The Variety of Two-dimensional Algebras
Over an Algebraically Closed Field

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Abstract. The work is devoted to the variety of two-dimensional algebras over algebraically closed fields. First we classify such algebras modulo isomorphism. Then we describe the degenerations and the closures of certain algebra series in the variety of two-dimensional algebras. Finally, we apply our results to obtain analogous descriptions for the subvarieties of flexible and bicommutative algebras. In particular, we describe rigid algebras and irreducible components for these subvarieties.

1 Introduction

In this paper, an algebra is simply a vector space over a field with a bilinear binary operation that does not need to be associative. Algebras of a fixed dimension form a variety with a natural action of a general linear group. Orbits under this action correspond to isomorphism classes of algebras. There are many classifications up to isomorphism for varieties of algebras of some fixed dimension satisfying some polynomial identities. For example, there exist such classifications of three-dimensional Novikov algebras [2], four-dimensional Leibniz algebras [9], six-dimensional Lie algebras [30] and many others.

In this paper we classify all two-dimensional algebras over an algebraically closed field up to isomorphism. It is not the first work devoted to this problem [1, 14, 28], but all of them are not convenient for our main goal: the geometric description of the algebraic variety of two-dimensional algebras. One of the advantages of our paper is that our approach deals uniformly with all possible characteristics while the authors of [1] do not consider characteristics 2 and 3 and the authors of [14] consider only the two-elements field in characteristic 2. The authors of [1], in fact, do not give an explicit classification of two-dimensional algebras up to isomorphism because they have other purposes. They describe the moduli space by proving that two-dimensional algebras can be divided into parts that can be naturally included into projective spaces of different dimensions. The authors claim that the classification up to isomorphism is easy and could be extracted from their proofs. The classification is really not very difficult, and we believe that one can extract it after reading the paper, taking parts of the classification from different places and carefully taking into account all the details. For us it was easier to produce this classification from scratch. Goze and Remm [14] produced a full classification. One of the problems is that this classification is spread
throughout the whole paper and is mixed with other formulas. It is tedious to collect all the parts of the classification from in one place and find all the additional conditions for these parts. At the same time, there are some inaccuracies. For example, the series $\mu_{10}$ parametrized by two scalars must be divided into two series parametrized by one scalar, the series $\mu_{11}$ admits nontrivial isomorphisms, and in the case of a commutative two-dimensional algebra with one idempotent $e$, it may be impossible to find $f$ linearly independent with $e$ such that $f^2$ and $e$ are linearly dependent. While Petersson [28] gave the full classification of two-dimensional algebras over any field, unfortunately, the answer is not given in terms of multiplication tables. The translation of this answer to the language of multiplication tables as well as its direct usage for the description of orbit closures is very difficult, and it seems to be easier to produce a new appropriate classification. Also, the consideration of arbitrary fields complicates the result and the extraction of the answer for an algebraically closed field becomes tedious. For these reasons, we give a classification that is valid over an algebraically closed field of arbitrary characteristic in Section 3. In the same section we also describe the automorphism groups for all algebras under consideration.

In the main part of our paper we develop the geometry of the variety of two-dimensional algebras. Namely, we describe the closures of orbits of some sets with respect to the Zariski topology. First, we describe all possible degenerations, i.e., closures of orbits of one point sets. Degenerations are an interesting subject that was studied in various papers [3–7, 11–13, 15–18, 20, 21, 25–27, 29]. One of the problems in this direction is to describe all degenerations in a variety of algebras of some fixed dimension satisfying some set of identities. For example, this problem was solved for two-dimensional pre-Lie algebras in [3], for three-dimensional Novikov algebras in [4], for four-dimensional Lie algebras in [7], for four-dimensional Zinbiel and nilpotent Leibniz algebras in [21], for nilpotent five- and six-dimensional Lie algebras in [15, 29], and for nilpotent five- and six-dimensional Malcev algebras in [20]. As an application of our results, one can easily recover the results of [3].

Another interesting notion concerning degenerations is the so-called level of an algebra defined in the end of Section 5. The algebras of the first level and the associative Lie and Jordan algebras of the second level are classified in [23, 24]. Gorbatsevich [11–13] defined the notion of an infinite level and described all anticommutative algebras that have an infinite level not greater than three. This notion is much easier in the sense that the infinite level of an algebra can be easily expressed in terms of the usual level. Algebras of low dimension play a special role in problems of such type, because they have small levels. The complete description of degenerations obtained by Gorbatsevich allows computing the level for all two-dimensional algebras.

Our next result is the description of orbit closures of certain series that appear in the classification up to isomorphism. Let $T$ be some subvariety of the variety of an $n$-dimensional algebra closed under the action of the general linear group. An $n$-dimensional algebra from $T$ is called rigid if its orbit is an open subset of $T$. Another important characteristic of a variety is its partition into irreducible components. The notion of a rigid algebra is closely related to this characteristic, because orbit closures of such algebras form irreducible components. For example, irreducible components and rigid algebras were classified for low dimensional associative algebras [26, 27]) and
for Jordan algebras [19]. Since the variety of two-dimensional algebras is simply $k^8$, it is clear that there is only one irreducible component and there are no rigid algebras in it. Thus, this problem is not relevant for the variety of all two-dimensional algebras itself. Nevertheless, it is relevant for subvarieties. In the last part we apply our results about the variety of all two-dimensional algebras to their subvarieties consisting of flexible and bicommutative algebras. We describe all degenerations and closures of orbits in these varieties. In reducible components and rigid algebras. Our results allow us to get such descriptions and classifications for varieties of two-dimensional algebras defined by any identities without any problems.

Let us give a resume of our motivations. The problems considered in this paper are classical and their solutions are interesting by themselves. In general the classification of all $n$-dimensional algebras is a wild problem and it is interesting to get the solution in particular cases where it is still possible. Our main motivation was the classification of all the algebras of the second level that we produced in [22] using the results of this paper. In fact, there are reasons to guess that our classification will also allow classifying algebras of the third and the fourth levels. Thus, our results are important for the classification of algebras of small levels and constitute a necessary part of it. Another application that we have in mind is the geometric description of subvarieties of the variety of two-dimensional algebras. There are some works, e.g., [3], devoted to this problem, and our work gives a powerful tool to solve it. Our results can be applied whenever the natural action of $\text{GL}(k^2)$ on $(k^2)^* \otimes (k^2)^* \otimes k^2$ appears, and we expect that they will have other applications, for example, in the theory of algebras with polynomial identities or in the geometric representation theory. Even when one deals with $n$-dimensional algebras for $n > 2$, it may be useful to consider two-dimensional subalgebras, and our results could be applied in this case. For example, the classification of $n$-dimensional algebras with an $(n-2)$-dimensional annihilator is fulfilled using our classification [8].

2 Definitions and Notation

Throughout the paper we fix an algebraically closed field $k$, a two-dimensional $k$-linear vector space $V$ and a basis $\{e_1, e_2\}$ of $V$. All spaces in this paper are considered over $k$, and we write simply dim, $\text{Hom}$ and $\otimes$ instead of $\text{dim}_k$, $\text{Hom}_k$ and $\otimes_k$.

An algebra $A$ is a set with a structure of a vector space and a binary operation that induces a bilinear map from $A \times A$ to $A$.

Since this paper is devoted to two-dimensional algebras, we give all definitions and notation only for this case, though everything in this section can be rewritten for any dimension.

The set $A_2 := \text{Hom}(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension 8. This space has a structure of the affine variety $k^8$. Indeed, any $\mu \in A_2$ is determined by eight structure constants $c^i_{jk} \in k$ ($i, j, k = 1, 2$) such that $\mu(e_i \otimes e_j) = c^i_{jk}e_1 + c^i_{jk}e_2$. A subset of $A_2$ is Zariski-closed if it can be defined by a set of polynomial equations in the variables $c^i_{jk}$.

The general linear group $\text{GL}(V)$ acts on $A_2$ by conjugation: for $x, y \in V$, $\mu \in A_2$ and $g \in \text{GL}(V)$,

$$(g \ast \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y).$$
Thus, $A_2$ is decomposed into $GL(V)$-orbits that correspond to the isomorphism classes of two-dimensional algebras. The classification of two-dimensional algebras up to isomorphism is equivalent to the classification of $GL(V)$-orbits.

Let $O(\mu)$ denote the orbit of $\mu \in A_2$ under the action of $GL(V)$ and let $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$. Let $A$ and $B$ be two two-dimensional algebras and $\mu, \lambda \in A_2$ represent $A$ and $B$, respectively. We say that $A$ degenerates to $B$ and write $A \to B$ if $\lambda \in O(\mu)$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of $\mu$ and $\lambda$. If $A \not\to B$, then the assertion $A \to B$ is called a proper degeneration. We write $A \not\to B$ if $\lambda \not\in O(\mu)$. Now let $A(\ast) := \{A(\alpha)\}_{\alpha \in I}$ be a set of two-dimensional algebras and $\mu_a \in A_2$ represent $A(\alpha)$ for $\alpha \in I$. If $\lambda \in \{O(\mu_a)\}_{\alpha \in I}$, then we write $A(\ast) \to B$ and say that $A(\ast)$ degenerates to $B$. In the opposite case we write $A(\ast) \not\to B$.

Let $A(\ast)$, $B$, $\mu_a (\alpha \in I)$, and $\lambda$ be as above. Let $c_{ij}^k (i, j, k = 1, 2)$ be the structure constants of $\lambda$ in the basis $e_1, e_2$. If we construct maps $a_i^k : k^* \to k$ $(i, j, k = 1, 2)$ and $f : k^* \to I$ such that $a_i^1(t) e_1 + a_i^2(t) e_2$ and $a_i^k(t) e_1 + a_i^k(t) e_2$ form a basis of $V$ for any $t \in k^*$, and the structure constants of $\mu(t)$ in this basis are polynomials $c_{ij}^k(t) \in k[t]$ such that $c_{ij}^k(0) = c_{ij}^k$, then $A(\ast) \to B$. Indeed, if there is some closed subset $\mathcal{R}$ containing $O(\mu_a)$ for all $\alpha \in I$, then it contains, in particular, $O(\mu(t))$ for all $t \in k^*$, and hence the element $\lambda_i \in A_2$ with structure constants $c_{ij}^k(t)$ belongs to $\mathcal{R}$ for any $t \in k^*$. Note that the assertion $\lambda_i \in \mathcal{R}$ is equivalent to the annihilation of some set polynomials in one variable in the point $t$. But if this set of polynomials vanishes for all $t \in k^*$, then each of these polynomials has infinitely many roots, and hence it equals zero. Thus, $t = 0$ annihilates all the required polynomials, too, i.e., $\lambda = \lambda_0 \in \mathcal{R}$. We will call $(a_i^1(t) e_1 + a_i^2(t) e_2, a_i^3(t) e_1 + a_i^3(t) e_2)$ and $f(t)$ a parametrized basis and a parametrized index for $A(\ast) \to B$, respectively. The case of degeneration between two algebras corresponds to the case $|I| = 1$. In this case we need only a parametrized basis, because $f(t)$ is the unique element of $I$ for any $t \in k^*$.

