1) we have
\begin{equation}
||f(x)| - |f(x)|^p| \leq \phi(x, \ldots, x)
\end{equation}
for all $x \in G$. From (2.11), for each $x \in G$, $|f(x)|$ satisfies
\[
|f(x)| \leq \gamma(x),
\]
which gives (2.8). For each \( x \in D \), \( f(x) \) satisfies either
\[
|f(x)| \leq \alpha(x)
\]
or
\[
\beta(x) \leq |f(x)| \leq \gamma(x).
\]
Now, we prove that \( f \) satisfies (2.12) for all \( x \in D \) or (2.13) for all \( x \in D \). Assume that there exist \( y_1, y_2 \in D \) such that
\[
|f(y_1)| \leq \alpha(y_1), \quad \beta(y_2) \leq |f(y_2)|.
\]
Putting \( x_1 = y_2 \) and \( x_2 = x_3 = \ldots = x_n = y_1 \) in (2.1) we have
\[
|f(y_2)|\left(1 - |f(y_1)|^{n-1}\right) \leq \phi(y_1, \ldots, y_1).
\]
On the other hand, from (2.14) we have
\[
|f(y_2)|\left(1 - |f(y_1)|^{n-1}\right) \geq \beta(y_2)\left(1 - \alpha(y_1)^{n-1}\right)
\]
which contradicts (2.15). Thus, we get (2.9) or (2.10). This completes the proof.  

Let \( \phi(x_2, \ldots, x_n) = \epsilon < c_n \) for all \( x_2, \ldots, x_n \in G \) in Theorem 2.2. Then we have the following.

**Corollary 2.3**  Let \( f : G \to K \) be a bounded function satisfying the inequality (1.5). Then \( f \) satisfies either
\[
|f(x)| \leq \alpha
\]
for all \( x \in G \), or
\[
\beta \leq |f(x)| \leq \gamma
\]
for all \( x \in G \).

In particular, if \( G \) is 2-divisible, \( K = \mathbb{R} \) and \( \phi(x_2, \ldots, x_n) = \epsilon < c_n \) for all \( x_2, \ldots, x_n \in G \), then we have the following.

**Corollary 2.4**  Assume that \( G \) is 2-divisible and \( f : G \to \mathbb{R} \) is a bounded function satisfying the inequality
\[
\left| f\left(\sum_{k=1}^{n} x_k\right) - \prod_{k=1}^{n} f(x_k)\right| \leq \epsilon
\]
for all \( x_1, \ldots, x_n \in G \). If \( n \) is even, then \( f \) satisfies either
\[
-\epsilon \leq f(x) \leq \alpha
\]
for all \( x \in G \), or
\[
\beta \leq f(x) \leq \gamma
\]
for all \( x \in G \). If \( n \) is odd, then \( f \) satisfies (2.20) for all \( x \in G \),
\[
-\alpha \leq f(x) \leq \alpha
\]
for all \( x \in G \), or
\[
-\gamma \leq f(x) \leq -\beta
\]
for all \( x \in G \).

**Proof** Replacing \( x_1, x_2 \) by \( x \) and putting \( x_3 = x_4 = \ldots = x_n = 0 \) in (2.18) we have
\[
f \left( \frac{x}{2} \right)^2 f(0)^n - \epsilon \leq f(x) \leq f \left( \frac{x}{2} \right)^2 f(0)^n + \epsilon
\]
for all \( x \in G \). We first consider the case when \( n \) is even or \( f(0) \geq 0 \). From (2.23) we have
\[
-\epsilon \leq f \left( \frac{x}{2} \right)^2 f(0)^n - \epsilon \leq f(x)
\]
for all \( x \in G \). Note that
\[
\epsilon = \alpha - \alpha^n < \alpha.
\]
From (2.16), (2.24), and (2.25) we have
\[
-\epsilon \leq f(x) \leq \alpha
\]
for all \( x \in G \), or from (2.17), (2.24), and (2.25) we have
\[
\beta \leq f(x) \leq \gamma
\]
for all \( x \in G \). Thus, if \( n \) is even, from (2.26) and (2.27) we get (2.19) or (2.20). Now, we consider the case when \( n \) is odd and \( f(0) < 0 \). From (2.24) we have
\[
f(x) \leq f \left( \frac{x}{2} \right)^2 f(0)^n + \epsilon \leq \epsilon
\]
for all \( x \in G \). Thus, from (2.16), (2.25), and (2.28), we have
\[
-\alpha \leq f(x) \leq \epsilon
\]
for all \( x \in G \), or from (2.17), (2.25), and (2.28) we have
\[
-\gamma \leq f(x) \leq -\beta
\]
for all \( x \in G \). Thus, if \( n \) is odd, from (2.26), (2.27), (2.29), and (2.30), we get (2.20), (2.21), or (2.22). This completes the proof.

