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The Variety of Two-dimensional Algebras Over an Algebraically Closed Field

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Abstract. The work is devoted to the variety of two-dimensional algebras over algebraically closed fields. First we classify such algebras modulo isomorphism. Then we describe the degenerations and the closures of certain algebra series in the variety of two-dimensional algebras. Finally, we apply our results to obtain analogous descriptions for the subvarieties of flexible and bicommutative algebras. In particular, we describe rigid algebras and irreducible components for these subvarieties.

1 Introduction

In this paper, an algebra is simply a vector space over a field with a bilinear binary operation that does not need to be associative. Algebras of a fixed dimension form a variety with a natural action of a general linear group. Orbits under this action correspond to isomorphism classes of algebras. There are many classifications up to isomorphism for varieties of algebras of some fixed dimension satisfying some polynomial identities. For example, there exist such classifications of three-dimensional Novikov algebras [2], four-dimensional Leibniz algebras [9], six-dimensional Lie algebras [30] and many others.

In this paper we classify all two-dimensional algebras over an algebraically closed field up to isomorphism. It is not the first work devoted to this problem [1, 14, 28], but all of them are not convenient for our main goal: the geometric description of the algebraic variety of two-dimensional algebras. One of the advantages of our paper is that our approach deals uniformly with all possible characteristics while the authors of [1] do not consider characteristics 2 and 3 and the authors of [14] consider only the two-elements field in characteristic 2. The authors of [1], in fact, do not give an explicit classification of two-dimensional algebras up to isomorphism because they have other purposes. They describe the moduli space by proving that two-dimensional algebras can be divided into parts that can be naturally included into projective spaces of different dimensions. The authors claim that the classification up to isomorphism is easy and could be extracted from their proofs. The classification is really not very difficult, and we believe that one can extract it after reading the paper, taking parts of the classification from different places and carefully taking into account all the details. For us it was easier to produce this classification from scratch. Goze and Remm [14] produced a full classification. One of the problems is that this classification is spread

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throughout the whole paper and is mixed with other formulas. It is tedious to collect all the parts of the classification from in one place and find all the additional conditions for these parts. At the same time, there are some inaccuracies. For example, the series μ_{10} parametrized by two scalars must be divided into two series parametrized by one scalar, the series μ_{11} admits nontrivial isomorphisms, and in the case of a commutative two-dimensional algebra with one idempotent e, it may be impossible to find f linearly independent with e such that f^2 and e are linearly dependent. While Petersson [28] gave the full classification of two-dimensional algebras over any field, unfortunately, the answer is not given in terms of multiplication tables. The translation of this answer to the language of multiplication tables as well as its direct usage for the description of orbit closures is very difficult, and it seems to be easier to produce a new appropriate classification. Also the consideration of arbitrary fields complicates the result and the extraction of the answer for an algebraically closed field becomes tedious. For these reasons, we give a classification that is valid over an algebraically closed field of arbitrary characteristic in Section 3. In the same section we also describe the automorphism groups for all algebras under consideration.

In the main part of our paper we develop the geometry of the variety of two-dimensional algebras. Namely, we describe the closures of orbits of some sets with respect to the Zariski topology. First, we describe all possible degenerations, *i.e.*, closures of orbits of one point sets. Degenerations are an interesting subject that was studied in various papers [3–7, 11–13, 15–18, 20, 21, 25–27, 29]. One of the problems in this direction is to describe all degenerations in a variety of algebras of some fixed dimension satisfying some set of identities. For example, this problem was solved for two-dimensional pre-Lie algebras in [3], for three-dimensional Novikov algebras in [4], for four-dimensional Lie algebras in [7], for four-dimensional Zinbiel and nilpotent Leibniz algebras in [21], for nilpotent five- and six-dimensional Lie algebras in [15, 29], and for nilpotent five- and six-dimensional Malcev algebras in [20]. As an application of our results, one can easily recover the results of [3].

Another interesting notion concerning degenerations is the so-called level of an algebra defined in the end of Section 5. The algebras of the first level and the associative Lie and Jordan algebras of the second level are classified in [23, 24]. Gorbatsevich [11–13] defined the notion of an infinite level and described all anticommutative algebras that have an infinite level not greater than three. This notion is much easier in the sense that the infinite level of an algebra can be easily expressed in terms of the usual level. Algebras of low dimension play a special role in problems of such type, because they have small levels. The complete description of degenerations obtained by Gorbatsevich allows computing the level for all two-dimensional algebras.

Our next result is the description of orbit closures of certain series that appear in the classification up to isomorphism. Let T be some subvariety of the variety of an n-dimensional algebra closed under the action of the general linear group. An n-dimensional algebra from T is called *rigid* if its orbit is an open subset of T. Another important characteristic of a variety is its partition into irreducible components. The notion of a rigid algebra is closely related to this characteristic, because orbit closures of such algebras form irreducible components. For example, irreducible components and rigid algebras were classified for low dimensional associative algebras [26, 27]) and for Jordan algebras [19]. Since the variety of two-dimensional algebras is simply $\mathbf{k}^8$, it is clear that there is only one irreducible component and there are no rigid algebras in it. Thus, this problem is not relevant for the variety of all two-dimensional algebras itself. Nevertheless, it is relevant for subvarieties. In the last part we apply our results about the variety of all two-dimensional algebras to their subvarieties consisting of flexible and bicommutative algebras. We describe all degenerations and closures of orbits in these varieties. In reducible components and rigid algebras. Our results allow us to get such descriptions and classifications for varieties of two-dimensional algebras defined by any identities without any problems.

