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# Grassmannian Trilogarithms

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**Abstract.** In the previous works of the first author, two completely different constructions of single valued Grassmannian trilogarithms were given. One of the constructions, in *Math. Res. Lett.* **2** (1995), 99–114, is very simple and provides Grassmannian $n$-logarithms for all $n$. However its motivic nature is hidden. The other construction in *Adv. in Math.* **114** (1995), 197–318, is very explicit and motivic, but the generalization for $n > 4$ is not known. In this paper we will compute explicitly the Grassmannian trilogarithm constructed in *Math. Res. Lett.* **2** (1995), 99–114 and prove that it differs from the motivic Grassmannian trilogarithm by an explicitly given product of logarithms. We also derive some general results about the Grassmannian polylogarithms.

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**Key Words.** Polylogarithms, motivic complexes.

## 1. Introduction

### 1.1. THE GRASSMANNIAN $n$-LOGARITHM

Let $V_m$ be an $m$-dimensional vector space over an arbitrary field $F$ with a given basis $e_0, \dots, e_{m-1}$. Let $\{z_i\}$ be the coordinate system in $V_m$ dual to the basis $\{e_i\}$ and $\widehat{G}^q_p$ the Grassmannian of $p$-dimensional subspaces in generic position with respect to the coordinate hyperplanes in $V_{p+q}$. The intersection of a hyperplane with the coordinate plane $z_i = 0$ provides a map $a_i$: $\widehat{G}^q_p \longrightarrow \widehat{G}^q_{p-1}$. The collection of the maps $\{a_i\}$ provides a truncated semisimplicial variety over $\mathbb{Z}$

$$\hat{G}^n_n \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \hat{G}^n_{n-1} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \hat{G}^n_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \hat{G}^n_1 \tag{1}$$

where $\widehat{G}^n_{n-k}$ sits in degree $2n-k$. Notice that $\widehat{G}^n_1 = (\mathbb{G}_m)^n$. Indeed, it consists of the one-dimensional subspaces in $V_{n+1}$ which do not lie in the hyperplanes $z_0 = 0, \dots, z_n = 0$. So $z_1/z_0, \dots, z_n/z_0$ are natural coordinates on $\widehat{G}^n_1$.

In [G4] we constructed a collection of $\mathbb{R}(n-1) = (2\pi i)^{n-1}\mathbb{R}$-valued differential $k$-forms $\mathcal{L}^{\mathrm{G}}_{k;n}$ on the complex Grassmannians $\widehat{G}^n_{n-k}(\mathbb{C})$ satisfying the cocycle condition

$$d\mathcal{L}^{\mathrm{G}}_{k;n} = \sum_{i=0}^{2n-k-1} (-1)^i a_i^* \mathcal{L}^{\mathrm{G}}_{k+1;n}, \quad -1 \leqslant k \leqslant n-2 \tag{2}$$

and such that

$$d\mathcal{L}^{\mathrm{G}}_{n-1;n}(h_0, \ldots, h_n) = -\pi_n\big(d\log(z_1/z_0) \wedge \cdots \wedge d\log(z_n/z_0)\big), \tag{3}$$

where $h_i = \{z_i = 0\}$ and for any real numbers $a$ and $b$

$$\pi_n(a + bi) = \begin{cases} a, & n \text{ odd}, \\ bi, & n \text{ even}. \end{cases}$$

A collection of forms as above is called (a *single valued*) Grassmannian $n$-logarithm. Its existence was conjectured in [BMS] (compare with [GGL]). The function $\mathcal{L}^{\mathrm{G}}_{0;n}$ is called the Grassmannian $n$-logarithm function and usually denoted by $\mathcal{L}^{\mathrm{G}}_n$.

*Remark*. The condition (2) for $k = -1$ is the $(2n + 1)$-term functional equation for the Grassmannian $n$-logarithm function:

$$\sum_{i=0}^{2n} (-1)^i a_i^* \mathcal{L}^{\mathrm{G}}_{0;n} = 0. \tag{4}$$

There are two other versions of the Grassmannian polylogarithms: the *real* Grassmannian polylogarithm function in [GM], on $\widehat{G}^{2n}_{2n}(\mathbb{R})$, and the *multivalued complex analytic* Grassmannian polylogarithms on $\widehat{G}^{n}_{\bullet}(\mathbb{C})$ in [HM1, HM2]. In general the real Grassmannian polylogarithm is expected to live on $\widehat{G}^{2n}_{\bullet}(\mathbb{R})$; it should be responsible for the combinatorial Pontryagin classes, see [GGL] and [Y]. We will not discuss them in our paper. The motivic construction of Grassmannian polylogarithms should in particular provide a coherent construction of all the three types of Grassmannian polylogs, as well as their étale, $p$-adic, etc., analogs.

The coinvariants of the natural action of the group $\mathrm{GL}(V_m)$ on the set of all $n$-tuples of vectors in $V_n$ are called the *configurations* of $n$ vectors in $V_m$. The configuration spaces of $m$ vectors in two vector spaces of the same dimension are canonically isomorphic. So we only need to specify the dimension of the vector space when talking about configuration of vectors. We denote by $C_m(V_n)$, or simply $C_m(n)$, the space of configurations of $m + 1$ vectors in generic position in $V_n$. Then there are the following *canonical* isomorphisms

$$\widehat{G}^q_p \cong C_{p+q-1}(V_p), \qquad \widehat{G}^q_p \cong C_{p+q-1}(V_q). \tag{5}$$

Namely, restricting the coordinate functions $z_i$ to a subspace $W \in \widehat{G}^q_p$ we get a configuration of vectors $z_0, \ldots, z_{p+q-1} \in W^*$. Projecting the vectors $e_i$ onto $V_{p+q}/W$ we get the second isomorphism.

## 1.2. CONSTRUCTION OF THE GRASSMANNIAN POLYLOGARITHMS $\mathcal{L}^{\mathrm{G}}_{k;n}$ [G4]

First we need the following construction. Let $X$ be a variety over $\mathbb{C}$ and $f_0, \ldots, f_{n-1}$ be $n$ complex-valued functions on $X(\mathbb{C})$. We attach to the above data the following singular $\mathbb{R}(n-1)$-valued differential $(n-1)$-form:

$$r_n(f_1, \ldots, f_n) := -\mathrm{Alt}_n \left\{ \sum_{k \geqslant 0} c_{k,n} \log|f_1| \bigwedge_{j=2}^{2k+1} d\log|f_j| \bigwedge_{j=2k+2}^{n} d\,i\arg f_j \right\}, \tag{6}$$

where $c_{k,n} := \binom{n}{2k+1}/n!$ and

$$\mathrm{Alt}_n F(x_1, \ldots, x_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The choice of the coefficients is dictated by the following property:

$$dr_n(f_1, \ldots, f_n) = -\pi_n(d\log f_1 \wedge \cdots \wedge d\log f_n) \tag{7}$$

Let $l_0, \ldots, l_{2n-k-1}$ be vectors in generic position in a complex vector space $V^*_{n-k}$. For $1 \leqslant i \leqslant 2n-k-1$ set $f_i := l_i/l_0$. They are $2n-k-1$ rational functions on $\mathbb{CP}^{n-k-1}$.

DEFINITION 1.1. The Grassmannian $k$-form of weight $n$ on $\widehat{G}^n_{n-k}(\mathbb{C})$ is defined by

$$\mathcal{L}^{\mathrm{G}}_{k;n}(l_0, \ldots, l_{2n-k-1}) = (2\pi i)^{k+1-n} \int_{\mathbb{CP}^{n-k-1}} r_{2n-k-1}(f_1, \ldots, f_{2n-k-1}).$$

For the precise meaning of the right-hand side, see [G4] or Section 3.1 below. It was proved in [G4] that this integral is convergent, so the definition makes sense.

The Grassmannian $n$-logarithm function $\mathcal{L}^{\mathrm{G}}_n$ can be descended onto the space of configurations of $2n$ points in $P(V^*_n) = \mathbb{CP}^{n-1}$ (see [G4]) or, what is the same, the space of configurations of $2n$ hyperplanes in $P(V_n)$. Let $h_0, \ldots, h_{2n-1}$ be $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. Choose rational functions $f_i$ such that $\mathrm{div}(f_i) = h_i - h_0$ for $1 \leqslant i \leqslant 2n-1$. Then the Grassmannian $n$-logarithm function $\mathcal{L}^{\mathrm{G}}_n$ is defined by

$$\mathcal{L}^{\mathrm{G}}_n(h_0, \ldots, h_{2n-1}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(f_1, \ldots, f_{2n-1}).$$

### 1.3. THE LIE-MOTIVIC CONSTRUCTION OF THE GRASSMANNIAN $n$-LOGARITHMS

A different construction of the the Grassmannian $n$-logarithms $L^{\mathrm{G}}_{\bullet;n}$ for $n = 2, 3$ was given in [G2, G3] and for $n = 4$ in [G1], see also [G6].

We will call these constructions *Lie-motivic* since they are obtained as a composition of a homomorphism from the Grassmannian complex (see Section 4 below) to a motivic complex, understood as the weight $n$ part of the cochain complex of the motivic *Lie algebra*, followed by the canonical regulator map to the real Deligne complex.

Let us explain in more details the notion of the Lie-motivic Grassmannian polylogarithm function. It is expected that there is a natural variation of $n$-framed mixed Tate motives over the Grassmannian $\widehat{G}^n_n$ responsible for the motivic

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Grassmannian $n$-logarithm function in the following way. Taking the Hodge realization of this variation we get a variation of $n$-framed Hodge-Tate structures over the Grassmannian. Let $\mathcal{H}_n$ be the group of $n$-framed Hodge-Tate structures. Then $\mathcal{H}_\bullet = \oplus_{n\geqslant 0}\mathcal{H}_n$ has a natural Hopf algebra structure (see [BGSV]). The coproduct on $\mathcal{H}_\bullet$ induces a graded Lie coalgebra structure on the quotient

$$\mathcal{L}(\mathcal{H})_\bullet := \frac{\mathcal{H}_\bullet}{\mathcal{H}_{>0}\cdot\mathcal{H}_{>0}}$$

There are two natural period maps

$$P^H: \mathcal{H}_\bullet \longrightarrow \mathbb{R}; \qquad p^L: \mathcal{H}_\bullet \longrightarrow \mathbb{R} \tag{8}$$

The first one is an algebra homomorphism, while the second kills the products: $p^L(\mathcal{H}_{>0}\cdot\mathcal{H}_{>0}) = 0$. Thus we get a canonical map $p^L: \mathcal{L}(\mathcal{H})_\bullet \longrightarrow \mathbb{R}$. Applying pointwise this map we get a function on the Grassmannian which we call the Lie-motivic Grassmannian polylogarithm $L_n^{\mathrm{G}}$.