We follow [29] for proving non-degenerations. Let $Q$ be a set of polynomial equations in the variables $x_{i,j}^k$ $(i, j, k = 1, 2)$. Suppose that $Q$ satisfies the following property: if $x_{i,j}^k = c_{ij}^k$ is a solution to all equations in $Q$, then also $x_{1,2}^k = c_{1,2}^k$ is a solution to all equations in $Q$ in the following cases:

1. there are $a_1, a_2 \in k^*$ such that $c_{ij}^k = \frac{a_i a_j}{a_k} c_{ij}^k$,
2. there is $\alpha \in k$ such that
   \begin{align*}
   &c_{11}^1 = c_{11}^1 + \alpha (c_{12}^1 + c_{21}^1) + \alpha^2 c_{12}^1, \\
   &c_{11}^2 = c_{12}^1 + \alpha c_{12}^2, \\
   &c_{12}^2 = c_{12}^2 + \alpha c_{12}^2, \\
   &c_{22}^2 = c_{22}^2, \\
   &c_{11}^2 = c_{11}^2 + \alpha (c_{22}^2 + c_{22}^2 - c_{11}^2) + \alpha^2 (c_{22}^2 - c_{12}^2 - c_{21}^2) - \alpha^3 c_{22}^1, \\
   &c_{21}^2 = c_{21}^2 + \alpha (c_{22}^2 - c_{21}^2) - \alpha^2 c_{22}^1, \\
   &c_{12}^2 = c_{12}^2 + \alpha (c_{22}^2 - c_{12}^1) - \alpha^2 c_{22}^1, \\
   &c_{22}^2 = c_{22}^2 - \alpha c_{22}^1.
   \end{align*}
Let $\mathcal{R} \subset \mathcal{A}_2$ be a set of all algebra structures whose structure constants satisfy all equations in $Q$. We will call such a set $\mathcal{R}$ a closed upper invariant set. Let $\{\mathcal{A}(\alpha)\}_{\alpha \in I}$ be a set of two-dimensional algebras such that $\mathcal{A}(\alpha)$ can be represented by a structure from $\mathcal{R}$ for any $\alpha \in I$. Let $B$ be a two-dimensional algebra represented by the structure $\lambda \in \mathcal{A}_2$. If $O(\lambda) \cap \mathcal{R} = \emptyset$, then $\mathcal{A}(\ast) \not\rightarrow B$. In this case we call $\mathcal{R}$ a separating set for $\mathcal{A}(\ast) \not\rightarrow B$.

Let us recall two more tools for proving degenerations and non-degenerations. First, if $A \rightarrow B$, then $\dim \text{Aut}(A) > \dim \text{Aut}(B)$. Note that if $\mathcal{A}(\ast) \rightarrow B$, then either $\dim \text{Aut}(A(\ast)) = \dim \text{Aut}(B)$ for infinitely many $\alpha \in I$ or $\dim \text{Aut}(A(\ast)) < \dim \text{Aut}(B)$ for some $\alpha \in I$, but it is possible that $\dim \text{Aut}(A(\ast)) \geq \dim \text{Aut}(B)$ for all $\alpha \in I$. Note also that $\dim \text{Aut}(A) = \dim \text{Der}(A)$. Secondly, if $A \rightarrow C$ and $C \rightarrow B$, then $A \rightarrow B$. If there is no $C$ such that $A \rightarrow C$ and $C \rightarrow B$ are proper degenerations, then the assertion $A \rightarrow B$ is called a primary degeneration. If there are no $C$ and $D$ such that $C \rightarrow A$, $B \rightarrow D$, $C \not\rightarrow D$, and one of the assertions $C \rightarrow A$ and $B \rightarrow D$ is a proper degeneration, then the assertion $A \not\rightarrow B$ is called a primary non-degeneration. It suffices to prove only primary degenerations and non-degenerations to describe degenerations in the variety under consideration. Note also that any algebra degenerates to the algebra with zero multiplication.

3 Algebraic Classification

The first of our aims is to classify all two-dimensional algebras over $\mathbf{k}$ modulo isomorphism. Our classification is based on the following lemma.

**Lemma 3.1** Let $A$ be a two-dimensional algebra. Then there exists a nonzero element $x \in A$ such that $x$ and $x^2$ are linearly dependent.

**Proof** The required assertion is equivalent to the existence of a one-dimensional subalgebra in $A$. Then the lemma follows from the discussion immediately following [1, Proposition 1].

Note that if $x \in A$ and $x^2$ are linearly dependent, then either $x^2 = 0$ or $x = \alpha e$ for some $\alpha \in \mathbf{k}$ and some $e \in A$ such that $e^2 = e$. If $x^2 = 0$, then $x$ is called a 2-nil element. An element $e$ such that $e^2 = e$ is called an idempotent.

**Corollary 3.2** Any two-dimensional $\mathbf{k}$-algebra belongs to one of the following disjoint classes.

(A) Algebras that do not have nonzero idempotents and have a unique one-dimensional subspace of 2-nil elements.

(B) Algebras that do not have nonzero idempotents and have two linearly independent 2-nil elements.

(C) Algebras that have a unique nonzero idempotent and do not have nonzero 2-nil elements.

(D) Algebras that have a unique nonzero idempotent and a nonzero 2-nil element.

(E) Algebras that have two different nonzero idempotents.

**Proof** The fact that the classes are disjoint is obvious. The fact that any two-dimensional algebra belongs to one of the classes follows easily from Lemma 3.1 and the remark after it.
To give the classification of two-dimensional algebras we need to introduction some notation. Let us consider the action of the cyclic group $C_2 = \langle \rho \mid \rho^2 \rangle$ on $k$ defined by the equality $\rho \alpha = -\alpha$ for $\alpha \in k$. Let us fix some set of representatives of orbits under this action and denote it by $k_{0}$. For example, if $k = \mathbb{C}$, then one can take $C_{20} = \{ \alpha \in \mathbb{C} \mid \text{Re}(\alpha) > 0 \} \cup \{ \alpha \in \mathbb{C} \mid \text{Re}(\alpha) = 0, \text{Im}(\alpha) \geq 0 \}$. Let us also consider the action of $C_2$ on $k^2$ defined by the equality $\rho(\alpha, \beta) = (1 - \alpha + \beta, \beta)$ for $(\alpha, \beta) \in k^2$. Let us fix some set of representatives of orbits under this action and denote it by $\mathcal{U}$. Let us also define $\mathcal{I} = \{(\alpha, \beta) \in k^2 \mid \alpha + \beta = 1 \}$.

Given $(\alpha, \beta, \gamma, \delta) \in k^4$, we define $D(\alpha, \beta, \gamma, \delta) = (\alpha + \gamma)(\beta + \delta) - 1$. We define

$$e_1(\alpha, \beta, \gamma, \delta) = (\beta, \delta), \quad e_2(\alpha, \beta, \gamma, \delta) = (\gamma, \alpha),$$

$$e_3(\alpha, \beta, \gamma, \delta) = \left( \frac{\beta \gamma - (\alpha - 1)(\delta - 1)}{D(\alpha, \beta, \gamma, \delta)} - 1, \frac{\alpha \delta - (\beta - 1)(\gamma - 1)}{D(\alpha, \beta, \gamma, \delta)} - 1 \right)$$
for $(\alpha, \beta, \gamma, \delta)$ such that $D(\alpha, \beta, \gamma, \delta) \neq 0$. Let us consider the set

$$X = \left\{ (e_1(\Gamma), e_2(\Gamma), e_3(\Gamma)) \mid \Gamma \in k^4, D(\Gamma) \neq 0, e_1(\Gamma), e_2(\Gamma), e_3(\Gamma) \notin \mathcal{I} \right\} \subset (k^2)^3.$$

One can show that the symmetric group $S_3$ acts on $X$ by the equality

$$e^\sigma(\Gamma) = \left\{ (e_1(\Gamma(\sigma), e_2(\Gamma(\sigma), e_3(\Gamma(\sigma))) \right\}$$
for $\sigma \in S_3$.

Indeed, suppose that $(e_1(\Gamma), e_2(\Gamma), e_3(\Gamma)) \in X$ for some $\Gamma = (\alpha, \beta, \gamma, \delta)$. We need to show that $e^\sigma(\Gamma) \in X$ for any $\sigma \in S_3$. We will check this for $\sigma$ interchanging 1 and 3; the other verifications are analogous. First, $e_1(\Gamma) \in \mathcal{I}$ is equivalent to the equality $\alpha + \beta + \delta + \gamma - 2 = D(\Gamma)$, which can be rewritten in the form $(\alpha + \gamma - 1)(\beta + \delta - 1) = 0$. It is clear that the last equality is not valid. Let us introduce

$$\Gamma' = \left( \alpha, \beta, \gamma - 1, \frac{\beta \gamma - (\alpha - 1)(\delta - 1)}{D(\Gamma)}, y, \frac{\alpha \delta - (\beta - 1)(y - 1)}{D(\Gamma)} \right).$$

It remains to check that $D(\Gamma') \neq 0$ and $e_3(\Gamma') = (\beta, \delta)$. The equality $D(\Gamma') = 0$ is equivalent to the equality $D(\Gamma) = (\alpha + \beta + \delta + \gamma - 2)(\alpha + \gamma) = 0$, which can be rewritten in the form $(\alpha + \gamma - 1)^2 = 0$. Hence, we get $D(\Gamma') \neq 0$. To prove that $e_3(\Gamma') = (\beta, \delta)$ we need to verify two equalities. We will consider only the first equality; the second one is analogous. Thus, it remains to show that

$$\frac{\beta \gamma - (\alpha - 1)(\delta - 1)}{D(\Gamma)} = \beta \left( \frac{\alpha + \gamma - 1}{D(\Gamma)} \right).$$

Multiplying by $D(\Gamma)$ and reducing all the equal terms, one sees that the last equality is valid. Note that there exists a set of representatives of orbits $\mathcal{V}$ under the action of $S_3$ on $X$ such that if $(e_1, e_2, e_3) \in \mathcal{V}$ and $e_1 \neq e_2$, then $e_3 \neq e_1, e_2$. Let us fix such $\mathcal{V}$ and define

$$\mathcal{V} = \left\{ \Gamma \in k^4 \mid D(\Gamma) \neq 0; e_1(\Gamma), e_2(\Gamma), e_3(\Gamma) \notin \mathcal{I}, (e_1(\Gamma), e_2(\Gamma), e_3(\Gamma)) \in \mathcal{V} \right\}.$$

For $\Gamma \in \mathcal{V}$, we also define $C(\Gamma) = \{ e_1(\Gamma), e_2(\Gamma), e_3(\Gamma) \} \subset k^2$.