Note that \( \alpha, \beta, \gamma \) satisfy
\[
0 < \alpha < \frac{n}{n - 1} \epsilon, \quad 1 - \sqrt[n]{\epsilon} < \beta < 1, \quad 1 < \gamma < 1 + \frac{\epsilon}{n - 1}.
\]

As a consequence of the Corollary 2.4 together with the inequality (2.31), we have the following.

**Corollary 2.5** Assume that \( G \) is \( 2 \)-divisible and \( f : G \to \mathbb{R} \) is a bounded function satisfying the inequality (2.18) for \( \epsilon < \epsilon_n \). If \( n \) is even, then \( f \) satisfies either
\[
-\epsilon \leq f(x) \leq \frac{n}{n - 1} \epsilon
\]
for all \( x \in G \), or
\[
-\sqrt[n]{\epsilon} \leq f(x) - 1 \leq \frac{\epsilon}{n - 1}
\]
for all $x \in G$. If $n$ is odd, then $f$ satisfies

$$\frac{-n}{n-1}\epsilon \leq f(x) \leq \frac{n}{n-1}\epsilon$$

for all $x \in G$,

$$-\sqrt{n}\epsilon \leq f(x) - 1 \leq \frac{\epsilon}{n-1}$$

for all $x \in G$, or

$$-\frac{\epsilon}{n-1} \leq f(x) + 1 \leq \sqrt{n}\epsilon$$

for all $x \in G$.

**Remark 2.6** From Corollary 2.5, if $n$ is even, every bounded solution of (2.18) tends to 0 or 1 as $\epsilon \to 0$, and if $n$ is odd, every bounded solution of (2.18) tends to 0, 1, or $-1$ as $\epsilon \to 0$.

If $n = 2$ and $0 < \epsilon < \frac{1}{4}$, then it is easy to see that

$$\alpha = \frac{1}{2}(1 - \sqrt{1-4\epsilon}), \quad \beta = \frac{1}{2}(1 + \sqrt{1-4\epsilon}), \quad \gamma = \frac{1}{2}(1 + \sqrt{1+4\epsilon}).$$

Thus, by Corollary 2.5 we obtain a improved version of the result of Albert and Baker for vector space [2].

**Corollary 2.7** Let $0 < \epsilon < \frac{1}{4}$. Assume that $G$ is a 2-divisible group and $f : G \to \mathbb{R}$ is a bounded function satisfying the inequality

$$|f(x+y) - f(x)f(y)| \leq \epsilon$$

for all $x, y \in G$. Then $f$ satisfies either

$$-\epsilon \leq f(x) \leq \frac{1}{2}(1 - \sqrt{1-4\epsilon})$$

for all $x \in G$ or

$$\frac{1}{2}(1 + \sqrt{1-4\epsilon}) \leq f(x) \leq \frac{1}{2}(1 + \sqrt{1+4\epsilon})$$

for all $x \in G$.

Finally, we investigate the unbounded solutions of the inequality (1.5).

**Theorem 2.8** Let $f : G \to \mathbb{K}$ satisfy the inequality (1.5). Assume that there exist $q_1, q_2, \ldots, q_n \in G$ such that

$$|f(q_1)(|f(q_2) \cdots f(q_n)| - 1)| > \phi(q_2, \ldots, q_n).$$

Then $f$ is unbounded and there exists an exponential function $m : G \to \mathbb{K}$ and $C \in \mathbb{K}$ with $C^{n-1} = 1$ such that

$$f(x) = Cm(x)$$

for all $x \in G$. 
Proof By Lemma 2.1, we can see that if $f$ satisfies (2.32), then $f$ is unbounded and $f(0) \neq 0$. Let $z_k \in G$, $k = 1, 2, 3, \ldots$ be a sequence such that $|f(z_k)| \to \infty$ as $k \to \infty$. Replacing $x_1$ by $z_k$, $k = 1, 2, 3, \ldots$, $x_2$ by $x$, putting $x_3 = \cdots = x_n = 0$ in (1.5) and dividing the result by $|f(z_k)|$ we have

$$
(2.34) \quad \left| f(x) f(0)^{n-2} - \frac{f(z_k + x)}{f(z_k)} \right| \leq \frac{\phi(x, 0, \ldots, 0)}{|f(z_k)|}.
$$