## 1.4. THE COMPARISON PROBLEM

Now a natural question arises:

PROBLEM 1.2. (a) What is the relation between the Grassmannian $n$-logarithms $\mathcal{L}_{\bullet;n}^{\mathrm{G}}$ and $L_{\bullet;n}^{\mathrm{G}}$? Do they coincide or not?

(b) Is it true that the Grassmannian $n$-logarithm $\mathcal{L}_{\bullet;n}^{\mathrm{G}}$ admits a motivic construction?

(c) Is it true that the Grassmannian $n$-logarithm $\mathcal{L}_{\bullet;n}^{\mathrm{G}}$ is Lie-motivic?

By the very definitions, one has $\mathcal{L}_{n-1;n}^{\mathrm{G}} = L_{n-1;n}^{\mathrm{G}}$.

It was known from [G4], and it is already a nontrivial fact, that the Grassmannian dilogarithms of both types coincide. We will recover this result in Section 4.5 below.

It was noticed by the first author during the preparation of [G4], and puzzled him very much, that the Grassmannian $n$-logarithms $\mathcal{L}_{\bullet;n}^{\mathrm{G}}$ for $n \geqslant 4$ should be different from $L_{\bullet;n}^{\mathrm{G}}$. The reason is that $\mathcal{L}_{\bullet;n}^{\mathrm{G}}$ satisfy some additional functional equations which should not be true for $L_{\bullet;n}^{\mathrm{G}}$ for $n \geqslant 4$. Namely, projection along the subspace generated by $e_j$ provides a map $b_j: \widehat{G}_p^{q+1} \longrightarrow \widehat{G}_p^q$. It was proved in [G4] that $\mathcal{L}_{p;q}^{\mathrm{G}}$ satisfies the property

$$\sum_{j=0}^{2q-p}(-1)^j b_j^* \mathcal{L}_{p;q}^{\mathrm{G}} = 0. \tag{9}$$

This property is valid for the motivic Grassmannian trilogarithm. However it should not be satisfied by $L_{n-2;n}^{\mathrm{G}}$ for $n \geqslant 4$, and because of this to construct the regulator map one needs to extend the $L^{\mathrm{G}}$ to a *bi-Grassmannian* $n$-logarithm, see [G7].

In this paper, we compute explicitly the Grassmannian trilogarithm $\mathcal{L}_{k;3}^{\mathrm{G}}$ and show that it is different from $L_{k;3}^{\mathrm{G}}$ for $k = 0$ and 1. The difference is explicitly computed and has a motivic origin. Therefore, the answer to part (b) of the problem is positive for $n = 3$. However it is not Lie-motivic, thus the answer to the question (c) is negative.

### 1.5. MAIN RESULT: COMPUTATION OF THE GRASSMANNIAN TRILOGARITHM $\mathcal{L}_3^{\mathrm{G}}$

Recall that the classical polylogarithms are defined by

$$Li_1(z) := -\log(1-z); \qquad Li_n(z) := \int_0^z Li_{n-1}(t)\frac{dt}{t}, \quad n \geqslant 2.$$

They admit the single-valued cousins (see [Z]). For the dilogarithm it is the Bloch–Wigner function

$$\widehat{\mathcal{L}}_2(z) := i\mathcal{L}_2(z) := \pi_2(Li_2(z)) + i\arg(1-z)\cdot\log|z|$$

and for the trilogarithm it is

$$\mathcal{L}_3(z) = \Re\mathrm{e}\{Li_3(z) - \log|z|\cdot Li_2(z)\} - \tfrac{1}{3}(\log|z|)^2\log|1-z|$$

which was used in the proof of Zagier's conjecture on $\zeta(3)$ in [G2] and [G3].

Choose a volume form $\omega \in \det V_n^*$ and set $\Delta_\omega(v_1, \dots, v_n) := \langle\omega, v_1\wedge\dots\wedge v_n\rangle \in F$. We will usually omit the subscript $\omega$.

The following result was proved in [G2, G3], see also [G5].

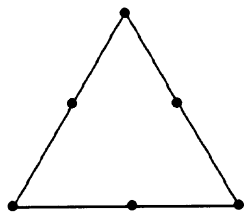
$$L_3^{\mathrm{G}}(l_0, \dots, l_5) = \tfrac{1}{90}\mathrm{Alt}_6\mathcal{L}_3\left(\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)}\right). \tag{10}$$

The next theorem is proved in Section 5:

THEOREM 1.3.

$$\begin{aligned}&\mathcal{L}_3^G(l_0, \dots, l_5)\\ &\quad = L_3^G(l_0, \dots, l_5) - \tfrac{1}{9}\mathrm{Alt}_6\Big(\log|\Delta(l_0, l_1, l_2)|\log|\Delta(l_1, l_2, l_3)|\log|\Delta(l_2, l_3, l_4)|\Big).\end{aligned}$$

Our next goal is to show that *the function $\mathcal{L}_3^{\mathrm{G}}$ does not satisfy the most interesting functional equation valid for the function $L_3^{\mathrm{G}}$*. Consider the following configuration of 6 points on the projective plane, called the special configuration.



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The set of the special configurations can be naturally identified with $\mathbb{P}^1\backslash\{0, \infty\}$ (see [G8]). Namely $z \in F^*$ corresponds to the configuration $g_3(z)$ given by the columns of the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & z \end{pmatrix}$$

If a function is defined at the points of a set $X$, we can extend it by linearity to a homomorphism from the free Abelian group generated by the points of $X$. We apply this construction to the functions $L_3^{\mathrm{G}}$ and $\mathcal{L}_3^{\mathrm{G}}$ and appropriate sets of the configurations of 6 points in $\mathbb{P}^2$. Then according to (10) one has

$$L_3^{\mathrm{G}}\left((l_0, \dots, l_5) - \tfrac{1}{90}\mathrm{Alt}_6 g_3\left(\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)}\right)\right) = 0. \tag{11}$$

However the function $\mathcal{L}_3^{\mathrm{G}}$ does not satisfy this functional equation. Indeed, we have the following result, which is proved in [G8], see also [G4].

THEOREM 1.4. *The restriction of the Grassmannian trilogarithm function* $\mathcal{L}_3^{\mathrm{G}}$ *to the special stratum coincides with the classical trilogarithm function* $\mathcal{L}_3$. *More precisely,* $\mathcal{L}_3^{\mathrm{G}}(g_3(z)) = \mathcal{L}_3(z)$.

Thus if the function $\mathcal{L}_3^{\mathrm{G}}$ satisfies the functional equation (11) it would coincide with $L_3^{\mathrm{G}}$ at the generic configuration, which is not true according to Theorem 1.3.

Notice that the function $\mathcal{L}_3^{\mathrm{G}}$ can be defined for an arbitrary configuration of points in $\mathbb{P}^2$ ([G4], [G8]), and one can show that it is continuous near the special stratum.

*Remark*. This shows that in the [G1, **S** 4] to define the motivic Lie coalgebra $\oplus G_*(F)$ of a field $F$ (cf. *loc. cit.*) one needs the functional equations for the Lie-motivic Grassmannian $n$-logarithm $L_n^{\mathrm{G}}$, instead of its relative $\mathcal{L}_n^{\mathrm{G}}$.

PROBLEM. Find explicit expression of the Lie-motivic Grassmannian $n$-logarithm via the functions $\mathcal{L}_n^{\mathrm{G}}$.

## 1.6. MAIN RESULT FROM THE POINT OF VIEW OF THE DELIGNE COHOMOLOGY

Recall that the Deligne cohomology $H^*_{\mathcal{D}}(X, \mathbb{R}(p))$ of a regular algebraic variety $X$ over $\mathbb{C}$ can be defined as the hypercohomology of $X$ with coefficients in the following complex of sheaves $\underline{\mathbb{R}}_{\mathcal{D}}(p)$ on $X(\mathbb{C})$:

$$\begin{array}{c} \left(\mathcal{D}_X^0 \xrightarrow{d} \mathcal{D}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}_X^p \xrightarrow{d} \mathcal{D}_X^{p+1} \xrightarrow{d} \cdots\right) \otimes \mathbb{R}(p-1) \\ \uparrow \pi_p \qquad \uparrow \pi_p \\ \Omega_X^p \xrightarrow{d} \Omega_X^{p+1} \xrightarrow{d} \cdots \end{array}$$

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Here $\mathcal{D}^i_X$ is the sheaf of $i$-currents on $X$. The group $\mathcal{D}^0_X$ is in degree 1. To compute the Deligne cohomology of $X$ let us replace $F^p\Omega^*$ by its Dolbeault resolution and denote by $\underline{\mathbb{R}}_{\mathcal{D}}(p)(X)$ the complex of the global sections of the complex of sheaves over $X(\mathbb{C})$. Then $H^*_{\mathcal{D}}(X, \mathbb{R}(p)) = H^*(\underline{\mathbb{R}}_{\mathcal{D}}(p)(X))$. Thus to calculate the Deligne cohomology $H^*_{\mathcal{D}}(\widehat{G}^n_\bullet, \mathbb{R}(n))$ of the semisimplicial Grassmannian (1) one needs to consider the cohomology of the total complex associated with the bicomplex of the shape

$$\underline{\mathbb{R}}_{\mathcal{D}}(n)(\widehat{G}^n_n) \longleftarrow \cdots \longleftarrow \underline{\mathbb{R}}_{\mathcal{D}}(n)(\widehat{G}^n_1). \tag{12}$$

A collection of differential forms on the simplicial Grassmannian (1) satisfies the two conditions (2) and (3) if and only if it represents a $2n$-cocycle in the bicomplex (12). By [G4, Lemma 2.3], $\mathcal{L}^{\mathrm{G}}_{\bullet;n}$ satisfies (2). We will see that by definition, for $n=2$ and 3, $L^{\mathrm{G}}_{\bullet;n}$ satisfies (2) and (3) and, moreover, $\mathcal{L}^{\mathrm{G}}_{n-1;n} = L^{\mathrm{G}}_{n-1;n}$. Thus one can expect that the difference between the 6-cocycles given by $L^{\mathrm{G}}_{\bullet;3}$ and $\mathcal{L}^{\mathrm{G}}_{\bullet;3}$ is a coboundary of a certain nice 5-chain in the bicomplex (12). We shall prove that this is indeed the case and calculate explicitly the 5-chain. It has a nonzero component only over $\widehat{G}^3_2$, and this component is given by the function

$$(l_0, \ldots, l_4) \longmapsto \tfrac{1}{9}\mathrm{Alt}_6\Big(\log|\Delta(l_0, l_1, l_2)| \log|\Delta(l_1, l_2, l_3)| \log|\Delta(l_2, l_3, l_4)|\Big). \tag{13}$$

Denote by $C_3$ this 5-chain in (12). It is also of motivic nature: the function (13) is a composition of the map

$$\begin{aligned}\widehat{G}^3_2(F) &\longrightarrow S^3F^*\\ (l_0, \ldots, l_4) &\longmapsto \tfrac{1}{9}\mathrm{Alt}_6\Big\{\Delta(l_0, l_1, l_2)\cdot\Delta(l_1, l_2, l_3)\cdot\Delta(l_2, l_3, l_4)\Big\},\end{aligned}$$

which is defined for an arbitrary field $F$, with the logarithm homomorphism

$$\begin{aligned}S^3\mathbb{C}^* &\longrightarrow \mathbb{R},\\ x_1\cdot x_2\cdot x_3 &\longmapsto \log|x_1|\log|x_2|\log|x_3|\end{aligned}$$

defined when $F = \mathbb{C}$.