Let us give a resume of our motivations. The problems considered in this paper are classical and their solutions are interesting by themselves. In general the classification of all n-dimensional algebras is a wild problem and it is interesting to get the solution in particular cases where it is still possible. Our main motivation was the classification of all the algebras of the second level that we produced in [22] using the results of this paper. In fact, there are reasons to guess that our classification will also allow classifying algebras of the third and the fourth levels. Thus, our results are important for the classification of algebras of small levels and constitute a necessary part of it. Another application that we have in mind is the geometric description of subvarieties of the variety of two-dimensional algebras. There are some works, *e.g.*, [3], devoted to this problem, and our work gives a powerful tool to solve it. Our results can be applied whenever the natural action of $\mathrm{GL}(\mathbf{k}^2)$ on $(\mathbf{k}^2)^* \otimes (\mathbf{k}^2)^* \otimes \mathbf{k}^2$ appears, and we expect that they will have other applications, for example, in the theory of algebras with polynomial identities or in the geometric representation theory. Even when one deals with n-dimensional algebras for $n > 2$, it may be useful to consider two-dimensional subalgebras, and our results could be applied in this case. For example, the classification of n-dimensional algebras with an $(n-2)$-dimensional annihilator is fulfilled using our classification [8].

2 Definitions and Notation

Throughout the paper we fix an algebraically closed field $\mathbf{k}$, a two-dimensional $\mathbf{k}$-linear vector space V and a basis $\{e_1, e_2\}$ of V. All spaces in this paper are considered over $\mathbf{k}$, and we write simply dim, Hom and $\otimes$ instead of $\dim_{\mathbf{k}}$, $\mathrm{Hom}_{\mathbf{k}}$ and $\otimes_{\mathbf{k}}$. An algebra A is a set with a structure of a vector space and a binary operation that induces a bilinear map from $A \times A$ to A.

Since this paper is devoted to two-dimensional algebras, we give all definitions and notation only for this case, though everything in this section can be rewritten for any dimension.

The set $\mathcal{A}_2 := \mathrm{Hom}(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension 8. This space has a structure of the affine variety $\mathbf{k}^8$. Indeed, any $\mu \in \mathcal{A}_2$ is determined by eight structure constants $c_{ij}^k \in \mathbf{k}$ $(i, j, k = 1, 2)$ such that $\mu(e_i \otimes e_j) = c_{ij}^1 e_1 + c_{ij}^2 e_2$. A subset of $\mathcal{A}_2$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k.

The general linear group $\mathrm{GL}(V)$ acts on $\mathcal{A}_2$ by conjugation: for $x, y \in V$, $\mu \in \mathcal{A}_2$ and $g \in \mathrm{GL}(V)$,

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y).$$

Thus, $\mathcal{A}_2$ is decomposed into $\mathrm{GL}(V)$-orbits that correspond to the isomorphism classes of two-dimensional algebras. The classification of two-dimensional algebras up to isomorphism is equivalent to the classification of $\mathrm{GL}(V)$-orbits.

Let $O(\mu)$ denote the orbit of $\mu \in \mathcal{A}_2$ under the action of $\mathrm{GL}(V)$ and let $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$. Let A and B be two two-dimensional algebras and $\mu, \lambda \in \mathcal{A}_2$ represent A and B, respectively. We say that A degenerates to B and write $A \to B$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ. If $A \not\cong B$, then the assertion $A \to B$ is called a *proper degeneration*. We write $A \not\to B$ if $\lambda \notin \overline{O(\mu)}$. Now let $A(*) := \{A(\alpha)\}_{\alpha \in I}$ be a set of two-dimensional algebras and $\mu_\alpha \in \mathcal{A}_2$ represent $A(\alpha)$ for $\alpha \in I$. If $\lambda \in \overline{\{O(\mu_\alpha)\}_{\alpha \in I}}$, then we write $A(*) \to B$ and say that $A(*)$ degenerates to B. In the opposite case we write $A(*) \not\to B$.

Let $A(*)$, B, μ_α ($\alpha \in I$), and λ be as above. Let c_{ij}^k ($i, j, k = 1, 2$) be the structure constants of λ in the basis e_1, e_2. If we construct maps $a_i^j : \mathbf{k}^* \to \mathbf{k}$ ($i, j = 1, 2$) and $f : \mathbf{k}^* \to I$ such that $a_1^1(t)e_1 + a_1^2(t)e_2$ and $a_2^1(t)e_1 + a_2^2(t)e_2$ form a basis of V for any $t \in \mathbf{k}^*$, and the structure constants of $\mu_{f(t)}$ in this basis are polynomials $c_{ij}^k(t) \in \mathbf{k}[t]$ such that $c_{ij}^k(0) = c_{ij}^k$, then $A(*) \to B$. Indeed, if there is some closed subset $\mathcal{R}$ containing $O(\mu_\alpha)$ for all $\alpha \in I$, then it contains, in particular, $O(\mu_{f(t)})$ for all $t \in \mathbf{k}^*$, and hence the element λ_t of $\mathcal{A}_2$ with structure constants $c_{ij}^k(t)$ belongs to $\mathcal{R}$ for any $t \in \mathbf{k}^*$. Note that the assertion $\lambda_t \in \mathcal{R}$ is equivalent to the annihilation of some set polynomials in one variable in the point t. But if this set of polynomials vanishes for all $t \in \mathbf{k}^*$, then each of these polinomials has infinitely many roots, and hence it equals zero. Thus, $t = 0$ annihilates all the required polynomials, too, *i.e.*, $\lambda = \lambda_0 \in \mathcal{R}$. We will call $(a_1^1(t)e_1 + a_1^2(t)e_2, a_2^1(t)e_1 + a_2^2(t)e_2)$ and $f(t)$ a *parametrized basis* and a *parametrized index* for $A(*) \to B$, respectively. The case of degeneration between two algebras corresponds to the case $|I| = 1$. In this case we need only a parametrized basis, because $f(t)$ is the unique element of I for any $t \in \mathbf{k}^*$.