Letting $k \to \infty$ in (2.34) we have

$$
(2.35) \quad f(x) f(0)^{n-2} = \lim_{k \to \infty} \frac{f(z_k + x)}{f(z_k)}
$$

for all $x \in G$. Thus, using (1.5) and (2.35) we have

$$
(2.36) \quad f(x + y) = \frac{1}{f(0)^{n-2}} \lim_{k \to \infty} \frac{f(z_k + x + y)}{f(z_k)} = \frac{1}{f(0)^{n-2}} \lim_{k \to \infty} \frac{f(z_k + x) f(y) f(0)^{n-2}}{f(z_k)} = f(y) \lim_{k \to \infty} \frac{f(z_k + x)}{f(z_k)} = f(y) f(x) f(0)^{n-2}
$$

for all $x, y \in G$. Putting $y = 0$ in (2.36) we have

$$
(2.37) \quad f(0)^{n-1} = 1.
$$

Dividing (2.36) by $f(0)^n$ and using (2.37) we have

$$
(2.38) \quad \frac{f(x + y)}{f(0)} = \frac{f(x)}{f(0)} \cdot \frac{f(y)}{f(0)}
$$

for all $x, y \in G$. From (2.38) we get (2.33). This completes the proof. 

\[\square\]

3 Distributions and Hyperfunctions

We briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions and the space $(\mathcal{S}_{1/2})'(\mathbb{R}^n)$ of Gelfand hyperfunctions. Here we use the notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We also denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Definition 3.1 A distribution $u$ is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ such that

$$
|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup \left| \partial^\alpha \varphi \right|
$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in $K$. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$. 

4 Distributional Solution of (1.6)

\textbf{Definition 3.2} We denote by $S_{1/2}^{1/2}(\mathbb{R}^n)$ the space of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ satisfying the following; there exist positive constants $A$ and $B$ such that

\begin{equation}
\|\varphi\|_{A,B} := \sup_{x \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}} A^{[|\alpha|]} B^{[|\beta|]} \|\partial^\alpha \varphi(x)\|_{1/2} < \infty.
\end{equation}

The topology on the space $S_{1/2}^{1/2}(\mathbb{R}^n)$ is defined by the seminorms $\| \cdot \|_{A,B}$ in the left-hand side of (3.1) and we denote by $(S_{1/2}^{1/2}(\mathbb{R}^n))'$ the dual space of $S_{1/2}^{1/2}(\mathbb{R}^n)$ and the elements of $(S_{1/2}^{1/2}(\mathbb{R}^n))'$ are called Gelfand hyperfunctions.

It is known that the space $S_{1/2}^{1/2}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ that can be continued to an entire function satisfying

\begin{equation}
|\varphi(x + iy)| \le C \exp(-a|x|^2 + b|y|^2)
\end{equation}

for some $a, b > 0$.

\textbf{Definition 3.3} Let $u_j \in \mathcal{D}'(\mathbb{R}^n)$ [resp. $S_{1/2}^{1/2}(\mathbb{R}^n)$] for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of $u_1$ and $u_2$, defined by

\[ \langle u_1 \otimes u_2, \varphi(x_1, x_2) \rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2) \rangle \rangle \]

for $\varphi(x_1, x_2) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, belongs to $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ [resp. $(S_{1/2}^{1/2}(\mathbb{R}^n))'$] if

\begin{equation}
\|u \otimes S - \underbrace{u \otimes \cdots \otimes u}_{\text{n-times}}\| \le \epsilon,
\end{equation}

where $\otimes$ is tensor product of distributions, $S(x_1, \ldots, x_n) = x_1 + \cdots + x_n$, the pullback $u \circ S$ is defined by

\[ \langle u \circ S, \varphi(x_1, \ldots, x_n) \rangle = \langle u, \int \varphi(x_1, \ldots, x_{n-1}, x - x_1 - \cdots - x_{n-1}) \, dx_1 \cdots dx_{n-1} \rangle, \quad \varphi \in C_c^\infty(\mathbb{R}^n), \]

and $\| \cdot \| \le \epsilon$ means that $|\langle \cdot, \varphi \rangle| \le \epsilon \|\varphi\|_{L^1}$ for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ [resp. $(S_{1/2}^{1/2}(\mathbb{R}^n))'$].