Denote by $\{\mathcal{L}^{\mathrm{G}}_{\bullet;3}\}$ and $\{L^{\mathrm{G}}_{\bullet;3}\}$ the 6-cocycles in the bicomplex (12) provided by the collection of forms $\mathcal{L}^{\mathrm{G}}_{\bullet;3}$ and $L^{\mathrm{G}}_{\bullet;3}$. Let $D$ be the differential in the total complex associated with the bicomplex (12).

THEOREM 1.5. $\{\mathcal{L}^{\mathrm{G}}_{\bullet;3}\} - \{L^{\mathrm{G}}_{\bullet;3}\} = D(C_3)$.

We expect a similar story for the Grassmannian $n$-logarithms in general: the forms $\mathcal{L}^{\mathrm{G}}_{\bullet;n}$ should have a motivic nature in the following precise sense:

(1) One should have an explicitly given homomorphism $\mathbb{L}_n$ from the weight $n$ Grassmannian complex to the weight $n$ part $(\Lambda^*L(F)^\vee_\bullet, \Delta)_{(n)}$ of the cochain complex of the motivic Lie algebra $L(F)_\bullet$ of an arbitrary field $F$. (In fact $\mathbb{L}_n$ should be a

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part of the homomorphism from the bi-Grassmannian complex to the cochain complex of $L(F)_\bullet$).

Composing this map with the regulator map we get a cocycle in the bicomplex (12), the Lie-motivic Grassmannian $n$-logarithm $L^{\mathrm{G}}_{\bullet;n}$. For $n = 2, 3, 4$ this program has been implemented in [G2], [G3] and [G1], but in general the homomorphism $\mathbb{L}_n$ is unknown. The story for $n = 2, 3$ is recalled in section 4 below.

(2) We expect a natural $(2n-1)$-chain $C_n$ of motivic origin in (12) such that $\{\mathcal{L}^{\mathrm{G}}_{\bullet;n}\} - \{L^{\mathrm{G}}_{\bullet;n}\} = D(C_n)$. Here $\{\mathcal{L}^{\mathrm{G}}_{\bullet;n}\}$ and $\{L^{\mathrm{G}}_{\bullet;n}\}$ are $2n$-cocycles in the bicomplex (12) provided by the forms $\mathcal{L}^{\mathrm{G}}_{\bullet;n}$ and $L^{\mathrm{G}}_{\bullet;n}$.

Our desire to understand better the structure of Grassmannian polylogarithms was motivated by the following reasons:

(i) The Grassmannian $n$-logarithm can be used for an explicit construction of the class $c_n \in H^{2n}(\mathrm{BGL}(\mathbb{C})_\bullet, \mathbb{R}_{\mathcal{D}}(n))$ which provides Beilinson's regulator for $\mathcal{L}^{\mathrm{G}}$ (see [G8]) and $L^{\mathrm{G}}$ (see [G2, G3]).

(ii) Explicit calculation of the Grassmannian $n$-logarithm $\mathcal{L}^{\mathrm{G}}_{\bullet;n}$ should give a clue for construction of the homomorphism $\mathbb{L}_n$ as well as the chain $C_n$.

## 2. Some Properties of the Differential Forms $r_m$

Here is another expression of the differential form $r_m$ which will be very useful in applications.

PROPOSITION 2.1. *The differential* $(m-1)$*-form* $r_m(f_1, \ldots, f_m)$ *can be expressed as*

$$\mathrm{Alt}_m\left\{\sum_{k=1}^{m}\frac{(-1)^{m-k-1}}{m!}\log|f_1|\bigwedge_{j=2}^{k} d\log f_j \bigwedge_{j=k+1}^{m} d\,\overline{\log f_j}\right\}. \tag{14}$$

*Proof.* By definition

$$\begin{aligned} &r_{n+1}(f_0, \ldots, f_n)\\ &\quad= \frac{-1}{2^n(n+1)!}\mathrm{Alt}_{n+1}\left\{\sum_{j\geqslant 0}\binom{n+1}{2j+1}\log|f_0|\cdot\bigwedge_{s=1}^{2j}\left(d\log f_s + d\,\overline{\log f_s}\right)\right.\\ &\qquad\left.\times\bigwedge_{s=2j+1}^{n}\left(d\log f_s - d\,\overline{\log f_s}\right)\right\}\\ &\quad= \frac{-1}{2^n(n+1)!}\mathrm{Alt}_{n+1}\left\{\sum_{j\geqslant 0}\binom{n+1}{2j+1}\sum_{k\geqslant 0}\sum_{l\geqslant 0}\binom{n-2j}{n-k-l}\binom{2j}{l}(-1)^{n-k-l}\times\right.\\ &\qquad\left.\times\log|f_0|\bigwedge_{s=1}^{k} d\log f_s\bigwedge_{s=k+1}^{n} d\,\overline{\log f_s}\right\}. \end{aligned}$$

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In the above, we've used the skew-symmetry property, for example,

$$\mathrm{Alt}_n\{d\,\overline{\log f_1}\wedge d\log f_2\cdots\}=\mathrm{Alt}_n\{d\log f_1\wedge d\,\overline{\log f_2}\cdots\}.$$

The coefficients in the summation of index $k$ is obtained as follows: for each $d\log f_s+d\,\overline{\log f_s}$ we can either choose $d\log f_s$ or $\overline{\log f_s}$ but not both, the same for $d\log f_s-d\,\overline{\log f_s}$. For any appropriately fixed $l$, there are $\binom{2j}{l}$ ways to choose $\overline{\log f_s}$ from the former and $\binom{n-2j}{n-k-l}$ ways from the later. Once $\overline{\log f_s}$ are chosen, $d\log f_s$ are determined. We can now show that

$$\sum_{0\leqslant 2j\leqslant n-1}\binom{n+1}{2j+1}\sum_{l\geqslant 0}\binom{n-2j}{n-k-l}\binom{2j}{l}(-1)^{n-k-l}=2^n(-1)^{n-k}$$

by comparing the coefficient of $x^{n-k}$ of the following polynomials in $x$:

$$\begin{aligned}&\sum_{p=0}^{n}\sum_{0\leqslant 2j\leqslant n}\binom{n+1}{2j+1}\sum_{l\geqslant 0}\binom{n-2j}{p-l}\binom{2j}{l}(-1)^{p-l}x^p\\&=\sum_{0\leqslant 2j\leqslant n}\binom{n+1}{2j+1}(1-x)^{n-2j}(1+x)^{2j}\\&=\frac{1}{2(1+x)}\left[(1-x+1+x)^{n+1}-(1-x-1-x)^{n+1}\right]\\&=\frac{2^n\left[1-(-x)^{n+1}\right]}{1+\chi}+x=2^n\sum_{m=0}^{n}(-x)^m.\end{aligned}$$

The proposition follows at once.

COROLLARY 2.2. *The* $(2n-1)$*-form* $r_{2n}(f_1,\ldots,f_{2n})$ *can be expressed by*

$$\mathrm{Alt}_{2n}\pi_{2n}\left\{\sum_{k=n+1}^{2n}\frac{2(-1)^{k-1}}{2n!}\log|f_1|\bigwedge_{j=2}^{k}d\log(f_j)\bigwedge_{j=k+1}^{2n}d\,\overline{\log(f_j)}\right\}$$

*and the* $(2n-2)$*-form* $r_{2n-1}(f_1,\ldots,f_{2n-1})$ *is*

$$\begin{aligned}\mathrm{Alt}_{2n-1}\mathrm{Re}\Bigg\{&\frac{(-1)^n}{(2n-1)!}\log|f_1|\bigwedge_{j=2}^{n}d\log(f_j)\bigwedge_{j=n+1}^{2n-1}d\,\overline{\log(f_j)}+\\&+\sum_{k=n+1}^{2n-1}\frac{2(-1)^k}{(2n-1)!}\log|f_1|\bigwedge_{j=2}^{k}d\log(f_j)\bigwedge_{j=k+1}^{2n-1}d\,\overline{\log(f_j)}\Bigg\}.\end{aligned}$$

*Proof.* We can use symmetry to bring (14) into the form in which at least $\lceil(m-1)/2\rceil$ holomorphic $d\log$ appear together with at most $\lceil(m-2)/2\rceil$ anti-holomorphic $d\,\overline{\log}$. □

EXAMPLES 2.3. We will need the following special cases later:

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(1) From Corollary 2.2 $r_3(f_1, f_2, f_3)$ is equal to

$$\frac{1}{6}\mathrm{Re}\Big(\mathrm{Alt}_3\Big\{\log|f_1| d\log f_2 \wedge d\overline{\log f_3} - 2\log|f_1| d\log f_2 \wedge d\log f_3\Big\}\Big).$$

(2) We have

$$\begin{aligned} &r_4(f_1, f_2, f_3, f_4) \\ &\quad = \tfrac{1}{12}\pi_4\Big(\mathrm{Alt}_4\Big\{\log|f_1| d\log f_2 \wedge d\log f_3 \wedge d\overline{\log f_4} - \\ &\qquad - \log|f_1| d\log f_2 \wedge d\log f_3 \wedge d\log f_4\Big\}\Big). \end{aligned}$$

## 3. Computation of the Grassmannian 1-Forms

### 3.1. THE SETUP

For any $0 \leqslant k \leqslant n-1$ we let $V_{n-k}$ be an $(n-k)$-dimensional complex vector space. Let $l_0, \dots, l_{2n-k-1}$ be vectors in generic position in the dual space $V_{n-k}^*$. Recall that the Grassmannian $k$-form of weight $n$ is

$$\mathcal{L}^{\mathrm{G}}_{k;n}(l_0, \cdots, l_{2n-k-1}) = (2\pi i)^{k+1-n}\int_{\mathbb{CP}^{n-k-1}} r_{2n-k-1}(f_1, \dots, f_{2n-k-1}). \tag{15}$$

where $f_i(t) = l_i/l_0$, $1 \leqslant i \leqslant 2n-k-1$, are $2n-k-1$ rational functions on $P(V_{n-k}) = \mathbb{CP}^{n-k-1}$. Our first goal is to explain the meaning of this integral. In the next subsection we give a recipe for its computation when $k = 1$.