We follow [29] for proving non-degenerations. Let Q be a set of polynomial equations in the variables $x_{i,j}^k$ ($i, j, k = 1, 2$). Suppose that Q satisfies the following property: if $x_{i,j}^k = c_{ij}^k$ is a solution to all equations in Q, then also $x_{i,j}^k = \tilde{c}_{ij}^k$ is a solution to all equations in Q in the following cases:

(1) there are $\alpha_1, \alpha_2 \in \mathbf{k}^*$ such that $\tilde{c}_{ij}^k = \frac{\alpha_i \alpha_j}{\alpha_k} c_{ij}^k$;

(2) there is $\alpha \in \mathbf{k}$ such that

$$\begin{aligned}
\tilde{c}_{11}^1 &= c_{11}^1 + \alpha(c_{12}^1 + c_{21}^1) + \alpha^2 c_{22}^1,\\
\tilde{c}_{21}^1 &= c_{21}^1 + \alpha c_{22}^1,\\
\tilde{c}_{12}^1 &= c_{12}^1 + \alpha c_{22}^1,\\
\tilde{c}_{22}^1 &= c_{22}^1,\\
\tilde{c}_{11}^2 &= c_{11}^2 + \alpha(c_{12}^2 + c_{21}^2 - c_{11}^1) + \alpha^2(c_{22}^2 - c_{12}^1 - c_{21}^1) - \alpha^3 c_{22}^1,\\
\tilde{c}_{21}^2 &= c_{21}^2 + \alpha(c_{22}^2 - c_{21}^1) - \alpha^2 c_{22}^1,\\
\tilde{c}_{12}^2 &= c_{12}^2 + \alpha(c_{22}^2 - c_{12}^1) - \alpha^2 c_{22}^1,\\
\tilde{c}_{22}^2 &= c_{22}^2 - \alpha c_{22}^1.
\end{aligned}$$

Let $\mathcal{R} \subset \mathcal{A}_2$ be a set of all algebra structures whose structure constants satisfy all equations in Q. We will call such a set $\mathcal{R}$ a *closed upper invariant set*. Let $\{A(\alpha)\}_{\alpha \in I}$ be a set of two-dimensional algebras such that $A(\alpha)$ can be represented by a structure from $\mathcal{R}$ for any $\alpha \in I$. Let B be a two-dimensional algebra represented by the structure $\lambda \in \mathcal{A}_2$. If $O(\lambda) \cap \mathcal{R} = \varnothing$, then $A(*) \not\to B$. In this case we call $\mathcal{R}$ a *separating set* for $A(*) \not\to B$.

Let us recall two more tools for proving degenerations and non-degenerations. First, if $A \to B$, then $\dim \operatorname{Aut}(A) > \dim \operatorname{Aut}(B)$. Note that if $A(*) \to B$, then either $\dim \operatorname{Aut}(A(\alpha)) = \dim \operatorname{Aut}(B)$ for infinitely many $\alpha \in I$ or $\dim \operatorname{Aut}(A(\alpha)) < \dim \operatorname{Aut}(B)$ for some $\alpha \in I$, but it is possible that $\dim \operatorname{Aut}(A(\alpha)) \geq \dim \operatorname{Aut}(B)$ for all $\alpha \in I$. Note also that $\dim \operatorname{Aut}(A) = \dim \operatorname{Der}(A)$. Secondly, if $A \to C$ and $C \to B$, then $A \to B$. If there is no C such that $A \to C$ and $C \to B$ are proper degenerations, then the assertion $A \to B$ is called a *primary degeneration*. If there are no C and D such that $C \to A$, $B \to D$, $C \not\to D$, and one of the assertions $C \to A$ and $B \to D$ is a proper degeneration, then the assertion $A \not\to B$ is called a *primary non-degeneration*. It suffices to prove only primary degenerations and non-degenerations to describe degenerations in the variety under consideration. Note also that any algebra degenerates to the algebra with zero multiplication.

3 Algebraic Classification

The first of our aims is to classify all two-dimensional algebras over $\mathbf{k}$ modulo isomorphism. Our classification is based on the following lemma.

Lemma 3.1 *Let A be a two-dimensional algebra. Then there exists a nonzero element $x \in A$ such that x and x^2 are linearly dependent.*

Proof The required assertion is equivalent to the existence of a one-dimensional subalgebra in A. Then the lemma follows from the discussion immediately following [1, Proposition 1]. ■

Note that if $x \in A$ and x^2 are linearly dependent, then either $x^2 = 0$ or $x = \alpha e$ for some $\alpha \in \mathbf{k}^*$ and some $e \in A$ such that $e^2 = e$. If $x^2 = 0$, then x is called a 2-*nil element*. An element e such that $e^2 = e$ is called an *idempotent*.