We denote by $\delta(x)$ the function on $\mathbb{R}^n$,

\[ \delta(x) = \begin{cases} qe^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \ge 1, \end{cases} \]

where

\[ q = \left( \int_{|x|<1} e^{-\frac{1}{1-|x|^2}} \, dx \right)^{-1}. \]
It is easy to see that \( \delta(x) \) an infinitely differentiable function with support \( \{ x : |x| \leq 1 \} \). Now we employ the function \( \delta_t(x) := t^{-\alpha} \delta(x/t) \), \( t > 0 \). Let \( u \in \mathcal{D}'(\mathbb{R}^n) \). Then for each \( t > 0 \), \( (u * \delta_t)(x) = (u_y, \delta_t(x - y)) \) is a smooth function in \( \mathbb{R}^n \) and \( (u * \delta_t)(x) \to u \) as \( t \to 0^+ \) in the sense of distributions, that is, for every \( \varphi \in C_c^\infty(\mathbb{R}^n) \),
\[
\langle u, \varphi \rangle = \lim_{t \rightarrow 0^+} \int (u * \delta_t)(x) \varphi(x) \, dx.
\]
We also employ the heat kernel
\[
E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0.
\]
In view of (3.2) it is easy to see that the heat kernel \( E_t(x) \) belongs to \( \mathcal{D}^{1/2}_t(\mathbb{R}^n) \) for each \( t > 0 \). It is well known that the heat kernel satisfies the semigroup property
\[
E_t * E_s = E_{t+s}
\]
for all \( t, s > 0 \), which will be useful. We first consider the inequality (4.1) in the space of Schwartz distributions.

**Theorem 4.1** Let \( u \in \mathcal{D}'(\mathbb{R}^n) \) satisfy the inequality (4.1). Then either \( u \) is a bounded measurable function satisfying
\[
\| u \|_{L^\infty} \leq \gamma,
\]
where \( \gamma > 1 \) is the root of the algebraic equation \( z^n - z = \epsilon \), or
\[
u = e^{\frac{\epsilon}{\gamma^2}} e^{-x}
\]
for some \( k \in \{0, 1, 2, \ldots, n - 2\} \), \( \epsilon \in \mathbb{C} \).