Let $\pi\colon Z \longrightarrow Y$ be a smooth map of manifolds with compact fibers and $\omega$ be a distribution on $Z$. Then one can define $\pi_*\omega$ so that $\langle \pi_*\omega, \varphi\rangle = \langle \omega, \pi^*\varphi\rangle$ for any smooth test form $\varphi$ on $Y$.

There is a canonical function on $V_{n-k}^* \times V_{n-k}$ whose value at the point $(l, t)$ is $l(t)$. The expression $r_{2n-k-1}(f_1(t), \dots, f_{2n-k-1}(t))$ is a differential form with logarithmic singularities on $\mathbb{CP}^{n-k-1} \times Y$ where

$$Y = \underbrace{V_{n-k}^* \times \cdots \times V_{n-k}^*}_{2n-k-1 \text{times}}.$$

It is proved in [G4] that it has integrable singularities, and thus provides a distribution on this manifold. The right-hand side of (15) is defined as $(2\pi i)^{k+1-n} \cdot \pi_*\big(r_{2n-k-1}(f_1(t), \dots, f_{2n-k-1}(t))\big)$, where $\pi$ is the canonical projection along $\mathbb{CP}^{n-k-1}$. Write $d = d_t + d_a$ for the differential on $P(V_{n-k}) \times Y$, where $d_t$ is the $P(V_{n-k})$- and $d_a$ is the $Y$-components of $d$.

Let $X$ and $Y$ be complex manifolds and $X$ is compact of complex dimension $d$. Let $\omega$ be a distribution on $X \times Y$. There is canonical projection $\pi\colon X \times Y \longrightarrow Y$. Denote by $\mathcal{D}^{(p_1,q_1)}(X)$ the space of distributions of the Dolbeault type $(p_1, q_1)$. The space $\mathcal{D}(X \times Y)$ of distributions on $X \times Y$ admits a decomposition $\mathcal{D}(X \times Y) = \oplus\mathcal{D}^{(p_1,q_1;p_2,q_2)}(X \times Y)$, where $(p_1, q_1)$ (resp. $(p_2, q_2)$) is the type of the distribution

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with respect to $X$ (resp. $Y$). If $\omega$ is of type $(p_1, q_1; p_2, q_2)$ then $\pi_*\omega$ is of type $(p_1 - d, q_1 - d; p_2, q_2)$. In particular $\pi_*\omega = 0$ if $p_1 < d$ or $q_1 < d$.

Let us present $r_{2n-k-1}(f_1(t), \dots, f_{2n-k-1}(t))$ as a sum of its Dolbeault components $\omega^{(p,2n-k-2-p)}$. Then

$$\omega^{(p,2n-k-2-p)} = \sum_\alpha \log|g_0^\alpha(t)| \bigwedge_{j=1}^{p} d\log g_j^\alpha(t) \bigwedge_{j=p+1}^{2n-k-2} d\,\overline{\log g_j^\alpha(t)},$$

where $g_i^\alpha(t)$ are some rational functions on $\mathbb{CP}^{n-k-1}$. Therefore the integral of $\omega^{(p,2n-k-2-p)}$ over $\mathbb{CP}^{n-k-1}$ is zero unless $n-k-1 \leqslant p \leqslant n-1$. If $p = n-1$ the integral is calculated as

$$\frac{\binom{n-1}{k}}{(2n-k-1)!}\mathrm{Alt}_{2n-k-1} \sum_\alpha \int_{\mathbb{CP}^{n-k-1}} \log|g_0^\alpha(t)| \times \\ \times \bigwedge_{j=1}^{k} d_a \log g_1^\alpha(t) \bigwedge_{j=k+1}^{n-1} d\log g_j^\alpha(t) \bigwedge_{j=n}^{2n-k-2d} \overline{\log g_j^\alpha(t)}.$$

In this paper we are mostly interested in the case $k = 1$.

### 3.2. THE KEY FORMULA

Our main task in this subsection is to calculate the Grassmannian 1-form $\mathcal{L}^{\mathrm{G}}_{1;n+1}(l_0, \dots, l_{2n})$ of weight $n+1$. (We increase the weight for ease of notation.)

Set $X := V_n^* \times \dots \times V_n^*$ and $Y := V_n$. Let $(l_0, \dots, l_n; t)$ be a point of the variety $X \times Y$. One has

$$\omega^n(X \times Y) = \oplus_{a+b=n}\omega^a(X) \otimes \omega^a(Y); \quad \omega = \sum \omega^{(a;b)}.$$

We will now write $d = d_a + d_t$ where $d_t$ is the $V_n$-components of $d$. Let us compute the $(1; n-1)$ component of the following differential form:

$$\sum_{i=0}^{n}(-1)^i d\log l_0(t) \wedge \dots \wedge d\log \widehat{l_i}(t) \wedge \dots \wedge d\log l_n(t).$$

One can define the $\mathrm{SL}(V_n)$-invariant Leray form

$$\alpha_{n-1}(l_1(t), \dots, l_n(t)) := \sum_{i=1}^{n}(-1)^{i-1} l_i(t) d_t l_1(t) \wedge \dots \wedge \widehat{d_t l_i(t)} \wedge \dots \wedge d_t l_n(t).$$

Let $p\colon V_n \setminus \{l_i(t) = 0\} \longrightarrow P(V_n)$ be the natural projection. Then one can check that the form

$$\frac{\alpha_{n-1}(l_1(t), \dots, l_n(t))}{l_1(t) \cdot \ldots \cdot l_n(t)}$$

is lifted from $P(V_n)$, i.e. it is equal to $p^*\omega$ for some form $\omega$ on $P(V_n)$.

PROPOSITION 3.1.

$$\frac{1}{n!}\mathrm{Alt}_{(l_0,\dots,l_n)}\Big(d\log l_0(t)\wedge\cdots\wedge d\log l_{n-1}(t)\Big)^{(1;n-1)}$$
$$=\frac{1}{(n-1)!}\mathrm{Alt}_{(l_0,\dots,l_n)}\Big(d\log\Delta(l_0,\dots,l_{n-1})\wedge d_t\log l_1(t)\wedge\cdots\wedge d_t\log l_{n-1}(t)\Big)$$
$$=\sum_{i=0}^{n}(-1)^i d\log\Delta(l_1,\dots,\widehat{l_i},\dots,l_n)\wedge\frac{\alpha_{n-1}(l_0(t),\dots,\widehat{l_i(t)},\dots,l_n(t))}{l_0(t)\cdot\ldots\widehat{l_i(t)}\ldots\cdot l_n(t)}.$$

*Remark* 3.2. Because $\Delta(l_1,\dots,l_n)$ is a function on $V_n^*\times\cdots\times V_n^*$, one sees that $d_a\log\Delta(l_1,\dots,l_n)=d\log\Delta(l_1,\dots,l_n)$.

EXAMPLE 3.3. In the simplest nontrivial case $n=2$ we get

$$d\log l_0(t)\wedge d\log l_1(t)-d\log l_0(t)\wedge d\log l_2(t)+d\log l_1(t)\wedge d\log l_2(t)$$
$$=d\log\Delta(l_0,l_1)\wedge\Big(d_t\log l_1(t)-d_t\log l_0(t)\Big)-$$
$$-\,d\log\Delta(l_0,l_2)\wedge\Big(d_t\log l_2(t)-d_t\log l_0(t)\Big)+$$
$$+\,d\log\Delta(l_1,l_2)\wedge\Big(d_t\log l_2(t)-d_t\log l_1(t)\Big).$$

In coordinates it looks as follows:

$$d\log\frac{a_1t_1+a_2t_2}{c_1t_1+c_2t_2}\wedge d\log\frac{b_1t_1+b_2t_2}{c_1t_1+c_2t_2}$$
$$=d\log(a_1b_2-a_2b_1)\wedge d_t\log\frac{b_1t_1+b_2t_2}{a_1t_1+a_2t_2}-$$
$$-\,d\log(a_1c_2-a_2c_1)\wedge d_t\log\frac{c_1t_1+c_2t_2}{a_1t_1+a_2t_2}+$$
$$+\,d\log(b_1c_2-b_2c_1)\wedge d_t\log\frac{c_1t_1+c_2t_2}{b_1t_1+b_2t_2}.$$

*Proof.* Choose a volume form $\omega\in\det V_n^*$. Denote by $\omega^{-1}\in\det V_n$ the dual volume form in $V_n^*$. Then for any vectors $l_1,\dots,l_n\in V_n^*$ we have $\Delta_{\omega^{-1}}(l_1,\dots,l_n)\in F$. It is easy to check that

$$\alpha_{n-1}(l_1(t),\dots,l_n(t))=\Delta_{\omega^{-1}}(l_1,\dots,l_n)\cdot i_E\omega, \tag{16}$$

where $E:=\sum t_i\partial_{t_i}$ is the Euler vector field in $V_n$. It follows from this that

$$\left(\sum_{i=0}^{n}(-1)^i\bigwedge_{0\leqslant j\leqslant n,j\neq i}d\log l_j(t)\right)^{(1;n-1)}$$
$$=\frac{\sum_{i=0}^{n}(-1)^i\Delta_{\omega^{-1}}(l_0,\dots,\widehat{l_i},\dots,l_n)\cdot d_al_i(t)}{l_0(t)\cdots l_n(t)}\wedge i_E\omega.$$

Now let us calculate (16). Using (17), we get

$$d\log\Delta_{\omega^{-1}}(l_0,\dots,l_{n-1})\frac{\alpha_{n-1}(l_0(t),\dots,l_{n-1}(t))}{l_0(t)\dots l_{n-1}(t)} = \frac{d\Delta_{\omega^{-1}}(l_0,\dots,l_{n-1})\wedge i_E\omega}{l_0(t)\dots l_{n-1}(t)}.$$