Corollary 3.2 *Any two-dimensional* $\mathbf{k}$*-algebra belongs to one of the following disjoint classes.*

- **(A)** *Algebras that do not have nonzero idempotents and have a unique one-dimensional subspace of* 2*-nil elements.*
- **(B)** *Algebras that do not have nonzero idempotents and have two linearly independent* 2*-nil elements.*
- **(C)** *Algebras that have a unique nonzero idempotent and do not have nonzero* 2*-nil elements.*
- **(D)** *Algebras that have a unique nonzero idempotent and a nonzero* 2*-nil element.*
- **(E)** *Algebras that have two different nonzero idempotents.*

Proof The fact that the classes are disjoint is obvious. The fact that any two-dimensional algebra belongs to one of the classes follows easily from Lemma 3.1 and the remark after it. ■

To give the classification of two-dimensional algebras we need to introduce some notation. Let us consider the action of the cyclic group $C_2 = \langle \rho \mid \rho^2 \rangle$ on $\mathbf{k}$ defined by the equality ${}^{\rho}\alpha = -\alpha$ for $\alpha \in \mathbf{k}$. Let us fix some set of representatives of orbits under this action and denote it by $\mathbf{k}_{\geq 0}$. For example, if $\mathbf{k} = \mathbb{C}$, then one can take $\mathbb{C}_{\geq 0} = \{\alpha \in \mathbb{C} \mid \mathrm{Re}(\alpha) > 0\} \cup \{\alpha \in \mathbb{C} \mid \mathrm{Re}(\alpha) = 0, \mathrm{Im}(\alpha) \geq 0\}$.

Let us also consider the action of C_2 on $\mathbf{k}^2$ defined by the equality ${}^{\rho}(\alpha, \beta) = (1 - \alpha + \beta, \beta)$ for $(\alpha, \beta) \in \mathbf{k}^2$. Let us fix some set of representatives of orbits under this action and denote it by $\mathcal{U}$. Let us also define $\mathcal{T} = \{(\alpha, \beta) \in \mathbf{k}^2 \mid \alpha + \beta = 1\}$.

Given $(\alpha, \beta, \gamma, \delta) \in \mathbf{k}^4$, we define $\mathcal{D}(\alpha, \beta, \gamma, \delta) = (\alpha + \gamma)(\beta + \delta) - 1$. We define

$$\mathcal{C}_1(\alpha, \beta, \gamma, \delta) = (\beta, \delta), \quad \mathcal{C}_2(\alpha, \beta, \gamma, \delta) = (\gamma, \alpha),$$
$$\mathcal{C}_3(\alpha, \beta, \gamma, \delta) = \Big(\frac{\beta\gamma - (\alpha - 1)(\delta - 1)}{\mathcal{D}(\alpha, \beta, \gamma, \delta)}, \frac{\alpha\delta - (\beta - 1)(\gamma - 1)}{\mathcal{D}(\alpha, \beta, \gamma, \delta)} \Big)$$

for $(\alpha, \beta, \gamma, \delta)$ such that $\mathcal{D}(\alpha, \beta, \gamma, \delta) \neq 0$. Let us consider the set

$$X = \Big\{ \big(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma) \big) \mid \Gamma \in \mathbf{k}^4, \mathcal{D}(\Gamma) \neq 0, \mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma) \notin \mathcal{T} \Big\} \subset (\mathbf{k}^2)^3.$$

One can show that the symmetric group S_3 acts on X by the equality

$${}^{\sigma}\big(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma) \big) = \big(\mathcal{C}_{\sigma^{-1}(1)}(\Gamma), \mathcal{C}_{\sigma^{-1}(2)}(\Gamma), \mathcal{C}_{\sigma^{-1}(3)}(\Gamma) \big) \text{ for } \sigma \in S_3.$$

Indeed, suppose that $\big(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma) \big) \in X$ for some $\Gamma = (\alpha, \beta, \gamma, \delta)$. We need to show that ${}^{\sigma}\big(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma) \big) \in X$ for any $\sigma \in S_3$. We will check this for σ interchanging 1 and 3; the other verifications are analogous. First, $\mathcal{C}_3(\Gamma) \in \mathcal{T}$ is equivalent to the equality $\alpha + \beta + \delta + \gamma - 2 = \mathcal{D}(\Gamma)$, which can be rewritten in the form $(\alpha + \gamma - 1)(\beta + \delta - 1) = 0$. It is clear that the last equality is not valid. Let us introduce

$$\Gamma' = \Big(\alpha, \frac{\beta\gamma - (\alpha - 1)(\delta - 1)}{\mathcal{D}(\Gamma)}, \gamma, \frac{\alpha\delta - (\beta - 1)(\gamma - 1)}{\mathcal{D}(\Gamma)} \Big).$$