**Proof** Convolving \( (\delta_t \otimes \cdots \otimes \delta_t)(x_1, \ldots, x_n) := \delta_t(x_1) \cdots \delta_t(x_n) \) in each side of (4.1) we have
\[
[(u \circ S) * (\delta_t \otimes \cdots \otimes \delta_t)](x_1, \ldots, x_n)
\]
\[
= \left\langle u_{c_t}, \int \delta_t(x_1 + \xi_1 + \cdots + \xi_n - \xi_1) \delta_t(x_2 - \xi_2) \cdots \delta_t(x_n - \xi_n) \, d\xi_2 \cdots d\xi_n \right\rangle
\]
\[
= \left\langle u_{c_t}, \int (\delta_{t_1} * \cdots * \delta_{t_n})(x_1 + x_2 + \xi_3 + \cdots + \xi_n - \xi_1) \times \delta_{t_3}(x_3 - \xi_3) \cdots \delta_{t_n}(x_n - \xi_n) \, d\xi_3 \cdots d\xi_n \right\rangle
\]
\[
= \left\langle u_{c_t}, \int (\delta_{t_1} * \cdots * \delta_{t_{n-1}})(x_1 + \cdots + x_{n-1} + \xi_n - \xi_1) \delta_{t_n}(x_n - \xi_n) \, d\xi_n \right\rangle
\]
\[
= \left\langle u_{c_t}, (\delta_{t_1} * \cdots * \delta_{t_n})(x_1 + \cdots + x_n - \xi_1) \right\rangle
\]
\[
= (u * \delta_{t_1} * \cdots * \delta_{t_n})(x_1 + \cdots + x_n).
\]
We also have
\[ [(u \otimes \cdots \otimes u) \ast (\delta_{t_1} \otimes \cdots \otimes \delta_{t_n})](x_1, \ldots, x_n) = (u \ast \delta_{t_1})(x_1) \cdots (u \ast \delta_{t_n})(x_n). \]
Thus, the inequality (4.1) is converted to the following inequality
\[
(4.4) \quad |(u \ast \delta_{t_1} \ast \cdots \ast \delta_{t_n})(x_1 + \cdots + x_n) - (u \ast \delta_{t_1})(x_1) \cdots (u \ast \delta_{t_n})(x_n)| \leq \epsilon
\]
for all \(x_1, \ldots, x_n \in \mathbb{R}^n, t_1, \ldots, t_n > 0.\) It follows from (4.4) that the limit
\[
f(x) := \lim_{t \to 0^+} (u \ast \delta_t)(x)
\]
eexists for all \(x \in \mathbb{R}^n.\) In (4.4), fixing \(x_2, \ldots, x_n\) and letting \(t_2, t_3, \ldots, t_n \to 0^+\) so that \((u \ast \delta_{t_j})(x_j) \to f(x_j)\) as \(t_j \to 0^+\) for all \(j = 2, 3, \ldots, n,\) we have
\[
(4.5) \quad |(u \ast \delta_{t_1})(x_1 + \cdots + x_n) - (u \ast \delta_{t_1})(x_1) f(x_2) \cdots f(x_n)| \leq \epsilon.
\]
Replacing \(x_n\) by \(x,\) letting \(x_1 = x_2 = \cdots = x_{n-1} = 0\) and \(t_1 \to 0^+,\) so that \((u \ast \delta_{t_1})(0) \to f(0)\) as \(n \to \infty\) in (4.5), we have
\[
(4.6) \quad \|u - f(0)^{n-1} f(x)\| \leq \epsilon.
\]
If \(f\) is bounded, then from (4.6) \(u\) is defined by a bounded measurable function, i.e.,
\[
\langle u, \varphi \rangle = \int h(x) \varphi(x) \, dx, \quad \varphi \in C_c^\infty(\mathbb{R}^n)
\]
for some bounded measurable function \(h.\) Now, using the heat kernel \(E_t\) instead of \(\delta_t\) and convolving \((E_{t_1} \otimes \cdots \otimes E_{t_n})(x_1, \ldots, x_n)\) in each side of (4.1), we have
\[
(4.7) \quad \|U(x_1 + \cdots + x_n, t_1 + \cdots + t_n) - U(x_1, t_1) \cdots U(x_n, t_n)\| \leq \epsilon
\]
for all \(x_1, x_2, \ldots, x_n \in \mathbb{R}^n, t_1, t_2, \ldots, t_n > 0,\) where \(U(x, t) = (u \ast E_t)(x)\). Using the same method as in the proof of Lemma 2.1 with (4.7), we can prove that
\[
(4.8) \quad \left| U(x_1, t_1) \left( \|U(x_2, t_2) \cdots U(x_n, t_n)\| - 1 \right) \right| \leq \epsilon
\]
for all \(x_1, x_2, \ldots, x_n \in \mathbb{R}^n, t_1, t_2, \ldots, t_n > 0.\) Letting \(x_1 = x_2 = \cdots = x_n = x, t_1 = t_2 = \cdots = t_n = t\) in (4.8) we have
\[
(4.9) \quad \|U(x, t)\| \leq \gamma
\]
for all \(x \in \mathbb{R}^n, t > 0.\) Letting \(t \to 0^+\) in (4.9), we get (4.2). Now, we consider the case when \(f\) is unbounded. Let \(c_k, k = 1, 2, 3, \ldots,\) be a sequence such that \(|f(c_k)| \to \infty\) as \(k \to \infty.\) Replacing \(x_2 = \cdots = x_n = c_k\) in (4.5) and dividing the result by \(|f(c_k)|^{n-1}\) and letting \(k \to \infty\) we have
\[
(4.10) \quad (u \ast \delta_{t_1})(x_1) = \lim_{n \to \infty} \frac{u \ast \delta_{t_1})(x_1 + (n - 1)c_k)}{|f(c_k)|^{n-1}}.
\]
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Multiplying both sides of (4.10) by \( f(x_2) \cdots f(x_n) \), and using (4.5) and (4.10), we have\(^{12}\)

\[
\lim_{k \to \infty} \frac{(u \ast \delta_{t_1})(x_1 + \cdots + x_n + (n-1)c_k)}{f(c_k)^{n-1}} = \lim_{k \to \infty} \frac{(u \ast \delta_{t_1})(x_1 + \cdots + x_n + (n-1)c_k)}{f(c_k)^{n-1}} = \frac{(u \ast \delta_{t_1})(x_1 + \cdots + x_n)}{f(c_k)^{n-1}} = (u \ast \delta_{t_1})(x_1 + \cdots + x_n)
\]