So it remains to show that

$$\sum_{i=0}^{n}(-1)^i l_i(t)d\Delta_{\omega^{-1}}(l_0,\dots,\widehat{l_i},\dots,l_n) = \sum_{i=0}^{n}(-1)^i\Delta_{\omega^{-1}}(l_0,\dots,\widehat{l_i},\dots,l_n)d_a l_i(t)$$

which follows by applying the differential $d_a$ to the identity

$$\sum_{i=0}^{n}(-1)^i\Delta_{\omega^{-1}}(l_0,\dots,\widehat{l_i},\dots,l_n)\cdot l_i(t)=0.$$

We now can finish the proof by observing that

$$\alpha_{n-1}(l_0,\dots,l_{n-1})=\frac{1}{(n-1)!}\mathrm{Alt}_n\Big\{l_0(t)d_tl_1(t)\wedge\dots\wedge d_tl_{n-1}(t)\Big\}.$$ □

COROLLARY 3.4. *The Grassmannian* 1*-form of weight* $n+1$ *is*

$$\mathcal{L}^{\mathrm{G}}_{1;n+1}(l_0,\dots,l_2n) = -\frac{(-2\pi i)^{1-n}}{(2n-1)!}\cdot\mathrm{Alt}_{2n+1}\times \times\pi_{2n}\left(d\log\Delta(l_1,\dots,l_n)\wedge\int_{\mathbb{CP}^{n-1}}\log|l_0(t)|\bigwedge_{j=2}^{n}d_t\log l_j(t)\wedge\bigwedge_{j=n+1}^{2n-1}d_t\overline{\log l_j(t)}\right).$$

*Proof.* Let $f_i=l_i/l_0$ and

$$b_n=\frac{2(-1)^n}{(2n)!}.$$

From

$$r_{2n}(f_1,\dots,f_{2n})=\sum_{i=0}^{2n}(-1)^i r_{2n}(l_0,\dots,\widehat{l_i},\dots,l_{2n})$$

one has

$$\mathcal{L}^{\mathrm{G}}_{1;n+1}(l_0,\dots,l_{2n})=(2\pi i)^{1-n}\int_{\mathbb{CP}^{n-1}}\frac{1}{(2n)!}\mathrm{Alt}_{2n+1}r_{2n}(l_0,\dots,l_{2n-1}).$$

Using Corollary 2.2 and observing that $(1/(2n)!)\mathrm{Alt}_{2n+1}\mathrm{Alt}_{2n}=\mathrm{Alt}_{2n+1}$ we can

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rewrite the integrand as

$$b_n \cdot \mathrm{Alt}_{2n+1}\pi_{2n}\left(\log|l_0(t)|\bigwedge_{j=1}^{n} d\log l_j(t)\bigwedge_{j=n+1}^{2n-1} d\,\overline{\log l_j(t)}\right)$$
$$= n\cdot b_n\cdot \mathrm{Alt}_{2n+1}\times$$
$$\times\ \pi_{2n}\left(\log|l_0(t)|d\log\Delta(l_1,\ldots,l_n)\bigwedge_{j=2}^{n} d_t\log l_j(t)\bigwedge_{j=n+1}^{2n-1} d_t\overline{\log l_j(t)}\right). \qquad \square$$

## 4. The Grassmannian and Polylogarithmic Complexes: A Review

### 4.1. THE GRASSMANNIAN COMPLEX

Let $C_m(n)$ be the configurations of $m+1$ vectors in generic position in $n$-dimensional vector space $V_n$ over $F$. Then there is a map

$$d'\colon C_{m+1}(n+1)\longrightarrow C_m(n),\qquad (v_0,\ldots,v_m)\longmapsto \sum_{i=0}^{m}(-1)^i(v_i|v_0,\ldots,\widehat{v_i},\ldots,v_m).$$

Here $(v_i|v_0,\ldots,\widehat{v_i},\ldots,v_m)$ means the configuration of $(v'_0,\ldots,\widehat{v'_i},\ldots,v'_m)$ in the space $V_n/\langle v_i\rangle$ where $v'_j$ is the image of $v_j$ in $V_n/\langle v_i\rangle$. It is straightforward to see that $(C_{*+n-1}(*),d')$ form a complex, called the ($n$-th) Grassmannian complex. It is isomorphic to the complex $(C_{*+n-1}(n),d)$ where

$$d'\colon C_{m+1}(n)\longrightarrow C_m(n),\qquad (v_0,\ldots,v_m)\longmapsto \sum_{i=0}^{m}(-1)^i(v_0,\ldots,\widehat{v_i},\ldots,v_m)$$

by the duality $*\colon C_{m+n-1}(m)\to C_{m+n-1}(n)$ obtained by comparing the two isomorphisms in (5).

### 4.2. THE POLYLOGARITHMIC COMPLEXES

The polylogarithmic complex $(B(F;n)_\bullet,\delta)$ is a candidate to the weight $n$ motivic complex of the field $F$. It was defined in [G2, G3] for $n\leqslant 3$ as follows. The groups $B_n(F)$ are quotients $B_n(F):=\mathbb{Z}[\mathbb{P}^1_F]/R_n(F)$, where the subgroups $R_n(F)$ reflect the known functional equations for the $n$-logarithms for $n=1,2,3$. For example, $B_1(F)=F^*\cong\mathbb{Z}[\mathbb{P}^1_F]/R_1(F)$ where $R_1(F):=\langle\{x\}+\{y\}-\{xy\}\colon\ x,y\in F^*;\{0\};\{\infty\}\rangle$. Consider the homomorphisms

$$\delta_n\colon\quad \mathbb{Z}[\mathbb{P}^1_F]\longrightarrow\begin{cases}F^*\wedge F^*, & \text{if } n=2,\\ B_2(F)\otimes F^*, & \text{if } n=3,\end{cases}$$
$$\{x\}\longmapsto\begin{cases}(1-x)\wedge x, & \text{if } n=2,\\ \{x\}_2\otimes x, & \text{if } n=3,\end{cases}$$
$$\{0\},\{1\},\{\infty\}\longmapsto 0,$$

where $\{x\}_n$ is the image of $\{x\}$ in $B_n(F)$. Then $\delta_n(R_n(F)) = 0$, so we get well defined homomorphisms $\delta\colon B_2(F) \longrightarrow \Lambda^2 F^*$ and $\delta\colon B_3(F) \longrightarrow B_2(F) \otimes F^*$. We get the polylogarithmic complexes

$$B_2(F) \xrightarrow{\delta} \Lambda^2 F^*, \qquad B_3(F) \xrightarrow{\delta} B_2(F) \otimes F^* \xrightarrow{\delta} \Lambda^3 F^*,$$

where $\delta(\{x\}_2 \otimes y) \longrightarrow (1-x) \wedge x \wedge y$. These complexes can be thought of as the weight 2 and 3 parts of the standard cochain complex of the motivic Lie algebra $L(F)_\bullet$, see [G3].

### 4.3. HOMOMORPHISMS FROM GRASSMANNIAN COMPLEXES TO THE POLYLOGARITHMIC ONES

There are two commutative diagrams:

$$\begin{array}{ccc} C_3(2) & \xrightarrow{d'} & C_2(1) \\ \downarrow{\scriptstyle \varphi_3(2)} & & \downarrow{\scriptstyle \varphi_3(1)} \\ B_2(F) & \xrightarrow{\delta} & \bigwedge^2 F^* \end{array} \quad \text{and} \quad \begin{array}{ccccc} C_5(3) & \xrightarrow{d'} & C_4(2) & \xrightarrow{d'} & C_3(1) \\ \downarrow{\scriptstyle \varphi_3(2)} & & \downarrow{\scriptstyle \varphi_4(2)} & & \downarrow{\scriptstyle \varphi_3(2)} \\ B_3(F) & \xrightarrow{\delta} & B_2(F) \otimes F^* & \xrightarrow{\delta} & \bigwedge^3 F^* \end{array},$$

where $\varphi_2(1)(v_0, v_1, v_2) = \frac{1}{2}\mathrm{Alt}_3\{\Delta(v_0) \wedge \Delta(v_1)\}$ and $\varphi_3(2)(v_0, v_1, v_2, v_3)$ is given by $\{r(v_0, v_1, v_2, v_3)\}_2$ which is the image of the cross-ratio

$$r(v_0, \ldots, v_3) = \frac{\Delta(v_0, v_2)\Delta(v_1, v_3)}{\Delta(v_0, v_3)\Delta(v_1, v_2)} \tag{18}$$

in $B_2(F)$. For the second commutative diagram, the map $\varphi_5(3)$ is the generalized cross-ratio

$$\varphi_5(3)(v_0, \ldots, v_5) = \tfrac{1}{90}\mathrm{Alt}_6\left\{\frac{\Delta(v_0, v_1, v_3)\Delta(v_1, v_2, v_4)\Delta(v_2, v_0, v_5)}{\Delta(v_0, v_1, v_4)\Delta(v_1, v_2, v_5)\Delta(v_2, v_0, v_3)}\right\}_3 \in B_3(F)$$

and

$$\begin{aligned} \varphi_4(2)(v_0, \ldots, v_4) &= \tfrac{1}{12}\mathrm{Alt}_5\Big\{r(v_0, v_2, v_3, v_4)_2 \otimes \Delta(v_3, v_4)\Big\}, \\ \varphi_3(1)(v_0, \ldots, v_3) &= -\tfrac{1}{6}\mathrm{Alt}_4\Big\{\Delta(v_0) \wedge \Delta(v_1) \wedge \Delta(v_2)\Big\}. \end{aligned} \tag{19}$$

*Remark* 4.1. The correct proof of the second commutative diagram was given in [G5]. Notice that our $\varphi_5(3)$, $\varphi_4(2)$ and $\varphi_3(1)$ are 1/6 of the corresponding maps in [G5]. We made these changes in order to have $\mathcal{L}^{\mathrm{G}}_{n-1;n} = L^{\mathrm{G}}_{n-1;n}$.