Let us consider the action of the cyclic group $C_2$ on $k^* \setminus \{1\}$ defined by the equality $\rho \alpha = \alpha^{-1}$ for $\alpha \in k^* \setminus \{1\}$. Let us fix some set of representatives of orbits under this
action and denote it by \( k^*_{A_1} \). For example, if \( k = \mathbb{C} \), then one can take \( C^*_{A_1} = \{ \alpha \in \mathbb{C}^* \mid \| \alpha \| > 1 \} \cup \{ \alpha \in \mathbb{C}^* \mid \| \alpha \| = 1, 0 < \arg(\alpha) \leq \pi \} \). For \( (\alpha, \beta, \gamma) \in k^2 \times k^*_{A_1} \) we define

\[
E(\alpha, \beta, \gamma) = \left\{ \left( \frac{\beta \gamma - 1}{\gamma}, \frac{1 - \beta \gamma}{\gamma} \right) \right\} \in k^2.
\]

Let \( \mathcal{F} \subset A_2 \) be the set formed by the algebra structures on the vector space \( V \) listed in Table 1. This section is devoted to the proof of the following theorem that gives a classification of two-dimensional algebras over \( k \) up to isomorphism.

**Theorem 3.3** Any non-trivial two-dimensional \( k \)-algebra can be represented by a unique structure from \( \mathcal{F} \).

In other words, Theorem 3.3 states that \( A_2 = \bigcup_{\mu \in \mathcal{F}} O(\mu) \cup \{k^2\} \) and that, if \( \mu, \lambda \in \mathcal{F} \) are different structures, then \( O(\mu) \cap O(\lambda) = \emptyset \). Whenever an algebra named \( A \) appears in this section, we suppose that it is represented by some structure from \( A_2 \) with structure constants \( c^k_{ij} \) \((i, j, k = 1, 2)\). According to Corollary 3.2, it suffices to consider each of the classes \( A - E \) separately. It is not difficult to show that the letter in the name of an algebra from \( \mathcal{F} \) corresponds to its class in each case. This will follow also from our proofs.

**Lemma 3.4** If \( A \) belongs to the class \( A \), then it can be represented by a unique structure from the set

\[
\left\{ A_1(\alpha) \right\}_{\alpha \in k} \cup \left\{ A_2 \right\} \cup \left\{ A_3 \right\} \cup \left\{ A_4(\alpha) \right\}_{\alpha \in k^*_{A_1}}.
\]

**Proof** Let us represent the algebra \( A \) by a structure such that \( e_2 e_2 = 0 \). It is easy to see that \( A \) belongs to the class \( A \) if and only if \( x_t = e_1 + t e_2 \) and \( x_t^2 \) are linearly independent for any \( t \in k \). Since

\[
x_t^2 = \left( c_{11}^1 + (c_{12}^1 + c_{21}^1) t \right) e_1 + \left( c_{11}^2 + (c_{12}^2 + c_{21}^2) t \right) e_2,
\]

\( x_t \) and \( x_t^2 \) are linearly independent if and only if

\[
0 \neq \left| \begin{array}{cc}
c_{11}^1 + (c_{12}^1 + c_{21}^1) t & c_{11}^2 + (c_{12}^2 + c_{21}^2) t \\
1 & t
\end{array} \right| = (c_{12}^1 + c_{21}^1) t^2 + (c_{11}^1 - c_{12}^1 - c_{21}^1) t - c_{11}^2.
\]

Since by our assumption \( x_t \) and \( x_t^2 \) are linearly independent for any \( t \in k \), we have \( c_{12}^1 + c_{21}^1 = 0, c_{11}^1 = c_{12}^1 + c_{21}^1, \) and \( c_{11}^2 \neq 0 \).

Now we have four cases.

- \( c_{12}^1 = 0, c_{11}^1 \neq 0 \). Considering the basis \( \frac{c_{11}^1}{c_{11}^2}, \frac{c_{12}^1}{c_{11}^2} \) of \( V \), one can check that \( A \) can be represented by \( A_1(\frac{c_{12}^1}{c_{11}^1}) \).

- \( c_{12}^1 = 0, c_{11}^1 = -c_{21}^1 \neq 0 \). Considering the basis \( \frac{c_{11}^1}{c_{12}^1}, \frac{c_{12}^1}{c_{11}^1} \) of \( V \), one can check that \( A \) can be represented by \( A_2 \).

- \( c_{12}^1 = c_{11}^1 = c_{21}^1 = 0 \). Considering the basis \( e_1, c_{11}^2 e_2 \) of \( V \), one can check that \( A \) can be represented by \( A_3 \).
\[ c_{12} \neq 0. \text{ Let } a \in k^* \text{ be such that } c_{11}^2 c_{12} a^2 = 1 \text{ and } c_{11} a \in k_{\geq 0}. \text{ Considering the basis } \]
\[ a(1 + \frac{c_{11}^2}{c_{12}} e_2, \frac{c_{11}^2}{c_{12}}) \text{ of } V, \text{ one can check that } A \text{ can be represented by } A_4(c_{11} a). \]

It remains to prove that any two different structures from the set (3.1) represent non-isomorphic algebras. First, note that \( \dim(A_2)^2 = \dim(A_3)^2 = 1 \) while \( \dim(A_4(a))^2 = \dim(A_4(a'))^2 = 2 \) for any \( a, a' \in k \). We also have \( A_2 \neq A_3 \), because \( A_3 \) has a nonzero annihilator.

Suppose that \( A \) is represented by the structure \( A_4(a) \) for some \( a \in k \). Then there exists \( x \in A \) such that \( x^2 = 0, xA + Ax \subset \langle x \rangle \), and \( axy = (1 - a)yx \) for any \( y \in A \). Such an element does not exist in \( A_4(\beta) \) for any \( \beta \in k \) nor in \( A_4(\beta) \) for any \( \beta \in k \setminus \{ a \} \).

Suppose that \( A \) is represented by the structure \( A_4(a) \) for some \( a \in k_{\geq 0} \). Suppose that the structure constants of \( A \) in the basis \( E_1, E_2 \) equal the structure constants of \( A_4(\beta) \) for some \( \beta \in k_{\geq 0} \). Since \( E_2 E_2 = 0 \) and \( E_2 E_1 = -E_1 E_2 \), it is easy to see that \( E_2 = e_2 \) and \( E_1 = ae_1 \) for some \( a \in k^* \). Then we obtain from the equality \( E_1 E_1 = \beta E_1 E_2 \) that \( a = \pm 1 \) and \( \beta = \mp a \). Since \( \beta \in k_{\geq 0} \), we have \( \beta = a \).

\[ \text{Lemma 3.5} \quad \text{If } A \text{ belongs to the class } B, \text{ then it can be represented by a unique structure from the set} \]
\[ \{ B_1(a) \}_{a \in k} \cup \{ B_2(a) \}_{a \in k} \cup \{ B_3 \}. \]

\[ \text{Proof} \quad \text{Let us represent the algebra } A \text{ by a structure such that } e_1 e_1 = e_2 e_2 = 0. \text{ For } s, t \in k, \text{ let us define } x_{s, t} = se_1 + te_2. \text{ Suppose that there are } s, t \in k^* \text{ such that } 0 = x_{s, t}^2 = st(e_1 e_2 + e_2 e_1). \text{ Then } e_1 e_2 + e_2 e_1 = 0 \text{ and } A \text{ is anticommutative. It is easy to see that any two-dimensional anticommutative algebra either has the trivial multiplication or can be represented by } B_3 \text{ (note that by our definition } A \text{ is anticommutative if and only if } x^2 = 0 \text{ for any } x \in A). \]

Suppose now that \( x_{s, t} \neq 0 \) for any \( s, t \in k^* \). Since \( A \) does not have idempotents, \( x_{s, t} \) and \( x_{s', t'} \) are linearly independent for \( s, t, s', t' \in k^* \). It is easy to check that \( x_{s, t}^2 \) and \( x_{s', t'}^2 \) are linearly dependent for \( s, t, s', t' \in k^* \). Hence, \( c_{12}^2 + c_{21}^2 = 0 \) or \( c_{12}^2 + c_{21}^2 = 0 \). Without loss of generality we may assume that \( c_{12}^2 + c_{21}^2 = 0 \). Since \( A \) is not anticommutative, we have \( c_{12}^2 + c_{21}^2 \neq 0 \) in this case.

If \( c_{12}^2 \neq 0 \), then, considering the basis \( \frac{c_{11}^2}{c_{12}}, \frac{c_{11}^2 + c_{21}^2}{c_{12}^2 + c_{21}^2} \) of \( V \), one can check that \( A \) can be represented by \( B_1\left( \frac{c_{11}^2}{c_{12}^2 + c_{21}^2} \right) \). If \( c_{21}^2 = 0 \), then, considering the basis \( c_{11}, \frac{c_{12}^2}{c_{12}^2 + c_{21}^2} \) of \( V \), one can check that \( A \) can be represented by \( B_2\left( \frac{c_{11}^2}{c_{12}^2 + c_{21}^2} \right) \).

It remains to prove that any two different structures from the set (3.2) represent non-isomorphic algebras. Since \( B_3 \) is anticommutative, it is not isomorphic to other algebras from (3.2). Note also that \( \dim(B_1(a))^2 = 2 > 1 = \dim(B_2(\beta))^2 \) for any \( a, \beta \in k \).