for all \( x_1, x_2, \ldots, x_n \in \mathbb{R}^n \), \( t_1, t_2, \ldots, t_n > 0 \). Putting \( x_2 = x_3 = \cdots = x_{n-1} = 0 \) in (4.11) we have

\[
(u \ast \delta_{t_1})(0) = (u \ast \delta_{t_1})(0) f(0)^{n-2} f(x) = (u \ast \delta_{t_1})(x)
\]

for all \( x \in \mathbb{R}^n \). Choosing \( t_1 > 0 \) such that \((u \ast \delta_{t_1})(0) \neq 0\) and putting (4.12) in (4.11) we have

\[
\int f(x_1) f(x_2) \cdots f(x_n) = f(x_1 + \cdots + x_n)
\]

for all \( x_1, x_2, \ldots, x_n \in \mathbb{R}^n \). Choosing a sequence \( s_k \), \( k = 1, 2, 3, \ldots \) so that \((u \ast \delta_{s_k})(0) \to 0 \) as \( k \to \infty \), replacing \( t_1 \) by \( s_k \) in (4.12) and letting \( k \to \infty \) we have\(^{13}\)

\[
\langle u, \varphi \rangle = \lim_{k \to \infty} \int (u \ast \delta_{s_k})(x) \varphi(x) dx = \lim_{k \to \infty} \int (u \ast \delta_{s_k})(0) f(0)^{n-2} f(x) \varphi(x) dx = \int f(x) \varphi(x) dx
\]

for all \( \varphi \in C_c^\infty(\mathbb{R}^n) \). Now, it is easy to see that the solution \( f \) of (4.13), being a measurable function, is given by

\[
f(x) = f(0)e^{c \cdot x} = e^{\frac{\|u\|_\infty}{\|u\|_\infty}} e^{c \cdot x}
\]

for some \( k \in \{0, 1, 2, \ldots, n-2\} \), \( c \in \mathbb{C}^n \). Thus, from (4.14) and (4.15), we get (4.3). This completes the proof.\(^{14}\)

Note that every locally integrable function \( f \) defines a distribution via the correspondence\(^{15}\)

\[
\varphi \mapsto \int f(x) \varphi(x) dx.
\]

As a direct consequence of the above result we obtain the following.

**Corollary 4.2** Let \( f : \mathbb{R}^n \to \mathbb{C} \) be a locally integrable function satisfying

\[
\|f(x_1 + \cdots + x_n) - f(x_1) \cdots f(x_n)\|_\infty(\mathbb{R}^n) \leq \epsilon.
\]

Then either \( f \) is a bounded measurable function satisfying

\[
\|f(x)\|_\infty \leq \gamma,
\]
where $\gamma > 1$ is the root of the algebraic equation $z^n - z = \epsilon$, or
\[
f(x) = e^{\frac{\Omega x}{\pi}} e^{\epsilon x}
\]
for almost every $x \in \mathbb{R}^n$, where $k \in \{0, 1, 2, \ldots, n-2\}$, $c \in \mathbb{C}^n$.

As a consequence of the method of proof of Theorem 4.1 we obtain the stability of the inequality (4.1) in the space $(S_{1/2}^{1/2}(\mathbb{R}^n))'$ of Gelfand hyperfunctions.

**Theorem 4.3** Let $u \in (S_{1/2}^{1/2}(\mathbb{R}^n))'$ satisfy the inequality (4.1). Then either $u$ is a bounded measurable function satisfying
\[
\|u\|_{L\infty} \leq \gamma,
\]
where $\gamma > 1$ is the root of the algebraic equation $z^n - z = \epsilon$, or
\[
u = e^{\frac{\Omega x}{\pi}} e^{\epsilon x}
\]
for some $k \in \{0, 1, 2, \ldots, n-2\}$, $c \in \mathbb{C}^n$.

**Proof** Let $u \in (S_{1/2}^{1/2}(\mathbb{R}^n))'$. Then using the heat kernel $E_t$ instead of $\delta_t$ and convolving $(E_{t_1} \otimes \cdots \otimes E_{t_n})(x_1, \ldots, x_n)$ in each side of (4.1) we have
\[
|U(x_1 + \cdots + x_n, t_1 + \cdots + t_n) - U(x_1, t_1) \cdots U(x_n, t_n)| \leq \epsilon
\]
for all $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$, $t_1, t_2, \ldots, t_n > 0$, where $U(x, t) = (u * E_t)(x)$. Using the same method as in the proof of Theorem 4.1, we get the result. $\blacksquare$

**References**


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