### 4.4. THE REGULATOR MAP ON THE POLYLOGARITHMIC COMPLEXES

Let $X$ be a variety over $\mathbb{C}$ and $F := \mathbb{C}(X)$. Let $\mathcal{A}^\bullet_\eta(X)$ is the de Rham complex of smooth forms at the generic point of $X$ over $\mathbb{C}$. Set $\alpha(f) = \log|f| d\log|1-f| -$

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$\log|1-f|d\log|f|$. Then there are the following commutative diagrams

$$\begin{array}{ccc} B_2(F) & \xrightarrow{\delta'} & \bigwedge^2 F^* \\ \downarrow{\scriptstyle r_2(1)} & & \downarrow{\scriptstyle r_2(2)} \\ \mathcal{A}^0_\eta(X) & \longrightarrow & \mathcal{A}^1_\eta(X) \end{array} \quad \text{and} \quad \begin{array}{ccccc} B_3(F) & \xrightarrow{\delta} & B_2(F)\otimes F^* & \xrightarrow{\delta} & \bigwedge^3 F^* \\ \downarrow{\scriptstyle r_3(1)} & & \downarrow{\scriptstyle r_3(2)} & & \downarrow{\scriptstyle r_3(1)} \\ \mathcal{A}^0_\eta(X) & \xrightarrow{\delta} & \mathcal{A}^1_\eta(X) & \xrightarrow{\delta} & \mathcal{A}^2_\eta(X) \end{array}, \tag{20}$$

where

$$\begin{aligned} &r_2(1)\colon \{f\}_2 \longmapsto \widehat{\mathcal{L}}_2(f) = i\mathcal{L}_2(f), \qquad r_2(2)\colon g_0\wedge g_1 \longmapsto r_2(g_0,g_1);\\ &r_3(1)\colon \{f\}_3 \longmapsto \mathcal{L}_3(f),\\ &r_3(2)\colon \{f\}_2\otimes g \longmapsto \widehat{\mathcal{L}}_2(f)di\arg g - \frac{\log|g|\alpha(f)}{3},\\ &r_3(3)\colon g_0\wedge g_1\wedge g_2 \longmapsto r_3(g_0,g_1,g_2). \end{aligned}$$

Composing the maps $\varphi$ and $r$ we get the Lie-motivic Grassmannian polylogarithms $L^G_{\bullet;n}$ for $n = 2, 3$.

### 4.5. THE GRASSMANNIAN DILOGARITHM

We deal with the left commutative diagram in (20). It is easy to see that

$$\begin{aligned} L^{\mathrm{G}}_{1;2}(l_0,l_1,l_2)&\colon= r_2(2)\circ\varphi_2(1)(l_0,l_1,l_2)\\ &= \tfrac{1}{2}\mathrm{Alt}_3\Big\{r_2(l_0,l_1)\Big\} = r_2(f_1,f_2) = \mathcal{L}^{\mathrm{G}}_{1;2}(l_0,l_1,l_2). \end{aligned}$$

Therefore, the difference between $\mathcal{L}^{\mathrm{G}}_2(l_0,\dots,l_3)$ and

$$L^{\mathrm{G}}_{0;2}(l_0,\dots,l_3)\colon= r_2(1)\circ\varphi_3(2)(l_0,\dots,l_3) = \widehat{\mathcal{L}}_2(r(l_0,\dots,l_3))$$

is a constant. But it is zero because it is skewsymmetric with respect to the permutations of the vectors $l_i$. Thus $L^{\mathrm{G}}_{0;2} = \mathcal{L}^{\mathrm{G}}_2$ and

$$\mathcal{L}^{\mathrm{G}}_2(l_0,\dots,l_3) = \widehat{\mathcal{L}}_2(r(l_0,\dots,l_3)). \tag{21}$$

## 5. Proof of Theorems 1.3 and 1.5

By the very definition

$$\begin{aligned} &L^{\mathrm{G}}_{2;3}(l_0,l_1,l_2,l_3)\\ &\quad= r_3(3)\circ\varphi_3(1)(l_0,l_1,l_2,l_3)\\ &\quad= -\tfrac{1}{6}\mathrm{Alt}_4\Big\{r_3(l_0,l_1,l_2)\Big\} = r_3(f_1,f_2,f_3) = \mathcal{L}^{\mathrm{G}}_{2;3}(l_0,l_1,l_2,l_3). \end{aligned}$$

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We now want to compare

$$L^{\mathrm{G}}_{1;3}(l_0,\dots,l_4) := r_3(2)\circ\varphi_4(2)(l_0,\dots,l_4)$$

and

$$\begin{aligned}\mathcal{L}^{\mathrm{G}}_{1;3}(l_0,\dots,l_4) &= (2\pi i)^{-1}\int_{\mathbb{CP}^1} r_4(f_1,f_2,f_3,f_4)\\ &= \frac{(2\pi i)^{-1}}{4!}\mathrm{Alt}_{(l_0,\dots,l_4)}\int_{\mathbb{CP}^1} r_4(l_0,l_1,l_2,l_3)\end{aligned}$$

Notice that the $(1,2)$-component of $r_4(l_0,l_1,l_2,l_3)$ is a $(1,2)$-form on $X\times V_2$, not on $X\times P(V_2)$. However, after the alternation we get a $(1,2)$-form on $X\times P(V_2)$.

We will write $\Delta(a,b) := \Delta(l_a,l_b)$ for the rest of the paper.

PROPOSITION 5.1.

$$\begin{aligned}\mathcal{L}^{\mathrm{G}}_{1;3}(l_0,\dots,l_4) = \mathrm{Alt}_5\Big\{\Big(&\tfrac{1}{12}\widehat{\mathcal{L}}_2(r(l_0,l_1,l_2,l_4))di\arg\Delta(1,4)-\\ &-\tfrac{1}{3}\log|\Delta(0,1)|\log|\Delta(1,4)|d\log|\Delta(2,4)|\Big)\Big\}.\end{aligned}$$

*Proof.* By Corollary 3.4

$$\begin{aligned}&6\cdot 2\pi i\mathcal{L}^{\mathrm{G}}_{1;3}(l_0,l_1,l_2,l_3,l_4)\\ &\quad=\mathrm{Alt}_5\pi_4\Big\{d\log\Delta(1,2)\int_{\mathbb{CP}^1}\log|l_0|d\log(l_2)\wedge d\overline{\log(l_3)}\Big\}\\ &\quad=-\mathrm{Alt}_5\pi_4\Big\{d\log\Delta(1,4)\int_{\mathbb{CP}^1}\log|l_0|d\log(l_1)\wedge d\overline{\log(l_2)}\Big\}\\ &\quad=2\mathrm{Alt}_5\Big\{d\log|\Delta(1,4)|\int_{\mathbb{CP}^1}\log|l_0|d\log|l_1|\wedge di\arg l_2\Big\}- \qquad (22)\\ &\qquad-2\mathrm{Alt}_5\Big\{di\arg\Delta(1,4)\int_{\mathbb{CP}^1}\log|l_0|d\log|l_1|\wedge d\log|l_2|\Big\} \qquad (23)\end{aligned}$$

To get (22) and (23) we use the following observations. Let $f$ and $g$ be holomorphic functions on a complex curve $X$ and $\varphi$ is a real valued function. Then since $\int_X\varphi d\log f\wedge d\log g = 0$ we have, taking the real and imaginary parts respectively,

$$\begin{aligned}\int_X\varphi d\log|f|\wedge d\log|g| &= \int_X\varphi d\arg f\wedge d\arg g,\\ \int_X\varphi d\log|f|\wedge d\arg g &= \int_X\varphi d\log|g|\wedge d\arg f.\end{aligned}$$

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One can easily show that (by using Examples 2.3(1))

$$\begin{aligned}
&2\pi i\mathcal{L}_2^{\mathrm{G}}(l_0, l_1, l_2, l_3)\\
&\quad = \int_{\mathbb{CP}^1} r_3(f_1, f_2, f_3) = 2\int_{\mathbb{CP}^1} \log|f_1| d\log|f_2| \wedge d\log|f_3|\\
&\quad = -\frac{2}{3!}\mathrm{Alt}_4 \int_{\mathbb{CP}^1} \log|l_0| d\log|l_1| \wedge d\log|l_2|.
\end{aligned} \tag{24}$$

This is a special case of Equation (38) in the Appendix.

Writing (23) as

$$-\mathrm{Alt}_5\Big\{di\arg\Delta(1,4)\int_{\mathbb{CP}^1}\Big(\mathrm{Alt}_{l_1,l_4}\Big\{\log|l_0| d\log|l_1| \wedge d\log|l_2|\Big\}\Big)\Big\}$$

and subtracting from this

$$\mathrm{Alt}_5\Big\{di\arg\Delta(1,4)\int_{\mathbb{CP}^1}\Big(\mathrm{Alt}_{l_1,l_2}\Big\{\log|l_0| d\log|l_1| \wedge d\log|l_4|\Big\}\Big)\Big\}$$

which is zero (to check this use the skewsymmetry with respect the alternations {2, 3} and {0, 3}), we see, using (24), that (23) is equal to

$$\begin{aligned}
&\mathrm{Alt}_5\Big\{\pi i\mathcal{L}_2^{\mathrm{G}}(l_0, l_1, l_2, l_4) di\arg\Delta(1,4)\Big\}\\
&\quad = \pi i \mathrm{Alt}_5\Big\{\widehat{\mathcal{L}}_2(r(l_0, l_1, l_2, l_4)) di\arg\Delta(1,4)\Big\}
\end{aligned}$$

by (21). To calculate (22) we compute, in two different ways, the expression

$$\mathrm{Alt}_5\Big\{d\log|\Delta(1,4)|\int_{\mathbb{CP}^1} d\log|l_1| \wedge d\mathcal{L}_2\Big(\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\Big)\Big\}. \tag{25}$$

(1) The integral over $\mathbb{CP}^1$, and hence the whole expression, is zero because

$$d\log|l_1| \wedge d\mathcal{L}_2\Big(\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\Big) = d\Big\{\log|l_1| \wedge d\mathcal{L}_2\Big(\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\Big)\Big\}$$

where both parts are understood as currents. Notice that $\log|z|$ and $\mathcal{L}_1(z)$ have integrable singularities and thus provide currents on $\mathbb{CP}^1$.