It remains to check that $\mathcal{D}(\Gamma') \neq 0$ and $\mathcal{C}_3(\Gamma') = (\beta, \delta)$. The equality $\mathcal{D}(\Gamma') = 0$ is equivalent to the equality $(\alpha + \beta + \delta + \gamma - 2)(\alpha + \gamma) = \mathcal{D}(\Gamma)$, which can be rewritten in the form $(\alpha + \gamma - 1)^2 = 0$. Hence, we get $\mathcal{D}(\Gamma') \neq 0$. To prove that $\mathcal{C}_3(\Gamma') = (\beta, \delta)$ we need to verify two equalities. We will consider only the first equality; the second one is analogous. Thus, it remains to show that

$$\frac{\beta\gamma - (\alpha - 1)(\delta - 1)}{\mathcal{D}(\Gamma)}\gamma - (\alpha - 1)\Big(\frac{\alpha\delta - (\beta - 1)(\gamma - 1)}{\mathcal{D}(\Gamma)} - 1 \Big)$$
$$= \beta\Big(\frac{(\alpha + \beta + \gamma + \delta - 2)(\alpha + \gamma)}{\mathcal{D}(\Gamma)} - 1 \Big).$$

Multiplying by $\mathcal{D}(\Gamma)$ and reducing all the equal terms, one sees that the last equality is valid. Note that there exists a set of representatives of orbits $\tilde{\mathcal{V}}$ under the action of S_3 on X such that if $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in \tilde{\mathcal{V}}$ and $\mathcal{C}_1 \neq \mathcal{C}_2$, then $\mathcal{C}_3 \neq \mathcal{C}_1, \mathcal{C}_2$. Let us fix such $\tilde{\mathcal{V}}$ and define

$$\mathcal{V} = \Big\{ \Gamma \in \mathbf{k}^4 \mid \mathcal{D}(\Gamma) \neq 0; \mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma) \notin \mathcal{T}, \big(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma) \big) \in \tilde{\mathcal{V}} \Big\}.$$

For $\Gamma \in \mathcal{V}$, we also define $\mathcal{C}(\Gamma) = \{\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma)\} \subset \mathbf{k}^2$.

Let us consider the action of the cyclic group C_2 on $\mathbf{k}^* \setminus \{1\}$ defined by the equality ${}^{\rho}\alpha = \alpha^{-1}$ for $\alpha \in \mathbf{k}^* \setminus \{1\}$. Let us fix some set of representatives of orbits under this action and denote it by $\mathbf{k}^*_{>1}$. For example, if $\mathbf{k} = \mathbb{C}$, then one can take $\mathbb{C}^*_{>1} = \{\alpha \in \mathbb{C}^* \mid |\alpha| > 1\} \cup \{\alpha \in \mathbb{C}^* \mid |\alpha| = 1, 0 < \arg(\alpha) \le \pi\}$. For $(\alpha, \beta, \gamma) \in \mathbf{k}^2 \times \mathbf{k}^*_{>1}$ we define

$$\mathcal{C}(\alpha, \beta, \gamma) = \Big\{ (\alpha\gamma, (1-\alpha)\gamma), \Big(\frac{\beta}{\gamma}, \frac{1-\beta}{\gamma} \Big) \Big\} \subset \mathbf{k}^2.$$

Let $\mathcal{F} \subset \mathcal{A}_2$ be the set formed by the algebra structures on the vector space V listed in Table 1. This section is devoted to the proof of the following theorem that gives a classification of two-dimensional algebras over $\mathbf{k}$ up to isomorphism.

Theorem 3.3 *Any non-trivial two-dimensional* $\mathbf{k}$*-algebra can be represented by a unique structure from* $\mathcal{F}$.

In other words, Theorem 3.3 states that $\mathcal{A}_2 = \bigcup_{\mu \in \mathcal{F}} O(\mu) \cup \{\mathbf{k}^2\}$ and that, if $\mu, \lambda \in \mathcal{F}$ are different structures, then $O(\mu) \cap O(\lambda) = \varnothing$. Whenever an algebra named A appears in this section, we suppose that it is represented by some structure from $\mathcal{A}_2$ with structure constants c_{ij}^k $(i, j, k = 1, 2)$. According to Corollary 3.2, it suffices to consider each of the classes **A**–**E** separately. It is not difficult to show that the letter in the name of an algebra from $\mathcal{F}$ corresponds to its class in each case. This will follow also from our proofs.

Lemma 3.4 *If A belongs to the class* **A***, then it can be represented by a unique structure from the set*

$$\big\{ \mathbf{A}_1(\alpha) \big\}_{\alpha \in \mathbf{k}} \cup \{\mathbf{A}_2\} \cup \{\mathbf{A}_3\} \cup \big\{ \mathbf{A}_4(\alpha) \big\}_{\alpha \in \mathbf{k}_{\ge 0}}. \tag{3.1}$$

Proof Let us represent the algebra A by a structure such that $e_2 e_2 = 0$. It is easy to see that A belongs to the class **A** if and only if $x_t = e_1 + te_2$ and x_t^2 are linearly independent for any $t \in \mathbf{k}$. Since

$$x_t^2 = \big(c_{11}^1 + (c_{12}^1 + c_{21}^1) t \big) e_1 + \big(c_{11}^2 + (c_{12}^2 + c_{21}^2) t \big) e_2,$$

x_t and x_t^2 are linearly independent if and only if

$$0 \ne \begin{vmatrix} c_{11}^1 + (c_{12}^1 + c_{21}^1) t & c_{11}^2 + (c_{12}^2 + c_{21}^2) t \\ 1 & t \end{vmatrix} = (c_{12}^1 + c_{21}^1) t^2 + (c_{11}^1 - c_{12}^2 - c_{21}^2) t - c_{11}^2.$$

Since by our assumption x_t and x_t^2 are linearly independent for any $t \in \mathbf{k}$, we have $c_{12}^1 + c_{21}^1 = 0$, $c_{11}^1 = c_{12}^2 + c_{21}^2$, and $c_{11}^2 \ne 0$.