Suppose that \( A \) is represented by the structure \( B_i(a) \) for some \( a \in k \) and \( i = 1, 2 \). Suppose that the structure constants of \( A \) in the basis \( E_1, E_2 \) equal the structure constants of \( B_i(\beta) \) for some \( \beta \in k \). Since \( E_1 E_1 = E_2 E_2 = 0 \), we have either \( E_1 = ae_1, E_2 = be_2 \) or \( E_1 = ae_2, E_2 = be_1 \) for some \( a, b \in k^* \). Since \( E_1 E_2 + E_2 E_1 = E_1 \), we have \( E_1 = ae_1 \) and \( E_2 = e_2 \). Then we get \( \beta = a \) from the equality \( E_1 E_1 = (1 - \beta)E_1 + (2 - i)E_2 \).
Lemma 3.6  If A belongs to the class C, then it can be represented by \( C(\alpha, \beta) \) for a unique pair \((\alpha, \beta) \in k \times k_{>0}\).

Proof  Let us represent the algebra A by a structure such that \( e_2 e_2 = e_2 \). It is easy to see that A belongs to the class C if and only if \( x_t = e_1 + t e_2 \) and \( x_t^2 \) are linearly independent for any \( t \in k \). Since

\[
x_t^2 = (c_1 + (c_1^2 + c_2^2) t) e_1 + (c_1^2 + (c_1^2 + c_2^2) t + t^2) e_2,
\]

\( x_t \) and \( x_t^2 \) are linearly independent if and only if

\[
0 \neq \begin{vmatrix}
1 & (c_1^2 + c_2^2) t + (c_1^2 + c_2^2) t + t^2 \\
1 & t
\end{vmatrix}
= (c_1^2 + c_2^2 - 1)t^2 + (c_1^2 + c_2^2) t - c_1^2.
\]

Since by our assumption \( x_t \) and \( x_t^2 \) are linearly independent for any \( t \in k \), we have \( c_1^2 + c_2^2 = 1 \), \( c_1^2 = c_2^2 = 2 \), and \( c_1^2 \neq 0 \).

Let \( a \) be such an element of \( k^* \) that \( c_1^2 a^2 = 1 \) and \( a(c_1^2 - c_2^2) \in k_{>0} \). Considering the basis \( \{ e_1 - c_1^2 e_1, e_2 \} \) of \( V \), one can check that A can be represented by \( C( c_1^2, a(c_1^2 - c_2^2) ) \).

Suppose that A is represented by the structure \( C(\alpha, \beta) \) for some pair \((\alpha, \beta) \in k \times k_{>0}\). Suppose that the structure constants of A in the basis \( E_1, E_2 \) equal the structure constants of \( C(\gamma, \delta) \) for some \((\gamma, \delta) \in k \times k_{>0}\). Since \( E_2 E_2 = E_2 \) and \( C(\alpha, \beta) \) has a unique idempotent, we have \( E_2 = e_2 \). We get \( E_1 = \pm e_1 \) from the equality \( E_1 E_1 = E_2 \). Then \( \gamma = \alpha \) and \( \delta = \pm \beta \). Since \( \beta, \delta \in k_{>0} \), we have \( (\gamma, \delta) = (\alpha, \beta) \).

Lemma 3.7  If A belongs to the class D, then it can be represented by a unique structure from the set

\[
\{ D_1(\alpha, \beta) \}_{(\alpha, \beta) \in k^* \cup \{ D_2(\alpha, \beta) \}_{(\alpha, \beta) \in k \times k^*} \cup \{ D_3(\alpha, \beta) \}_{(\alpha, \beta) \in k^* \times k^*} \}
\]

Proof  Let us represent the algebra A by a structure such that \( e_1 e_1 = e_1 \) and \( e_2 e_2 = 0 \).

Let us consider the following cases.

- \( c_1^2 + c_2^2 \neq 0 \). If \( c_1^2 + c_2^2 \neq 0 \), then one can check that

\[
\frac{1}{c_1^2 + c_2^2} \left( \frac{c_1^2 + c_2^2}{c_1^2 + c_2^2} - 1 \right) e_2
\]

is an idempotent that is not equal to \( e_1 \). Thus, \( c_1^2 + c_2^2 = 0 \). If \( \left( \frac{c_1^2}{c_1^2 + c_2^2}, c_2^2 \right) \in \mathbb{U} \), then, considering the basis \( e_1, \frac{c_1^2}{c_1^2 + c_2^2}, e_2 \) of \( V \), one can check that A can be represented by \( D_1(\frac{c_1^2}{c_1^2 + c_2^2}, c_2^2) \). If \( \left( \frac{c_1^2}{c_1^2 + c_2^2}, c_2^2 \right) \notin \mathbb{U} \), then

\[
\left( \frac{c_1^2}{c_1^2 + c_2^2} + c_2^2, c_2^2 \right) \in \mathbb{U}
\]

and, considering the basis \( e_1, e_1 - \frac{c_1^2}{c_1^2 + c_2^2}, e_2 \) of \( V \), one can check that A can be represented by \( D_1(\frac{c_1^2}{c_1^2 + c_2^2} + c_2^2, c_2^2) \).
• \( c_{12}^2 = -c_{21}^2 \neq 0 \). Considering the basis \( e_1, \frac{e_2}{c_{12}^2} \) of \( V \), one can check that \( A \) can be represented by \( D_3(c_{12}, c_{21}) \). Since \( e_1 + e_2 \) is not idempotent, \( (c_{12}^2, c_{21}^2) \not\in T \).

• \( c_{12}^2 = c_{21}^2 \neq 0 \). Then one can check that \( A \) is represented by \( D_2(c_{12}, c_{21}) \). Since \( e_1 + e_2 \) is not idempotent, \( (c_{12}^2, c_{21}^2) \not\in T \).

It remains to prove that any two different structures from the set (3.3) represent non-isomorphic algebras.

Suppose that \( A \) is represented by the structure \( D_1(\alpha, \beta) \) for some pair \( (\alpha, \beta) \in \mathcal{U} \). Note that \( e_2^2 = (e_1 - e_2)^2 = 0 \) in \( D_1(\alpha, \beta) \) while the structures \( D_2(y, \delta) \) and \( D_3(y, \delta) \) have a unique one-dimensional subspace of 2-nil elements for any pair \( (y, \delta) \in k^2 \).

Suppose now that the structure constants of \( A \) in the basis \( E_1, E_2 \) equal the structure constants of \( D_1(\alpha, \beta) \) for some pair \( (y, \delta) \in \mathcal{U} \). Since \( E_1 \) is an idempotent and \( D_1(\alpha, \beta) \) has a unique idempotent, we have \( E_1 = e_1 \). Since \( E_2E_2 = 0 \), we have either \( E_2 = ae_2 \) or \( E_2 = a(e_1 - e_2) \) for some \( a \in k^* \). We obtain \( a = 1 \) in both cases from the equality \( E_1E_2 + E_2E_1 = E_1 \). Then we have \( \delta = \beta \) and either \( y = \alpha \) or \( y = 1 - \alpha + \beta \). Since \( (\alpha, \beta), (y, \delta) \in \mathcal{U} \), we have \( (y, \delta) = (\alpha, \beta) \).

Suppose that \( A \) is represented by the structure \( D_2(\alpha, \beta) \) for some pair \( (\alpha, \beta) \in k^2 \). Note that \( A \) has an element \( x \) such that \( x^2 = 0 \) and \( xA + Ax \in (x) \) while \( D_3(x, \delta) \) does not have such an element for any pair \( (y, \delta) \in k^2 \) because any square zero element of \( D_3(y, \delta) \) is linearly dependent with \( e_2 \). Suppose now that the structure constants of \( A \) in the basis \( E_1, E_2 \) equal the structure constants of \( D_2(y, \delta) \) for some pair \( (y, \delta) \in k^2 \).

Since \( E_1 \) is an idempotent and \( D_2(\alpha, \beta) \) has a unique idempotent, we have \( E_1 = e_1 \). Since \( E_2E_2 = 0 \), we have \( E_2 = ae_2 \) for some \( a \in k^* \). Then it is easy to see that \( (y, \delta) = (\alpha, \beta) \).

Finally, suppose that \( A \) is represented by the structure \( D_3(\alpha, \beta) \) for some pair \( (\alpha, \beta) \in k^2 \). Suppose that the structure constants of \( A \) in the basis \( E_1, E_2 \) equal the structure constants of \( D_3(y, \delta) \) for some pair \( (y, \delta) \in k^2 \). Since \( E_1 \) is an idempotent and \( D_3(\alpha, \beta) \) has a unique idempotent, we have \( E_1 = e_1 \). Since \( E_2E_2 = 0 \), we have \( E_2 = ae_2 \) for some \( a \in k^* \). Then it is easy to see that \( a = 1 \) and \( (y, \delta) = (\alpha, \beta) \).

As a consequence of the proofs of Lemmas 3.4–3.7 we can describe the automorphism groups of the algebras of the classes \( \text{A–D} \).

**Corollary 3.8**

(i) \( \text{Aut}(A_1(\alpha)) \cong \text{Aut}(A_2) \) is isomorphic to the additive group of \( k \).

(ii) \( \text{Aut}(A_3) \) is isomorphic to the subgroup of \( GL_2(k) \) formed by matrices of the form 
\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix},
\]
where \( a \in k^* \) and \( b \in k \).

(iii) \( \text{Aut}(A_4(\alpha)) \cong C_2 \) if \( a = 0 \) and \( \text{char } k \neq 2 \); \( \text{Aut}(A_4(\alpha)) \) is trivial if either \( a \in k^* \) or \( a = 0 \) and \( \text{char } k = 2 \).

(iv) \( \text{Aut}(B_3(\alpha)) \) is trivial; \( \text{Aut}(B_3(\alpha)) \cong k^* \).

(v) \( \text{Aut}(B_3) \) is isomorphic to the subgroup of \( GL_2(k) \) formed by matrices of the form 
\[
\begin{pmatrix}
1 & 0 \\
0 & a
\end{pmatrix},
\]
where \( a \in k^* \) and \( b \in k \).

(vi) \( \text{Aut}(C(\alpha, \beta)) \cong C_2 \) if \( \beta = 0 \) and \( \text{char } k \neq 2 \); \( \text{Aut}(C(\alpha, \beta)) \) is trivial if either \( \beta \in k^* \) or \( \beta = 0 \) and \( \text{char } k = 2 \).

(vii) \( \text{Aut}(D_1(\alpha, \beta)) \cong C_2 \) if \( \beta = 2\alpha - 1 \) and \( \text{Aut}(C(\alpha, \beta)) \) is trivial if \( \beta \neq 2\alpha - 1 \).