(2) Using formulas $\Delta(2,4)l_0(t) - \Delta(0,4)l_2(t) = \Delta(0,2)l_4(t)$ and

$$d\mathcal{L}_2(f) = \log|f| d\arg(1-f) - \log|1-f| d\arg f$$

we see that $d\mathcal{L}_2(\Delta(2,4)l_0/\Delta(0,2)l_4)$ is equal to

$$\begin{aligned}
&\log\left|\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\right| d\arg\Big(\frac{\Delta(0,4)l_2}{\Delta(0,2)l_4}\Big) - \log\left|\frac{\Delta(0,4)l_2}{\Delta(0,2)l_4}\right| d\arg\Big(\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\Big)\\
&\quad = \log\left|\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\right| d\arg\frac{l_2}{l_4} - \log\left|\frac{\Delta(0,4)l_2}{\Delta(0,2)l_4}\right| d\arg\frac{l_0}{l_4}.
\end{aligned}$$

Since this expression is skewsymmetric with respect to the transposition {2, 0} exchanging the indices 2 and 0 we can write (25) as

$$0 = 2\mathrm{Alt}_5\left\{d\log|\Delta(1,4)|\int_{\mathbb{CP}^1} d\log|l_1|\wedge\log\left|\frac{\Delta(2,4)l_0}{\Delta(0,2)l_4}\right|di\arg\frac{l_2}{l_4}\right\}$$

$$= 2\mathrm{Alt}_5\left\{d\log|\Delta(1,4)|\int_{\mathbb{CP}^1}\log\left|\frac{l_0}{l_4}\right|d\log|l_1|\wedge di\arg\frac{l_2}{l_4}\right\} - \qquad (26)$$

$$- 2\mathrm{Alt}_5\left\{\log|\Delta(0,2)|d\log|\Delta(1,4)|\int_{\mathbb{CP}^1} d\log|l_1|\wedge di\arg\frac{l_2}{l_4}\right\}. \qquad (27)$$

We got last line by using transposition $\{0,3\}$. Now (26) is exactly (22) since the other three possible terms vanish due to alternation $\{0,3\}$ (or $\{2,3\}$). Therefore (22) is equal to

$$2\mathrm{Alt}_5\left\{\log|\Delta(0,2)|d\log|\Delta(1,4)|\int_{\mathbb{CP}^1} d\log|l_1|\wedge di\arg\frac{l_2}{l_4}\right\} -$$

$$= -2\mathrm{Alt}_5\left\{\log|\Delta(0,2)|d\log|\Delta(1,4)|\int_{\mathbb{CP}^1}\log|l_1|\wedge d\left(di\arg\frac{l_2}{l_4}\right)\right\} \qquad (28)$$

$$= -4\pi i\mathrm{Alt}_5\left\{\log|\Delta(0,2)|d\log|\Delta(1,4)|\log\left|\frac{\Delta(1,2)}{\Delta(1,4)}\right|\right\}.$$

We got this line by noting that $d(di\arg f) = 2\pi i\delta(f)$. Notice that

$$4\pi i\mathrm{Alt}_5\left\{\log|\Delta(0,2)|d\log|\Delta(1,4)|\log|\Delta(1,4)|\right\} = 0$$

since the expression is unchanged under the transposition $\{0,2\}$, we see that (28) equals to

$$-4\pi i\cdot\mathrm{Alt}_5\left(\log|\Delta(0,1)|d\log|\Delta(2,4)|\log|\Delta(1,4)|\right)$$

by transposition $\{1,2\}$ followed by $\{2,4\}$. This finishes the proof of the proposition. □

To calculate $L^{\mathrm{G}}_{1;3}(l_0,\ldots,l_4)$ we need

PROPOSITION 5.2. $\mathrm{Alt}_5\left(\log|\Delta(1,4)|\alpha(r(l_0,l_1,l_2,l_4))\right)$ *is equal to*

$$4d\mathrm{Alt}_5\left\{\log|\Delta(2,4)|\log|\Delta(1,4)|\log|\Delta(0,2)|\right\} +$$

$$+ 12\mathrm{Alt}_5\left\{\log|\Delta(1,4)|\log|\Delta(0,1)|d\log|\Delta(2,4)|\right\}.$$

*Proof.* Here is the algebraic reason behind this lemma. There is the following exact sequence of $\mathbb{Q}$-vector spaces

$$F^*_{\mathbb{Q}}\otimes\Lambda^2F^*_{\mathbb{Q}}\xrightarrow{\kappa_1}S^2F^*_{\mathbb{Q}}\otimes F^*_{\mathbb{Q}}\xrightarrow{\kappa_2}S^3F^*_{\mathbb{Q}},$$

$$\kappa_1: a\otimes b\wedge c\longmapsto a\cdot b\otimes c - a\cdot c\otimes b, \qquad \kappa_2: a\cdot b\otimes c\longmapsto a\cdot b\cdot c.$$

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It is a special case of the Koszul complex. The map $\kappa_2$ admits a natural splitting

$$\kappa_2' : a \cdot b \cdot c \longmapsto \tfrac{1}{3}\Big(a \cdot b \otimes c + a \cdot c \otimes b + b \cdot c \otimes a\Big).$$

If $F = \mathbb{C}(X)$ then there is a map

$$S^2F^* \otimes F^* \longrightarrow \mathcal{A}^1(X); \qquad f_1 \cdot f_2 \otimes f_3 \longmapsto \log|f_1| \log|f_2| d \log|f_3|.$$

Now the proposition is an immediate corollary of the following lemma. □

LEMMA 5.3.

$$\begin{aligned} &-\kappa_1 \mathrm{Alt}_5\Big\{\Delta(1,4) \otimes (1 - r(l_0, l_1, l_2, l_4)) \wedge r(l_0, l_1, l_2, l_4)\Big\} \\ &\quad = 12\kappa_2'\Big(\mathrm{Alt}_5\Big\{\Delta(2,4) \cdot \Delta(1,4) \cdot \Delta(0,2)\Big\}\Big) + \\ &\qquad + 12\mathrm{Alt}_5\Big\{\Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(2,4)\Big\}. \end{aligned} \tag{29}$$

*Proof.* Let us show that that (29) equals to

$$\begin{aligned} &4\mathrm{Alt}_5\Big\{\Delta(1,4) \cdot \Delta(0,2) \otimes \Delta(0,1) - \Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(0,2) + \\ &\qquad + \Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(1,2)\Big\} \end{aligned} \tag{30}$$

Indeed, using (18) we get

$$(1 - r(l_0, l_1, l_2, l_4)) \wedge r(l_0, l_1, l_2, l_4) = \tfrac{1}{2}\mathrm{Alt}_{(l_0, l_1, l_2, l_4)}\Big\{\Delta(0,1) \wedge \Delta(0,2)\Big\}$$

Using this we write (29) as a sum of the following 12 terms:

$$\begin{aligned} \mathrm{Alt}_5\Big(&-\Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(0,2) + \Delta(1,4) \cdot \Delta(0,2) \otimes \Delta(0,1) + \\ &+ \Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(1,2) - \Delta(1,4) \cdot \Delta(1,2) \otimes \Delta(0,1) - \\ &- \Delta(1,4) \cdot \Delta(0,2) \otimes \Delta(1,2) + \Delta(1,4) \cdot \Delta(1,2) \otimes \Delta(0,2) - \\ &- \Delta(1,4) \cdot \Delta(0,2) \otimes \Delta(0,4) + \Delta(1,4) \cdot \Delta(0,4) \otimes \Delta(0,2) + \\ &+ \Delta(1,4) \cdot \Delta(0,2) \otimes \Delta(2,4) - \Delta(1,4) \cdot \Delta(2,4) \otimes \Delta(0,2) - \\ &- \Delta(1,4) \cdot \Delta(0,4) \otimes \Delta(2,4) + \Delta(1,4) \cdot \Delta(2,4) \otimes \Delta(0,4)\Big) \end{aligned}$$

Notice that *a priori* (29) is a sum of 24 terms corresponding to the 24 terms in (18). However 12 of them disappear after the alternation. For instance, $\mathrm{Alt}_5(\Delta(1,4) \cdot \Delta(0,1) \otimes \Delta(0,4)) = 0$ since the involution $\{2,3\}$ does not change the expression.

Computing the sign of the appropriate permutation, we see that this sum is equal to (30). Indeed, the terms 2, 5, 7 and 9 provide the first summand, the terms 1, 6, 8, 10 the second summand, and the rest the third summand.

The third term in (30) vanishes by using the involution $\{0, 4\}$. This involution also bring the first summand in (30) into the following form $4\mathrm{Alt}_5\big\{\Delta(1,4)\cdot\Delta(0,1)\otimes\Delta(2,4)\big\}$.

The second term in (30) contributes

$$8\mathrm{Alt}_5\Big\{\Delta(1,4)\cdot\Delta(0,1)\otimes\Delta(2,4)\Big\}+ \\ +12\kappa_2'\Big(\mathrm{Alt}_5\Big\{\Delta(2,4)\cdot\Delta(1,4)\cdot\Delta(0,2)\Big\}\Big).$$

The lemma, and hence Proposition 5.2, are proved. □

Notice that $\{r(l_1, l_2, l_3, l_4)\}_2$ is skewsymmetric with respect to the permutations of $l_i$'s. So applying involution $\{1, 3\}$ to (19) one gets

$$\varphi_4(2)(l_0,\dots,l_4)=\tfrac{1}{12}\mathrm{Alt}_5\Big\{\{r(l_0,l_1,l_2,l_4)\}_2\otimes\Delta(l_1,l_4)\Big\}.$$

Therefore using Propositions 5.1 and the formula for $r_3(2)$ at the end of the Section 4.4, we get

$$\begin{aligned}&\mathcal{L}^{\mathrm{G}}_{1;3}(l_0,\dots,l_4)-L^{\mathrm{G}}_{1;3}(l_0,\dots,l_4)\\ &\quad=\mathcal{L}^{\mathrm{G}}_{1;3}(l_0,\dots,l_4)-r_3(2)\circ\varphi_4(2)(l_0,\dots,l_4)\\ &\quad=\tfrac{1}{9}d\Big(\mathrm{Alt}_5\Big\{\log|\Delta(2,4)|\log|\Delta(1,4)|\log|\Delta(0,2)|\Big\}\Big).\end{aligned}\tag{31}$$

Thus using this and formula (2) in the case $k=0$, $n=3$ we conclude that

$$\mathcal{L}^{\mathrm{G}}_{0;3}(l_0,\dots,l_5)-L^{\mathrm{G}}_{0;3}(l_0,\dots,l_5)+ \\ +\tfrac{1}{9}\mathrm{Alt}_6\Big\{\log|\Delta(5,2,4)|\log|\Delta(5,1,4)|\log|\Delta(5,0,2)|\Big\}$$

is a constant (notice the change of the sign before 1/9). Since it is skewsymmetric with respect to the permutations of the vectors $l_0$, ..., $l_5$, it must be zero. Finally, notice that

$$\begin{aligned}&\mathrm{Alt}_6\Big\{\log|\Delta(5,2,4)|\log|\Delta(5,1,4)|\log|\Delta(5,0,2)|\Big\}\\ &\quad=\mathrm{Alt}_6\Big\{\log|\Delta(0,1,2)|\log|\Delta(1,2,3)|\log|\Delta(2,3,4)|\Big\}\end{aligned}$$

Theorems 1.3 and 1.5 are proved.