Now we have four cases.

- $c_{12}^1 = 0$, $c_{11}^1 \ne 0$. Considering the basis $\frac{e_1}{c_{11}^1}, \frac{c_{11}^2 e_2}{(c_{11}^1)^2}$ of V, one can check that A can be represented by $\mathbf{A}_1\big(\frac{c_{21}^2}{c_{11}^1}\big)$.
- $c_{12}^1 = 0$, $c_{12}^2 = -c_{21}^2 \ne 0$. Considering the basis $\frac{e_1}{c_{12}^2}, \frac{c_{11}^2 e_2}{(c_{12}^2)^2}$ of V, one can check that A can be represented by $\mathbf{A}_2$.
- $c_{12}^1 = c_{12}^2 = c_{21}^2 = 0$. Considering the basis $e_1, c_{11}^2 e_2$ of V, one can check that A can be represented by $\mathbf{A}_3$.

- $c_{12}^1 \neq 0$. Let $a \in \mathbf{k}^*$ be such that $c_{11}^2 c_{12}^1 a^2 = 1$ and $c_{11}^1 a \in \mathbf{k}_{\geq 0}$. Considering the basis $a\big(e_1 - \frac{c_{21}^2}{c_{12}^1} e_2\big), \frac{e_2}{c_{12}^1}$ of V, one can check that A can be represented by $\mathbf{A}_4(c_{11}^1 a)$.

It remains to prove that any two different structures from the set (3.1) represent non-isomorphic algebras. First, note that $\dim(\mathbf{A}_2)^2 = \dim(\mathbf{A}_3)^2 = 1$ while dim $(\mathbf{A}_1(\alpha))^2 = \dim(\mathbf{A}_4(\alpha))^2 = 2$ for any $\alpha \in \mathbf{k}$. We also have $\mathbf{A}_2 \not\cong \mathbf{A}_3$, because $\mathbf{A}_3$ has a nonzero annihilator.

Suppose that A is represented by the structure $\mathbf{A}_1(\alpha)$ for some $\alpha \in \mathbf{k}$. Then there exists $x \in A$ such that $x^2 = 0$, $xA + Ax \subset \langle x \rangle$, and $\alpha xy = (1-\alpha)yx$ for any $y \in A$. Such an element does not exist in $\mathbf{A}_4(\beta)$ for any $\beta \in \mathbf{k}$ nor in $\mathbf{A}_1(\beta)$ for any $\beta \in \mathbf{k} \setminus \{\alpha\}$.

Suppose that A is represented by the structure $\mathbf{A}_4(\alpha)$ for some $\alpha \in \mathbf{k}_{\geq 0}$. Suppose that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{A}_4(\beta)$ for some $\beta \in \mathbf{k}_{\geq 0}$. Since $E_2E_2 = 0$ and $E_2E_1 = -E_1$, it is easy to see that $E_2 = e_2$ and $E_1 = ae_1$ for some $a \in \mathbf{k}^*$. Then we obtain from the equality $E_1E_1 = \beta E_1 + E_2$ that $a = \pm 1$ and $\beta = \pm\alpha$. Since $\alpha, \beta \in \mathbf{k}_{\geq 0}$, we have $\beta = \alpha$. ∎

Lemma 3.5 *If A belongs to the class $\mathbf{B}$, then it can be represented by a unique structure from the set*

$$\{\mathbf{B}_1(\alpha)\}_{\alpha\in\mathbf{k}} \cup \{\mathbf{B}_2(\alpha)\}_{\alpha\in\mathbf{k}} \cup \{\mathbf{B}_3\}. \tag{3.2}$$

Proof Let us represent the algebra A by a structure such that $e_1e_1 = e_2e_2 = 0$. For $s, t \in \mathbf{k}$, let us define $x_{s,t} = se_1 + te_2$. Suppose that there are $s, t \in \mathbf{k}^*$ such that $0 = x_{s,t}^2 = st(e_1e_2 + e_2e_1)$. Then $e_1e_2 + e_2e_1 = 0$ and A is anticommutative. It is easy to see that any two-dimensional anticommutative algebra either has the trivial multiplication or can be represented by $\mathbf{B}_3$ (note that by our definition A is anticommutative if and only if $x^2 = 0$ for any $x \in A$).

Suppose now that $x_{s,t}^2 \neq 0$ for any $s, t \in \mathbf{k}^*$. Since A does not have idempotents, $x_{s,t}$ and $x_{s,t}^2$ are linearly independent for $s, t \in \mathbf{k}^*$. It is easy to check that $x_{s,t}$ and $x_{s,t}^2$ are linearly dependent for $s = c_{12}^1 + c_{21}^1$, $t = c_{12}^2 + c_{21}^2$. Hence, $c_{12}^1 + c_{21}^1 = 0$ or $c_{12}^2 + c_{21}^2 = 0$. Without loss of generality we may assume that $c_{12}^2 + c_{21}^2 = 0$. Since A is not anticommutative, we have $c_{12}^1 + c_{21}^1 \neq 0$ in this case.