(viii) \( \text{Aut}(D_2(\alpha, \beta)) \cong k^* \) and \( \text{Aut}(D_3(\alpha, \beta)) \) is trivial if \( \alpha + \beta \neq 1 \).

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In particular,
\[
\dim \text{Aut}(A_4(\alpha)) = \dim \text{Aut}(B_3(\alpha)) = \dim \text{Aut}(C_3(\alpha, \beta)) \\
= \dim \text{Aut}(D_1(\alpha, \beta)) = \dim \text{Aut}(D_2(\alpha, \beta)) = 0,
\]
\[
\dim \text{Aut}(A_3(\alpha)) = \dim \text{Aut}(A_2(\alpha)) = \dim \text{Aut}(B_2(\alpha)) = \dim \text{Aut}(D_2(\alpha, \beta)) = 1,
\]
\[
\dim \text{Aut}(A_3) = \dim \text{Aut}(B_3) = 2.
\]

Proof (i)–(iii). Any structure of the class A has a unique subspace of 2-nil elements generated by \(e_2\). Thus, any automorphism of such an algebra sends \(e_1\) and \(e_2\) to \(ae_1 + be_2\) and \(ce_1\), respectively, where \(a, c \in \mathbb{k}^*\) and \(b \in \mathbb{k}\). It is easy to check that \(a = c = 1\) for \(A_1(\alpha)\) and \(A_2(\alpha); c = a^2\) for \(A_3(\alpha)\); \(a = 1, b = 0\), and \(c = 1\) for \(A_4(\alpha)\); and \(a = c = 1, b = 0\) for \(A_4(\alpha)\) if \(\alpha \neq 0\).

(iv) It follows from the proof of Lemma 3.5 that any automorphism of the algebra \(B_i(\alpha),\) where \(i \in \{1, 2\},\) sends \(e_1\) and \(e_2\) to \(ae_1\) and \(e_2\), respectively, for some \(a \in \mathbb{k}^*\). It is easy to see that \(a = 1\) for \(i = 1\) and \(a\) can be arbitrary for \(i = 2\).

(v) Since \(B_3(V, V)\) is generated by \(e_2\), any automorphism of \(B_3(\alpha)\) sends \(e_1\) and \(e_2\) to \(ae_1 + be_2\) and \(ce_1\), respectively, where \(a, c \in \mathbb{k}^*\) and \(b \in \mathbb{k}\). It is easy to see that such a map is an automorphism if and only if \(\alpha = 1\).

(vi) It follows from the proof of Lemma 3.6 that any automorphism of the algebra \(C(\alpha, \beta)\) sends \(e_1\) and \(e_2\) to \(\pm e_1\) and \(e_2\), respectively. It is easy to see that the map that sends \(e_1\) and \(e_2\) to \(e_1\) and \(e_2\) respectively is an automorphism if and only if \(\beta = 0\) or \(\alpha = 0\).

(vii) It follows from the proof of Lemma 3.7 that any automorphism of the algebra \(D_1(\alpha, \beta)\) sends \(e_1\) to \(e_1\) and sends \(e_2\) either to \(e_2\) or \(e_1 - e_2\). It is easy to see that the map that sends \(e_1\) and \(e_2\) to \(e_1\) and \(e_1 - e_2\), respectively, is an automorphism if and only if \(\beta = 2\alpha - 1\).

(viii) It follows from the proof of Lemma 3.7 that any automorphism of the algebra \(D_2(\alpha, \beta)\) sends \(e_1\) and \(e_2\) to \(e_1\) and \(ae_2\), respectively for some \(a \in \mathbb{k}^*\). It follows from the same proof that any automorphism of the algebra \(D_3(\alpha, \beta)\) is trivial. \(\blacksquare\)

We will finish the proof of Theorem 3.3 in the next section devoted to the algebras of the class E.

4 Algebras of the Class E

In this section we consider the algebras of the class E. It is clear that such an algebra is isomorphic to \(E_i(\Gamma)\) for some \(\Gamma \in \mathbb{k}^4\). First, we describe isomorphisms inside this set and, thus, finish the proof of Theorem 3.3.

**Lemma 4.1** \(E_i(\Gamma_1) \cong E_i(\Gamma_2)\) if and only if one of the following conditions holds.

(i) \(\Gamma_1 = \Gamma_2;\)

(ii) \(C_i(\Gamma_1) = C_2(\Gamma_2)\) and \(C_3(\Gamma_1) = C_3(\Gamma_2);\)

(iii) \(C_i(\Gamma_1), C_2(\Gamma_1), C_3(\Gamma_1), C_3(\Gamma_2) \in \mathcal{I}, C_i(\Gamma_1) \neq C_2(\Gamma_1), C_1(\Gamma_2) \neq C_2(\Gamma_2);\)

(iv) \(C_1(\Gamma_2), C_2(\Gamma_2) \notin \mathcal{I}, \mathcal{D}(\Gamma_2) \neq 0,\) and there is some \(\alpha \in S_3\) such that \(C_i(\Gamma_1) = C_{\alpha(i)}(\Gamma_2)\) for \(i \in \{1, 2, 3\}\).
Proof Suppose that $g \in \text{GL}(V)$ is such that $g \star \mathbf{E}_1(\Gamma_1) = \mathbf{E}_1(\Gamma_2)$. Then $ge_1$ and $ge_2$ are two linearly independent idempotents of $\mathbf{E}_1(\Gamma_2)$. Let us describe all nonzero idempotents of this algebra. Let $\Gamma_2 = (\alpha, \beta, y, \delta)$ and $u = xe_1 + ye_2$ be some element of $V$. Then $\mathbf{E}_1(\Gamma_2)(u, u) = u$ if and only if
\[
x = x^2 + (\alpha + y)xy, \quad y = (\beta + \delta)xy + y^2.
\]
The solutions $(x, y) = (0, 0), (1, 0), (0, 1)$ give the obvious idempotents $0, e_1$ and $e_2$. All the other pairs $(x, y)$ satisfying the obtained equations are the solutions of the system of linear equations
\[
\begin{align*}
(\beta + \delta)x + y &= 1, \\
x + (\alpha + y)y &= 1
\end{align*}
\]
with the additional conditions $x \neq 0$ and $y \neq 0$. Let us consider the following cases.

- $\mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, i.e., $\alpha + y = \beta + \delta = 1$. In this case $\mathbf{E}_1(\Gamma_2)(u, u) = u$ if and only if either $u = 0$ or $x + y = 1$. Thus, $ge_1 = ae_1 + (1 - a)e_2$ and $ge_2 = be_1 + (1 - b)e_2$ for two different $a, b \in \mathbb{K}$. One can check that in this case
\[
\Gamma_1 = (1 - b)\alpha + b\delta, \alpha + (1 - a)\gamma, b\beta + (1 - b)\gamma, (1 - a)\alpha + a\delta).
\]
If $\mathcal{C}_1(\Gamma_2) \neq \mathcal{C}_2(\Gamma_2)$, then the formula above gives all the possible $\Gamma_1$ with $\mathcal{C}_1(\Gamma_1), \mathcal{C}_2(\Gamma_1) \in \mathcal{T}$, and $\mathcal{C}_1(\Gamma_1) \neq \mathcal{C}_2(\Gamma_1)$.

- One of the following three conditions holds.
  - $\mathcal{C}_1(\Gamma_2) \neq \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, i.e., $\beta + \delta = 1, \alpha + y = 1$.
  - $\mathcal{C}_1(\Gamma_2) \neq \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, i.e., $\beta + \delta = 1, \alpha + y = 1$.
  - $\mathcal{C}_2(\Gamma_2) \neq \mathcal{T}, \mathcal{D}(\Gamma_2) = 0$, i.e., $\beta + \delta, \alpha + y = 1$ and our system has zero determinant.

It is easy to see that in all of these cases our system of linear equations does not have solutions satisfying the additional conditions, i.e., $e_1$ and $e_2$ are all the nonzero idempotents of $\mathbf{E}_1(\Gamma_2)$. Thus, either $ge_1 = e_1$, $ge_2 = e_2$, and $\Gamma_1 = \Gamma_2$ or $ge_1 = e_2$,
\[
ge_2 = e_1, \mathcal{C}_1(\Gamma_1) = \mathcal{C}_2(\Gamma_1), \text{ and } \mathcal{C}_2(\Gamma_1) = \mathcal{C}_2(\Gamma_2).
\]

Now we can finish the proof of Theorem 3.3.
The Variety of Two-dimensional Algebras Over an Algebraically Closed Field

- If \( c_1(\Gamma) \in \mathcal{I} \) and \( c_2(\Gamma) \notin \mathcal{I} \), then \( E_1(\Gamma) \cong E_2(1 - \beta, \gamma, \beta, \alpha) = E_2(\beta, \gamma, \alpha) \) by Lemma 4.1.
- If \( c_1(\Gamma) \notin \mathcal{I} \) and \( c_2(\Gamma) \in \mathcal{I} \), then \( E_1(\Gamma) = E_2(\gamma, \beta, \delta) \).
- If \( c_1(\Gamma), c_2(\Gamma) \notin \mathcal{I} \), \( D(\Gamma) = 0 \) and \( \alpha + \gamma \notin k^*_\alpha \), then
  \[
  E_1(\Gamma) = E_3(\gamma(\beta + \delta), \beta(\alpha + \gamma), \alpha + \gamma).
  \]
- If \( c_1(\Gamma), c_2(\Gamma) \notin \mathcal{I} \), \( D(\Gamma) = 0 \), and \( \alpha + \gamma \notin k^*_\alpha \), then, by Lemma 4.1,
  \[
  E_1(\Gamma) \cong E_3(\beta(\alpha + \gamma), \gamma(\beta + \delta), \beta + \delta).
  \]
- If \( c_1(\Gamma), c_2(\Gamma) \notin \mathcal{I} \) and \( D(\Gamma) \neq 0 \), then there is a unique \( \sigma \in S_3 \) such that \( ^{-1}(c_1(\Gamma), c_2(\Gamma), c_3(\Gamma)) \in \mathcal{V} \) and we have \( E_1(\Gamma) \cong E_1(\Gamma') \) by Lemma 4.1, where \( \Gamma' \in \mathcal{V} \) is such that \( c_i(\Gamma') = c_{\sigma(i)}(\Gamma) \) for \( i \in \{1, 2, 3\} \).