## 6. Appendix: Formulas for the Grassmannian *n*-Logarithm Function

To simplify the Grassmannian $n$-logarithm we need

LEMMA 6.1. *Let $X$ be an $n$-dimensional complex manifold. Let $f_1,\dots,f_{2n}$ be any $2n$ rational functions on $X$. Then*

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(i) *For every* $0 \leqslant j \leqslant n-1$

$$\mathrm{Alt}_{2n}\left\{\bigwedge_{k=1}^{2j+1} d\log|f_j| \bigwedge_{k=2j+2}^{2n} d\arg f_k\right\} = 0. \tag{32}$$

(ii) *For every* $0 \leqslant j \leqslant n-1$

$$\mathrm{Alt}_{2n}\left\{\bigwedge_{k=1}^{2j} d\log|f_j| \bigwedge_{k=2j+1}^{2n} d\arg f_k\right\} = b_{j,n} d\log|f_1| \wedge \cdots \wedge d\log|f_{2n}|, \tag{33}$$

*where* $b_{j,n} = (2n)!\binom{n}{j}\Big/\binom{2n}{2j}$.

*Proof.* (i) Because $\dim X = n$, for any $0 \leqslant i \leqslant n-1$ one has

$$d\log|f_1| \wedge \cdots \wedge d\log|f_i| \wedge d\log f_{i+1} \wedge \cdots \wedge d\log f_{2n} = 0. \tag{34}$$

Denote by $x_j$ the left side of (32) multiplied by $\sqrt{-1}^{2n-2j-1}$. Taking the imaginary part of (34) and alternating $f_1, \ldots, f_{2n}$ we get

$$\sum_{j}^{n-1} \binom{2n-i}{2j+1-i} x_j = 0, \tag{35}$$

where the sum is over $j$ such that $2j+1 \geqslant i$. Indeed, denote the expression inside of Alt in (32) by $T_j$. An alternation of (34) contributes to $T_j$ if and only if $2j+1 \geqslant i$, so that we can specify $2j+1-i$ terms from $f_{i+1}, \ldots, f_{2n}$ and make them contribute the log part of $T_j$, which, together with $\log|f_1|, \ldots, \log|f_i|$, contribute the log part in $T_j$.

Let $s_i$ be the left-hand side of (35) considered for arbitrary $i$. Let us multiply it by $\binom{2n}{i}t^i$ and take a sum over $0 \leqslant i \leqslant 2n-1$. Since $s_i = 0$ for any $0 \leqslant i \leqslant n-1$ we have $\sum_{i=0}^{2n-1} s_i \binom{2n}{i} t^i = t^n A(t)$ for some polynomial $A(t)$ in $t$ whose coefficients are $\mathbb{Q}$-linear combinations of $x_i$'s. Using the identity

$$\binom{2n-i}{p-i}\binom{2n}{i} = \binom{2n}{p}\binom{p}{i}. \tag{36}$$

we have

$$\begin{aligned} t^n A(t) &= \sum_{i=0}^{2n-1} \sum_{j}^{n-1} \binom{2n-i}{2j+1-i}\binom{2n}{i} x_j t^i \\ &= \sum_{j=0}^{n-1} x_j \binom{2n}{2j+1} \sum_{i=0}^{2j+1} \binom{2j+1}{i} t^i = \sum_{j=0}^{n-1} x_j \binom{2n}{2j+1} (1+t)^{2j+1}. \end{aligned}$$

Replacing $t$ by $t-1$ we get

$$\sum_{j=0}^{n-1} x_j \binom{2n}{2j+1} t^{2j+1} = (t-1)^n A(t-1).$$

The left-hand side is an odd polynomial of degree $2n-1$; it has a zero of order $n$ at $t=1$; therefore it must have a zero of order $n$ at $t=-1$, so its degree is at least $2n$. Thus it is a zero polynomial.

(ii) Denote the left side of (33) by $y_j$. Taking the real part of Equation (34) and alternating we get

$$\sum_{j=0}^{n} \binom{2n-i}{2j-i} (-1)^{n-j} y_j = 0, \quad 0 \leqslant i \leqslant n-1. \tag{37}$$

By definition it is clear that $b_{n,n} = (2n)!$. Multiplying (37) by $\binom{2n}{i} t^i$ and taking sum over $0 \leqslant i \leqslant 2n$ we have

$$\sum_{i=0}^{2n} \sum_{j \geqslant \lfloor i/2 \rfloor}^{n} (-1)^{n-j} y_j \binom{2n-i}{2j-i} \binom{2n}{i} t^i = t^n B(t)$$

for some polynomial $B(t)$. Using combinatorial identity (36) we write it as

$$\sum_{j=0}^{n} (-1)^{n-j} y_j \binom{2n}{2j} \sum_{i=0}^{2j} \binom{2j}{i} t^i = \sum_{j=0}^{n} (-1)^{n-j} y_j \binom{2n}{2j} (t+1)^{2j} = t^n B(t).$$

Changing $t$ to $t-1$ and noticing that the left-hand side is an even polynomial we get

$$\sum_{j=0}^{n} (-1)^{n-j} y_j \binom{2n}{2j} t^{2j} = (t-1)^n B(t-1) = (t^2-1)^n C(t).$$

Therefore $C(t) = y_n$ is a constant. Thus we finally have $y_j / y_n = b_{j,n}/(2n)! = \binom{n}{j} / \binom{2n}{2j}$ $(0 \leqslant j \leqslant n)$. The lemma is proved. □

Recall that the Grassmannian $n$-logarithm is defined by

$$\mathcal{L}_n^{\mathrm{G}}(l_0, \ldots, l_{2n-1}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(f_1, \ldots, f_{2n-1})$$

where $f_i = l_i / l_0$.

PROPOSITION 6.2. *The Grassmannian n-logarithm* $\mathcal{L}_n^{\mathrm{G}}(l_0, \ldots, l_{2n-1})$ *can be expressed by*

$$-\frac{(-2\pi i)^{1-n}}{(2n-1)!} \mathrm{Alt}_{2n-1} \int_{\mathbb{CP}^{n-1}} \mathrm{Re}\left\{ \log|f_1| \bigwedge_{j=2}^{n} d\log(f_j) \bigwedge_{j=n+1}^{2n-1} d\,\overline{\log(f_j)} \right\}$$

*or*

$$
\begin{aligned}
&-\frac{(-4)^{n-1}((n-1)!)^2}{(2\pi i)^{n-1}(2n-2)!}\int_{\mathbb{CP}^{n-1}}\log|f_1|\bigwedge_{j=2}^{2n-1}d\log|f_j|\\
&=\frac{(-4)^{n-1}((n-1)!)^2}{(2\pi i)^{n-1}(2n-2)!(2n-1)!}\mathrm{Alt}_{2n}\left\{\int_{\mathbb{CP}^{n-1}}\log|l_0|\bigwedge_{j=1}^{2n-2}d\log|l_j|\right\}.
\end{aligned}
\tag{38}
$$

*Proof.* The first expression follows directly from Corollary 2.2. Now we prove the second. By definition (6)

$$
\begin{aligned}
r_{2n-1}(f_1,\dots,f_{2n-1}) = -\sum_{k=0}^{n=1}(-1)^{n-k-1}c_{k,2n-1}&\\
\mathrm{Alt}_{2n-1}\left\{\log|f_1|\bigwedge_{j=2}^{2k+1}d\log|f_j|\bigwedge_{j=2k+2}^{2n-1}d\arg f_j\right\}&
\end{aligned}
\tag{39}
$$

where $c_{k,2n-1}=\binom{2n-1}{2k+1}/(2n-1)!$. Now let us look at the terms in the expansion of (39) which correspond to the term

$$
\log|f_1|d\log|f_2|\wedge\cdots\wedge d\log|f_{2n-1}|. \tag{40}
$$

By Lemma 6.1(ii), each term inside the sum of (39) with $\log|f_1|$ as the first factor contributes to (40) as many as $(-1)^{n-k}c_{k,2n-1}b_{k,n-1}$ times. So the total contribution to (40) from (39) is

$$
\begin{aligned}
d_n &= \sum_{k=0}^{n-1}(-1)^{n-k}c_{k,2n-1}b_{k,n-1} = \sum_{k=0}^{n-1}\frac{(-1)^{n-k}}{2k+1}\binom{n-1}{k}\\
&= -\int_0^1(t^2-1)^{n-1}\,dt = (-1)^n\frac{\Gamma(n)\Gamma(\frac{1}{2})}{2\cdot\Gamma(n+\frac{1}{2})}\\
&= (-1)^n\frac{2^{2n-2}((n-1)!)^2}{(2n-1)!}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{L}_n^{\mathrm{G}}(l_0,\dots,l_{2n-1})\\
&\quad=(2\pi i)^{1-n}\int_{\mathbb{CP}^{n-1}}r_{2n-1}(f_1,\dots,f_{2n-1})\\
&\quad=-\frac{(-4)^{n-1}((n-1)!)^2}{(2\pi i)^{n-1}(2n-1)!}\sum_{i=1}^{2n-1}\sigma_{1i}\left(\int_{\mathbb{CP}^{n-1}}\log|f_1|\bigwedge_{j=2}^{2n-1}d\log|f_j|\right)
\end{aligned}
\tag{41}
$$

where $\sigma_{11}=\mathrm{id}$ and for $i\neq 1$

$$
\sigma_{1i}F(f_1,\dots,f_{2n-1}) = -F(f_i,f_2,\dots,f_{i-1},f_1,f_{i+1},\dots,f_{2n-2}).
$$

Now we observe that for any $2 \leqslant i \leqslant 2n-1$ we have

$$\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j| - \sigma_{1i}\left(\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|\right)$$
$$= (-1)^i \int_{\mathbb{CP}^{n-1}} d(\log|f_1|\log|f_i|) \wedge d\log|f_2| \wedge \cdots d\widehat{\log|f_i|}\cdots \wedge d\log|f_{2n-2}| = 0.$$

Therefore

$$\sum_{i=1}^{2n-1} \sigma_{1i}\left(\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|\right)$$
$$= (2n-1)\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|$$

which together with equation (41) yields the second equality. To prove the last equality in our proposition it suffices to observe that

$$(2n-1)!\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|$$
$$= \mathrm{Alt}_{2n-1}\int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|$$
$$= -\mathrm{Alt}_{2n}\int_{\mathbb{CP}^{n-1}} \log|l_0| \bigwedge_{j=1}^{2n-2} d\log|l_j|. \qquad \square$$

*Remark* 6.3. This result improves Proposition 3.2 of [G4].

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