If $c_{12}^2 \neq 0$, then, considering the basis $\frac{e_1}{c_{12}^2}, \frac{e_2}{c_{12}^1 + c_{21}^1}$ of V, one can check that A can be represented by $\mathbf{B}_1\big(\frac{c_{21}^1}{c_{12}^1 + c_{21}^1}\big)$. If $c_{12}^2 = 0$, then, considering the basis $e_1, \frac{e_2}{c_{12}^1 + c_{21}^1}$ of V, one can check that A can be represented by $\mathbf{B}_2\big(\frac{c_{21}^1}{c_{12}^1 + c_{21}^1}\big)$.

It remains to prove that any two different structures from the set (3.2) represent non-isomorphic algebras. Since $\mathbf{B}_3$ is anticommutative, it is not isomorphic to other algebras from (3.2). Note also that $\dim(\mathbf{B}_1(\alpha))^2 = 2 > 1 = \dim(\mathbf{B}_2(\beta))^2$ for any $\alpha, \beta \in \mathbf{k}$.

Suppose that A is represented by the structure $\mathbf{B}_i(\alpha)$ for some $\alpha \in \mathbf{k}$ and $i = 1, 2$. Suppose that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{B}_i(\beta)$ for some $\beta \in \mathbf{k}$. Since $E_1E_1 = E_2E_2 = 0$, we have either $E_1 = ae_1$, $E_2 = be_2$ or $E_1 = ae_2$, $E_2 = be_1$ for some $a, b \in \mathbf{k}^*$. Since $E_1E_2 + E_2E_1 = E_1$, we have $E_1 = ae_1$ and $E_2 = e_2$. Then we get $\beta = \alpha$ from the equality $E_1E_2 = (1-\beta)E_1 + (2-i)E_2$. ∎

Lemma 3.6 *If A belongs to the class $\mathbf{C}$, then it can be represented by $\mathbf{C}(\alpha,\beta)$ for a unique pair $(\alpha,\beta)\in\mathbf{k}\times\mathbf{k}_{\geq 0}$.*

Proof Let us represent the algebra A by a structure such that $e_2e_2 = e_2$. It is easy to see that A belongs to the class $\mathbf{C}$ if and only if $x_t = e_1 + te_2$ and x_t^2 are linearly independent for any $t\in\mathbf{k}$. Since

$$x_t^2 = (c_{11}^1 + (c_{12}^1 + c_{21}^1)t)e_1 + (c_{11}^2 + (c_{12}^2 + c_{21}^2)t + t^2)e_2,$$

x_t and x_t^2 are linearly independent if and only if

$$0 \neq \begin{vmatrix} c_{11}^1 + (c_{12}^1 + c_{21}^1)t & c_{11}^2 + (c_{12}^2 + c_{21}^2)t + t^2 \\ 1 & t \end{vmatrix}$$
$$= (c_{12}^1 + c_{21}^1 - 1)t^2 + (c_{11}^1 - c_{12}^2 - c_{21}^2)t - c_{11}^2.$$

Since by our assumption x_t and x_t^2 are linearly independent for any $t\in\mathbf{k}$, we have $c_{12}^1 + c_{21}^1 = 1$, $c_{11}^1 = c_{12}^2 + c_{21}^2$, and $c_{11}^2 \neq 0$.

Let a be such an element of $\mathbf{k}^*$ that $c_{11}^2a^2 = 1$ and $a(c_{12}^2 - c_{11}^1c_{21}^1)\in\mathbf{k}_{\geq 0}$. Considering the basis $a(e_1 - c_{11}^1e_2)$, e_2 of V, one can check that A can be represented by $\mathbf{C}\big(c_{21}^1, a(c_{12}^2 - c_{11}^1c_{21}^1)\big)$.

Suppose that A is represented by the structure $\mathbf{C}(\alpha,\beta)$ for some pair $(\alpha,\beta)\in\mathbf{k}\times\mathbf{k}_{\geq 0}$. Suppose that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{C}(\gamma,\delta)$ for some $(\gamma,\delta)\in\mathbf{k}\times\mathbf{k}_{\geq 0}$. Since $E_2E_2 = E_2$ and $\mathbf{C}(\alpha,\beta)$ has a unique idempotent, we have $E_2 = e_2$. We get $E_1 = \pm e_1$ from the equality $E_1E_1 = E_2$. Then $\gamma = \alpha$ and $\delta = \pm\beta$. Since $\beta,\delta\in\mathbf{k}_{\geq 0}$, we have $(\gamma,\delta) = (\alpha,\beta)$. ■

Lemma 3.7 *If A belongs to the class $\mathbf{D}$, then it can be represented by a unique structure from the set*

$$\{\mathbf{D}_1(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{U}} \cup \{\mathbf{D}_2(\alpha,\beta)\}_{(\alpha,\beta)\in\mathbf{k}^2\setminus\mathcal{T}} \cup \{\mathbf{D}_3(\alpha,\beta)\}_{(\alpha,\beta)\in\mathbf{k}^2\setminus\mathcal{T}}. \tag{3.3}$$

Proof Let us represent the algebra A by a structure such that $e_1e_1 = e_1$ and $e_2e_2 = 0$.

Let us consider the following cases.