By Lemma 4.1, the structures from the set
\[
\left\{ E_1(\Gamma) \right\}_{\Gamma \in \mathcal{V}} \cup \{ E_2(\alpha, \beta, \gamma) \}_{(\alpha, \beta, \gamma) \in k^3 \times k \times \mathcal{I}} \cup \{ E_3(\alpha, \beta, \gamma) \}_{(\alpha, \beta, \gamma) \in k^3 \times k^*_\alpha} \cup \{ E_4 \} \cup \{ E_5(\alpha) \}_{\alpha \in k}
\]
are pairwise non-isomorphic.

As a consequence of the proof of Lemma 4.1 we can describe the automorphism groups of algebras of the class \( E \).

**Corollary 4.2**

(i) For \( \Gamma \in \mathcal{V} \),
(a) \( \text{Aut}(E_1(\Gamma)) \) is trivial if \( c_1(\Gamma) \neq c_2(\Gamma) \).
(b) \( \text{Aut}(E_1(\Gamma)) \cong C_2 \) if \( c_1(\Gamma) = c_2(\Gamma) \neq (-1, -1) \).
(c) \( \text{Aut}(E_1(-1, -1, -1)) \cong S_3 \) if \( \text{char} k \neq 3 \).

(ii) \( \text{Aut}(E_4) \) and \( \text{Aut}(E_2(\alpha, \beta, \gamma)) \) are trivial for \( (\alpha, \beta, \gamma) \in k^3 \times k \times \mathcal{I} \).

(iii) For \( (\alpha, \beta, \gamma) \in k^3 \times k^*_\alpha \), \( \text{Aut}(E_3(\alpha, \beta, \gamma)) \cong C_2 \) if \( \gamma = -1 \) and \( \alpha = \beta \) and \( \text{Aut}(E_3(\alpha, \beta, \gamma)) \) is trivial otherwise.

(iv) \( \text{Aut}(E_5(\alpha)) \) is isomorphic to the subgroup of \( \text{GL}_2(k) \) formed by matrices of the form \( \begin{pmatrix} a & b \\ a & 1 \end{pmatrix} \), where \( a, b \in k, a \neq b \).

In particular, we have
\[
\dim \text{Aut}(E_1(\Gamma)) = \dim \text{Aut}(E_2(\alpha, \beta, \gamma)) = \dim \text{Aut}(E_3(\alpha, \beta, \gamma)) = \dim \text{Aut}(E_4) = 0, \quad \dim \text{Aut}(E_5(\alpha)) = 2.
\]

**Proof**

(i) Any automorphism of \( E_1(\Gamma) \) must send \( e_1 \) and \( e_2 \) to \( e_{\sigma(1)} \) and \( e_{\sigma(2)} \), respectively, for some \( \sigma \in S_3 \), where \( e_1 \) is defined in the proof of Lemma 4.1. Such a map is an automorphism if and only if \( c_1(\Gamma) = c_{\sigma(i)}(\Gamma) \) for \( i = 1, 2 \). If \( c_1(\Gamma) \neq c_2(\Gamma) \), then we have also \( c_3(\Gamma) \neq c_1(\Gamma), c_2(\Gamma) \) and, hence, only the identical element of \( S_3 \) determines an automorphism. If \( c_1(\Gamma) = c_2(\Gamma) \neq (-1, -1) \), then one can check that \( c_3(\Gamma) \neq c_1(\Gamma), c_2(\Gamma) \) and, hence, only the identical element of \( S_3 \) and the element that swaps \( e_1 \) and \( e_2 \) determine automorphisms. If \( c_1(\Gamma) = c_2(\Gamma) = (-1, -1) \), then any \( \sigma \in S_3 \) determines an automorphism. Note that \( (-1, -1, -1, -1) \in \mathcal{V} \) if and only if \( \text{char} k \neq 3 \).
(ii) Follows directly from the proof of Lemma 4.1.
(iii) It follows from the proof of Lemma 4.1 that an automorphism of $E_3(\alpha, \beta, y)$ is either trivial or swaps $e_1$ and $e_2$. The last mentioned map is an automorphism if and only if $y = -1$ and $\alpha = \beta$.
(iv) It follows directly from the proof of Lemma 4.1 that automorphisms of $E_3(\alpha)$ are exactly the linear maps that send $e_1$ and $e_2$ to $ae_1 + (1 - a)e_2$ and $be_1 + (1 - b)e_2$ for two different $a, b \in k$.

Now we will discuss some facts about degenerations of the form $A \to B$, where $A$ is an algebra of the class $E$. First of all, let us prove the following lemma.

**Lemma 4.3**  
(i) For any $\Gamma \in \mathcal{V}$ and $(\beta, y) \in \mathcal{C}(\Gamma)$, there exists a degeneration $E_1(\Gamma) \to D_2(\beta, y)$.

(ii) For any $(\alpha, \beta, y) \in k^3 \setminus k \times T$, there exists a degeneration $E_2(\alpha, \beta, y) \to D_2(\beta, y)$.

(iii) For any $(\alpha, \delta, e) \in k^2 \times k^*_1$ and $(\beta, y) \in \mathcal{C}(\alpha, \delta, e)$, there exists a degeneration $E_3(\alpha, \delta, e) \to D_2(\beta, y)$.

**Proof** The parametrized basis $E_{1i}^i = e_i$, $E_{2i}^j = te_j$ gives the degeneration $E_1(\Gamma) \to D_2(\mathcal{C}_1(\Gamma))$ for any $\Gamma \in k^3$. If $\Gamma \in \mathcal{V}$ and $1 \leq i \leq 3$, then, by Lemma 4.1, there exists $\Gamma' \in k^4$ such that $E_1(\Gamma) \cong E_1(\Gamma')$ and $\mathcal{C}_1(\Gamma) = \mathcal{C}_1(\Gamma')$. Hence, $E_1(\Gamma) \to D_2(\mathcal{C}_1(\Gamma'))$. We also have $E_2(\alpha, \beta, y) = E_1(1 - \alpha, \beta, \alpha, y) \to D_2(\beta, y)$ for $(\alpha, \beta, y) \in k^3 \setminus k \times T$ and

$$E_3(\alpha, \delta, e) = E_1\left(1 - \alpha e, \frac{\delta}{e}, \alpha e, \frac{1 - \delta}{e}\right)$$

$$\cong E_1\left(1 - \frac{\delta}{e}, \alpha e, \frac{\delta}{e}, (1 - \alpha)e\right) \to D_2(\alpha e, (1 - \alpha)e), D_2\left(\frac{\delta}{e}, \frac{1 - \delta}{e}\right)$$

for $(\alpha, \delta, e) \in k^2 \times k^*_1$.

For $\Gamma = (\alpha, \beta, y, \delta) \in k^4$, let us define the following subset of $A_2$.

$$G(\Gamma) = \{ \mu \}$$

where

$$\begin{cases}
  c_{12} = 0; c_{21} = y c_{22}; c_{12} = \alpha c_{22}; \\
  (1 - y - \delta(\alpha + y)) c_{12} - (1 - \alpha - \beta(\alpha + y)) c_{21} = (\beta(1 - y) - \delta(1 - \alpha)) c_{11}; \\
  (1 - \alpha - \beta(\alpha + y))^2 c_{11} c_{22} = (\beta c_{11}^1 - c_{12}^1) (\mathcal{D}(\Gamma) c_{21}^2 + ((\alpha - 1)(\delta - 1) - \beta y) c_{11}^1); \\
  (1 - y - \delta(\alpha + y))^2 c_{11} c_{22} = (\delta c_{11}^1 - c_{12}^1) (\mathcal{D}(\Gamma) c_{21}^2 + ((\alpha - 1)(\delta - 1) - \beta y) c_{11}^1); \\
  (1 - \alpha - \beta(\alpha + y)) (1 - \alpha - \beta(\alpha + y)) c_{11} c_{22} = (\beta c_{11}^1 - c_{12}^1) (\mathcal{D}(\Gamma) c_{21}^2 + ((\beta - 1)(\gamma - 1) - a \delta) c_{11}^1); \\
  (1 - \alpha - \beta(\alpha + y)) (1 - \gamma - \delta(\alpha + y)) c_{11} c_{22} = (\delta c_{11}^1 - c_{12}^1) (\mathcal{D}(\Gamma) c_{21}^2 + ((\beta - 1)(\gamma - 1) - a \delta) c_{11}^1).
\end{cases}$$
5 Degenerations of Two-dimensional Algebras

In this section we describe all degenerations of two-dimensional algebras. Note that the results are valid for algebras over an algebraically closed field of arbitrary characteristic.

**Theorem 5.1.** \(A_2\) has the graph of primary degenerations presented in Figure 1.

**Proof.** All primary degenerations that do not follow from Lemma 4.3 are presented in Table 2. Table 3 describes separating sets for all required non-degenerations and, thus, finishes the proof of the theorem.

The verification of degenerations is an easy direct calculation in each case. An example clarifying how to do this can be found in the proof of [21, Theorem 3]. The verification of Table 3 is more difficult. To clarify how one can achieve it, let us consider the first row of the table. It is easy to prove that \(A_4(a)\) belongs to the presented separating set that we denote by \(\mathcal{R}\). The fact that \(\mathcal{R}\) is upper invariant can be checked by a direct calculation. What exactly one has to check is explained in Section 2. Let us prove that the orbits of \(B_2(y)\), \(D_2(\beta, y)\), and \(E_5(\beta)\) do not intersect \(\mathcal{R}\). Let \(\lambda\) be one of these structures. Suppose that the structure constants \(c^i_j (i, j, k = 1, 2)\) of \(\lambda\) in the basis \(f_1, f_2\) satisfy the defining equations of \(\mathcal{R}\). Then \(c_{12}^1 = c_{22}^2 = 0\), and hence \(\lambda(f_2, f_2) = 0\). Since \(\mathcal{R}\) is invariant under the basis rescaling, we may assume that \(f_2 \in \{e_1, e_2\}\) if \(\lambda = B_2(y)\), \(f_2 = e_2\) if \(\lambda = D_2(\beta, y)\), and \(f_2 = e_2 - e_1\) if \(\lambda = E_5(\beta)\). Now in view of the upper invariance of \(\mathcal{R}\), we may assume that \((f_1, f_2) \in \{(e_1, e_2), (e_2, e_1)\}\) if \(\lambda = B_2(\beta, y)\), \((f_1, f_2) = (e_1, e_2)\) if \(\lambda = D_2(\beta, y)\), and \((f_1, f_2) = (e_1, e_2 - e_1)\) if \(\lambda = E_5(\beta)\). We have the following.

- \(c_{12}^1 + c_{22}^1 = 1 \neq 0\) if \(\lambda = B_2(\beta, y)\), \(f_1 = e_1, f_2 = e_2\).
- \(c_{12}^1 + c_{22}^2 = 1 \neq 0\) if \(\lambda = B_2(\beta, y)\), \(f_1 = e_2, f_2 = e_1\).
- \(\alpha^2 c_{12}^1 c_{22}^2 = 0 \neq (c_{12}^1)^2\) if either \(\lambda = D_2(\beta, y)\), \(f_1 = e_1, f_2 = e_2\) or \(\lambda = E_5(\beta)\), \(f_1 = e_1, f_2 = e_2 - e_1\).

Thus, the structure constants of \(\lambda\) in any basis do not satisfy the defining equations of \(\mathcal{R}\), i.e., \(O(\lambda) \cap \mathcal{R} = \emptyset\). The other nondegenerations can be considered in the same manner.

Let us recall that \(n\)-dimensional algebra \(A\) has a level \(m\) if
- there exists a sequence of \(n\)-dimensional algebras \(A_0, \ldots, A_m\) such that \(A_0 = k^n\), \(A_m = A\), and, for \(0 \leq i \leq m - 1\), one has \(A_{i+1} \supseteq A_i\) and \(A_{i+1} \neq A_i\);
- if \(A_0, \ldots, A_{m+1}\) is a sequence of algebras such that \(A_0 = k^n\), \(A_{m+1} = A\), and \(A_{i+1} \supseteq A_i\) for \(0 \leq i \leq m\), then \(A_{i+1} \supseteq A_i\) for some \(1 \leq i \leq m\).

Theorem 5.1 gives us the following partition of \(A_2\) to levels:
6 Closures for Orbits of Infinite Series

In this section we describe closures of orbits for infinite series from our classification. To make this description nicer and more complete, we introduce two additional series and one additional algebra. For \( \alpha \in k \), we introduce \( D_3^x(\alpha) = D_2(\alpha, -\alpha) \) and \( D_3^y(\alpha) = D_3(\alpha, -\alpha) \). Note that \( D_3^x(*) \subset D_2(*) \) and \( D_3^y(*) \subset D_3(*) \). Also we define \( A_4' = A_4(0) \in A_4(*) \). Henceforth for a symbol \( X \), we denote by \( X(*) \) the set formed by all \( X(\Gamma) \) that are defined. For example, \( D_3(\cdot) = \{ D_2(\Gamma) \mid \Gamma \in k^2 \} \), \( E_3(\cdot) = \{ E_3(\Gamma) \mid \Gamma \in k^2 \times k^* \} \).

**Theorem 6.1** For each row of Table 4, the second column contains all isomorphism classes of algebras whose orbits lie in the closure of the orbit of the series of algebras contained in the first column of the same row.

**Proof** All required degenerations that do not follow from Theorem 5.1 are proved in Table 5. Table 6 describes separating sets for all required non-degenerations and, thus, finishes the proof of the theorem.

Note that the structures \( B_1(\alpha) \), \( D_1(\alpha, \beta) \), and \( E_3(\alpha, \beta, y) \) do not lie in the corresponding separating sets, but the structures

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ast B_1(\alpha), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ast D_1(\alpha, \beta), \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \ast E_3(\alpha, \beta, y)
\]

satisfy the required conditions. \( \blacksquare \)

Theorems 5.1 and 6.1 give a lattice of subsets for \( A_2 \). This lattice is presented in Figure 2. In this figure the leftmost set coincides with \( A_2 \) and sets placed in one column have the same dimension equal to the number standing above them. Two sets of dimensions \( i \) and \( i + 1 \) are connected by an edge if and only if the set of dimension \( i + 1 \) contains the set of dimension \( i \). Moreover, if \( X, Y \subset A_2 \) correspond to two vertices of the diagram, then \( X \cap Y \) is equal to the union of all \( Z \subset A_2 \) corresponding to vertices of the diagram such that there exist paths from \( X \) to \( Z \) and from
Y to Z going from left to right. For example, \( \overline{O(D_1(*))} \cap \overline{O(C(*))} = \overline{O(A_1')} \) and \( \overline{O(D_2(*))} \cap \overline{O(D_2(*))} = \overline{O(B_2(*))} \cup \overline{O(D_2')(*)} \).

7 Subvarieties Defined by Identities

Now we are going to apply the results of previous sections to develop the varieties of two-dimensional flexible and bicommutative algebras. In particular, we will describe the varieties of commutative and anticommutative algebras. Since there exists only one nontrivial two-dimensional anticommutative algebra, the last mentioned problem is not of big interest. Note also that in the same way one can recover the results of [3], where the analogous problems were solved for two-dimensional Novikov and pre-Lie algebras. Since the classifications of flexible and bicommutative algebras depend on the characteristic of the ground field, we assume everywhere in this section that \( \text{Char} k \neq 2 \).

7.1 Flexible Algebras

By definition, an algebra is called flexible if it satisfies the identity \((x y)x = x(y x)\). It is clear that all commutative and anticommutative algebras are flexible. Using Theorem 3.3, one can verify that any two-dimensional flexible algebra is either (anti)commutative or \( E_3(a) \). For \( a, \beta \in k \), let us introduce the algebras

\[
\begin{align*}
D_2'(a) &= D_2(a, a), \\
E_2'(a) &= E_2(\frac{1}{2}, a, a), \\
E_2(a) &= E_3(\frac{1}{2}, \frac{1}{2}, a), \\
E_2'(a, \beta) &= E_4(a, \beta, a, \beta).
\end{align*}
\]

It follows from our classification that any nontrivial two-dimensional commutative algebra can be represented by a unique structure from the set

\[
\left\{ A_1(\frac{1}{2}), A_3, B_2(\frac{1}{2}), C(\frac{1}{2}, 0), D_2(\frac{1}{2}, 0), E_3(\frac{1}{2}) \right\}
\]

\[
\cup \left\{ D_2'(a), E_2'(a) \right\}_{a \in k \setminus \{\frac{1}{2}\}} \cup \left\{ E_2'(a, \beta) \right\}_{a \in k \setminus \{\frac{1}{2}\}} \cup \left\{ E_2'(a, \beta) \right\}_{a \in k \setminus \{\frac{1}{2}\}}
\]

and that any noncommutative flexible algebra can be represented by a structure from the set \( \{ B_3 \} \cup \{ E_4 \}_{a \in k \setminus \{\frac{1}{2}\}} \). Using Theorem 5.1, we get the graph of primary degenerations for the variety of two-dimensional flexible algebras presented in Figure 3.

It is easy to see that the variety of commutative algebras is simply \( k^6 \), i.e., irreducible. Then it is clear that the variety of flexible algebras has two irreducible components. The first component is \( \{ \overline{O(E_4(a))} \}_{a \in k} = \{ E_3(a), B_3, k^2 \}_{a \in k} \). The second component, formed by all commutative algebras, is equal to the closure of the orbit of the algebra series \( E_3'(\ast) \).

This variety of flexible algebras does not have rigid algebras, and has the lattice of subsets presented in Figure 4. The lattice satisfies the same properties as the lattice from the previous section. To prove this it is enough to use Theorem 6.1 and its proof. The only difference is that one must use the parametrized indices \((\frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}), (\frac{1}{2} + \frac{1}{2} - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}), \frac{1}{2} \) in the degenerations \( D_2(*) \to A_1(\frac{1}{2}) \),

\[
D_2'(a) = D_2(a, a), \\
E_2'(a) = E_2(\frac{1}{2}, a, a), \\
E_2(a) = E_3(\frac{1}{2}, \frac{1}{2}, a), \\
E_2'(a, \beta) = E_4(a, \beta, a, \beta).
\]
E₂(*) → C(1/2, 0), and E₃(*) → D₄(1/2, 0), respectively, to obtain the degenerations
D₃₂(*) → A₁(1/2), E₃₂(*) → C(1/2, 0), and E₃₅(*) → D₄(1/2, 0).

7.2 Bicommutative Algebras

The variety of bicommutative algebras (see, for example, [10]) is defined by the identities
x(yz) = y(xz) and (xy)z = (xz)y. It follows from Theorem 3.3 that any non-trivial bicommutative algebra is isomorphic to a unique algebra from the set

\{ A₁, B₂(0), B₂(1), D₁(0, 0), D₂(1, 1), D₂(0, 0), E₁(0, 0, 0, 0) \}.

Using Theorem 5.1, we get the graph of primary degenerations for the variety of two-dimensional bicommutative algebras. This graph is presented in Figure 5.

Thus, the irreducible components in the variety of two-dimensional bicommutative algebras are

\[
\overline{O(D₁(0, 0))} = \{ D₁(0, 0), D₂(0, 0), B₂(1), A₃, k² \},
\]

\[
\overline{O(E₁(0, 0, 0, 0))} = \{ E₁(0, 0, 0, 0), D₂(0, 0), D₂(1, 1), A₃, k² \}.
\]

These components are generated by the rigid bicommutative algebras D₁(0, 0) and E₁(0, 0, 0, 0) and all have dimension 4.

Appendix A. Tables.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁(a), a ∈ k</td>
</tr>
<tr>
<td>A₂</td>
</tr>
<tr>
<td>A₃</td>
</tr>
<tr>
<td>A₄(a), a ∈ k₄¹</td>
</tr>
<tr>
<td>B₁(a), a ∈ k</td>
</tr>
<tr>
<td>B₂(a), a ∈ k</td>
</tr>
<tr>
<td>C₁(a, b), (a, b) ∈ k × k₁₀</td>
</tr>
<tr>
<td>D₁(a, b), (a, b) ∈ k₄₂</td>
</tr>
<tr>
<td>D₂(a, b), (a, b) ∈ k¹ × k₄</td>
</tr>
<tr>
<td>D₃(a, b), (a, b) ∈ k² × k₂</td>
</tr>
<tr>
<td>E₁(a, b, y, δ), (a, b, y, δ) ∈ V₁</td>
</tr>
<tr>
<td>E₂(a, b, y), (a, b, y) ∈ k³ × k²</td>
</tr>
<tr>
<td>E₃(a, b, y), (a, b, y) ∈ k³ × k³</td>
</tr>
<tr>
<td>E₄(a)</td>
</tr>
<tr>
<td>E₅(a)</td>
</tr>
</tbody>
</table>
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Table 2

degenerations parametrized bases

| A₁(α) → A₁ | E₁ = te₁ | E₂ = t²e₂ |
| A₁(α) → E₂(α) | E₁ = e₁ | E₂ = c₁ + t⁻¹e₂ |
| A₂ → A₃ | E₁ = te₁ | E₂ = t²e₂ |
| A₂ → B₃ | E₁ = e₁ | E₂ = t⁻¹e₂ |
| A₄(α) → A₂ | E₁ = te₁ - e₂ | E₂ = t²e₂ |
| B₁(y) → A₂ | E₁ = e₁ + te₂ | E₂ = -t²e₂ |
| B₁(y) → B₂(y) | E₁ = te₁ | E₂ = e₂ |
| B₂(y) → A₃ | E₁ = e₁ + te₂ | E₂ = te₁ |
| C(α, β) → A₁(α) | E₁ = te₁ + e₂ | E₂ = t²e₂ |
| D₁(α, β) → B₂(α) | E₁ = te₁ | E₂ = e₂ |
| D₁(α, β) → B₂(1 - α + β) | E₁ = te₂ | E₂ = e₁ - e₂ |
| D₁(α, β) → D₂(β, -β) | E₁ = e₁ | E₂ = te₂ |
| D₂(β, γ) → A₁ | E₁ = te₁ + te₂ | E₂ = t²e₁ + (β + γ)t²e₂ |
| D₃(β, γ) → A₂ | E₁ = t⁻¹βγe₁ - e₂ | E₂ = te₂ |
| D₃(α, β, γ) → A₁(α) | E₁ = te₁ + e₂ | E₂ = (1 - β + γ)t²e₂ |
| E₃(α, δ, c) → B₂⁻¹ \( \frac{1 - \delta - (1 - a)e}{1 - c} \) | E₁ = te₁ | E₂ = c - c₁ |
| E₁ → B₃ | E₁ = e₁ - e₂ | E₂ = te₂ |
| E₄ → E₅(α) | E₁ = αe₁ + (1 - α)e₂ | E₂ = (α - t)e₁ + (1 - α + t)e₂ |

Table 3

<table>
<thead>
<tr>
<th>non-degenerations separating sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁(α) ∩ B₁(γ), D₁(β, γ), E₁(β)</td>
</tr>
<tr>
<td>B₁(y) ∩ B₁(β) (β ≠ γ), D₁(β, δ), E₁(α)</td>
</tr>
<tr>
<td>C(α, β) ∩ B₁(γ), B₂(γ), D₂(γ, δ), E₂(γ) (γ = α)</td>
</tr>
<tr>
<td>D₁(α, β) ∩ B₁(γ) (γ ≠ (α, 1 - α + β)), B₂(γ), D₂(γ, δ) (γ, δ ≠ (β, β)), E₁(γ)</td>
</tr>
<tr>
<td>D₁(δ, γ) ∩ B₂(γ), D₂(δ, c) (δ, c) = (β, γ)), E₁(α)</td>
</tr>
<tr>
<td>E₁(Γ) ∩ B₁(γ), B₂(γ), D₂(β, γ) (β, γ) ≠ C(Γ), E₁(α)</td>
</tr>
<tr>
<td>E₁(a, β, γ) ∩ B₁(γ), D₁(β, δ), E₁(δ) (δ = α)</td>
</tr>
<tr>
<td>E₁(a, δ, c) ∩ B₁(γ) (γ ≠ ( \frac{1 - (1 - a)e}{1 - c} )), B₂(γ), D₂(β, γ) (β, γ) ≠ C(δ, α, c), E₁(γ)</td>
</tr>
<tr>
<td>E₁ ∩ A₁</td>
</tr>
<tr>
<td>Table 4</td>
</tr>
<tr>
<td>---------------------------------------------</td>
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<tr>
<td>$A_1(<em>) \rightarrow A_2$, $A_2$, $A_1$, $B_1$, $E_2(</em>), k^2$</td>
</tr>
<tr>
<td>$A_4(<em>) \rightarrow A_2$, $A_2$, $A_4(</em>)$, $B_1$, $E_4(*), k^2$</td>
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<tr>
<td>$B_1(<em>) \rightarrow A_2$, $A_2$, $A_1$, $B_1$, $E_2(</em>), k^2$</td>
</tr>
<tr>
<td>$B_2(<em>) \rightarrow A_2$, $A_2$, $B_2(</em>), B_1$, $k^2$</td>
</tr>
<tr>
<td>$C(<em>) \rightarrow A_2$, $A_2$, $A_4(</em>)$, $B_1$, $E_4(*), k^2$</td>
</tr>
<tr>
<td>$D_1(<em>) \rightarrow A_2$, $A_2$, $A_4(</em>)$, $B_1$, $E_2(<em>), D_2(</em>)$, $D_2'(*), k^2$</td>
</tr>
<tr>
<td>$D_2(<em>) \rightarrow A_2$, $A_2$, $B_2(</em>), B_1$, $D_2(*)$, $k^2$</td>
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<tr>
<td>$D_2'(<em>) \rightarrow A_2$, $A_2$, $B_2(</em>), D_2'(*), k^2$</td>
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<td>$D_3(<em>) \rightarrow A_2$, $A_2$, $A_4(</em>)$, $B_1$, $B_2(<em>), D_2(</em>)$, $E_2(*)$, $k^2$</td>
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<tr>
<td>$E_0(*) \rightarrow A_2$</td>
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<tr>
<td>$E_2(<em>) \rightarrow A_2$, $A_3$, $A_4(</em>)$, $B_1$, $B_2(<em>), B_3$, $E_4(</em>), k^2$</td>
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<td>$E_4(<em>) \rightarrow B_1$, $E_4(</em>), k^2$</td>
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<table>
<thead>
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<th>Table 5</th>
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<tbody>
<tr>
<td>degenerations</td>
</tr>
<tr>
<td>$A_1(*) \rightarrow A_2$, $E_1^t = t e_1, E_2^t = t^2 e_2$</td>
</tr>
<tr>
<td>$A_4(*) \rightarrow A_4(a)$, $E_1^t = t e_1 + (1 - a) e_2, E_2^t = t^2 e_2$</td>
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<tr>
<td>$A_4(*) \rightarrow E_4$, $E_1^t = t e_1, E_2^t = t e_1 + e_2$</td>
</tr>
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<td>$B_1(*) \rightarrow A_4$, $E_1^t = -t^{-1} e_1 + t e_2, E_2^t = -t^2 e_2$</td>
</tr>
<tr>
<td>$B_2(*) \rightarrow A_2$, $E_1^t = e_1 + t e_2, E_2^t = -t^2 e_2$</td>
</tr>
<tr>
<td>$C(*) \rightarrow A_4(a)$, $E_1^t = t e_1 + a e_2, E_2^t = t^2 e_2$</td>
</tr>
<tr>
<td>$D_1(*) \rightarrow B_1(a)$, $E_1^t = t e_1, E_2^t = e_2$</td>
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<td>$D_1(*) \rightarrow D_2(a)$, $E_1^t = e_1, E_2^t = t e_2$</td>
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<td>$D_2(*) \rightarrow A_1(a)$, $E_1^t = e_1 + e_2, E_2^t = t e_2$</td>
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<td>$D_2'(*) \rightarrow A_2$, $E_1^t = e_1 - t e_2, E_2^t = t e_2$</td>
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<td>$D_3(*) \rightarrow A_4(a)$, $E_1^t = a e_1 + \frac{1}{\alpha^2} e_2, E_2^t = e_2$</td>
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<td>$D_3(*) \rightarrow B_1(a)$, $E_1^t = -e_2, E_2^t = t e_1$</td>
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<td>$D_3'(*) \rightarrow A_4$, $E_1^t = e_1 - \frac{1}{\alpha} e_2, E_2^t = e_2$</td>
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<tr>
<td>$E_3(*) \rightarrow C(a, \beta)$, $E_1^t = t^{-1} e_1 - t^{-1} e_2, E_2^t = e_2$</td>
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<td>$E_3(*) \rightarrow D_3(a, \beta)$, $E_1^t = e_1, E_2^t = t e_2$</td>
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<tr>
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<td>$E_3(*) \rightarrow B_3$, $E_1^t = t e_1, E_2^t = e_2 - e_1$</td>
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Table 6

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<th>non-degenerations</th>
<th>separating sets</th>
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<td>( B_1(+) \neq D'_2(a) )</td>
<td>( { \beta \mid c_{12}^1 + c_{12}^2 + c_{21} = c_{11} = 0 } )</td>
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<td>( C(+) \neq B_2(a) \cdot D_2(a, \beta) \ (a + \beta \neq 1) )</td>
<td>( { \beta \mid c_{12}^1 + c_{12}^2 + c_{12}^3 + c_{21} = c_{11} } )</td>
</tr>
<tr>
<td>( D_1(+) \neq A_4(\alpha) \ (a \neq 0), D_2(a, \beta) \ (a + \beta = 0), E_5(\alpha) )</td>
<td>( { \beta \mid c_{12}^1 + c_{12}^2 + c_{12}^3 = c_{11} = 0 } )</td>
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<tr>
<td>( D'_1(+) \neq B_2(\alpha) )</td>
<td>( { \beta \mid c_{12}^1 + c_{12}^2 + c_{12}^3 + c_{21}^3 = 0 } )</td>
</tr>
<tr>
<td>( E_2(+) \neq D_1(\alpha, \beta), E_1(\Gamma) \ (\Gamma \in \mathbb{V}), E_3(\alpha, \beta, y) \ (y \neq 1) )</td>
<td>( { \beta \mid c_{12}^1 + c_{12}^2 + c_{12}^3 = 0 } )</td>
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<tr>
<td>( E_3(+) \neq C(\alpha, \beta), E_1(\Gamma) \ (\Gamma \in \mathbb{V}), E_2(\alpha, \beta, y) \ (\beta + y \neq 1) )</td>
<td>( { \beta \mid c_{12}^1 = 0 } )</td>
</tr>
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</table>

Appendix B. Figures.

Figure 1

Figure 2
Acknowledgment  We would like to thank the referee for his exhaustive review as well as for his useful comments and Vesselin Drensky for his useful comments about the classification of bicommutative algebras.
The Variety of Two-dimensional Algebras Over an Algebraically Closed Field

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