- $c_{12}^1 + c_{21}^1 \neq 0$. If $c_{12}^2 + c_{21}^2 \neq 0$, then one can check that

  $$\frac{1}{c_{12}^2 + c_{21}^2}\Big(e_1 + \frac{c_{12}^2 + c_{21}^2 - 1}{c_{12}^1 + c_{21}^1}e_2\Big)$$

  is an idempotent that is not equal to e_1. Thus, $c_{12}^2 + c_{21}^2 = 0$. If $\big(\frac{c_{12}^1}{c_{12}^1+c_{21}^1}, c_{12}^2\big)\in\mathcal{U}$, then, considering the basis e_1, $\frac{e_2}{c_{12}^1+c_{21}^1}$ of V, one can check that A can be represented by $\mathbf{D}_1\big(\frac{c_{12}^1}{c_{12}^1+c_{21}^1}, c_{12}^2\big)$. If $\big(\frac{c_{12}^1}{c_{12}^1+c_{21}^1}, c_{12}^2\big)\notin\mathcal{U}$, then

  $$\Big(\frac{c_{21}^1}{c_{12}^1 + c_{21}^1} + c_{12}^2, c_{12}^2\Big)\in\mathcal{U}$$

  and, considering the basis e_1, $e_1 - \frac{e_2}{c_{12}^1+c_{21}^1}$ of V, one can check that A can be represented by $\mathbf{D}_1\big(\frac{c_{21}^1}{c_{12}^1+c_{21}^1} + c_{12}^2, c_{12}^2\big)$.

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- $c_{12}^1 = -c_{21}^1 \neq 0$. Considering the basis $e_1, \frac{e_2}{c_{12}^1}$ of V, one can check that A can be represented by $\mathbf{D}_3(c_{12}^2, c_{21}^2)$. Since $e_1 + e_2$ is not idempotent, $(c_{12}^2, c_{21}^2) \notin \mathfrak{T}$.
- $c_{12}^1 = c_{21}^1 \neq 0$. Then one can check that A is represented by $\mathbf{D}_2(c_{12}^2, c_{21}^2)$. Since $e_1 + e_2$ is not idempotent, $(c_{12}^2, c_{21}^2) \notin \mathfrak{T}$.

It remains to prove that any two different structures from the set (3.3) represent non-isomorphic algebras.

Suppose that A is represented by the structure $\mathbf{D}_1(\alpha, \beta)$ for some pair $(\alpha, \beta) \in \mathfrak{U}$. Note that $e_2 e_2 = (e_1 - e_2)^2 = 0$ in $\mathbf{D}_1(\alpha, \beta)$ while the structures $\mathbf{D}_2(\gamma, \delta)$ and $\mathbf{D}_3(\gamma, \delta)$ have a unique one-dimensional subspace of 2-nil elements for any pair $(\gamma, \delta) \in \mathbf{k}^2$. Suppose now that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{D}_1(\gamma, \delta)$ for some pair $(\gamma, \delta) \in \mathfrak{U}$. Since E_1 is an idempotent and $\mathbf{D}_1(\alpha, \beta)$ has a unique idempotent, we have $E_1 = e_1$. Since $E_2 E_2 = 0$, we have either $E_2 = a e_2$ or $E_2 = a(e_1 - e_2)$ for some $a \in \mathbf{k}^*$. We obtain $a = 1$ in both cases from the equality $E_1 E_2 + E_2 E_1 = E_1$. Then we have $\delta = \beta$ and either $\gamma = \alpha$ or $\gamma = 1 - \alpha + \beta$. Since $(\alpha, \beta), (\gamma, \delta) \in \mathfrak{U}$, we have $(\gamma, \delta) = (\alpha, \beta)$.

Suppose that A is represented by the structure $\mathbf{D}_2(\alpha, \beta)$ for some pair $(\alpha, \beta) \in \mathbf{k}^2$. Note that A has an element x such that $x^2 = 0$ and $xA + Ax \subset \langle x \rangle$ while $\mathbf{D}_3(\gamma, \delta)$ does not have such an element for any pair $(\gamma, \delta) \in \mathbf{k}^2$ because any square zero element of $\mathbf{D}_3(\gamma, \delta)$ is linearly dependent with e_2. Suppose now that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{D}_2(\gamma, \delta)$ for some pair $(\gamma, \delta) \in \mathbf{k}^2$. Since E_1 is an idempotent and $\mathbf{D}_2(\alpha, \beta)$ has a unique idempotent, we have $E_1 = e_1$. Since $E_2 E_2 = 0$, we have $E_2 = a e_2$ for some $a \in \mathbf{k}^*$. Then it is easy to see that $(\gamma, \delta) = (\alpha, \beta)$.

Finally, suppose that A is represented by the structure $\mathbf{D}_3(\alpha, \beta)$ for some pair $(\alpha, \beta) \in \mathbf{k}^2$. Suppose that the structure constants of A in the basis E_1, E_2 equal the structure constants of $\mathbf{D}_3(\gamma, \delta)$ for some pair $(\gamma, \delta) \in \mathbf{k}^2$. Since E_1 is an idempotent and $\mathbf{D}_3(\alpha, \beta)$ has a unique idempotent, we have $E_1 = e_1$. Since $E_2 E_2 = 0$, we have $E_2 = a e_2$ for some $a \in \mathbf{k}^*$. Then it is easy to see that $a = 1$ and $(\gamma, \delta) = (\alpha, \beta)$. ∎

As a consequence of the proofs of Lemmas 3.4–3.7 we can describe the automorphism groups of the algebras of the classes **A**–**D**.

Corollary 3.8 (i) $\operatorname{Aut}(\mathbf{A}_1(\alpha)) \cong \operatorname{Aut}(\mathbf{A}_2)$ *is isomorphic to the additive group of* $\mathbf{k}$.

(ii) $\operatorname{Aut}(\mathbf{A}_3)$ *is isomorphic to the subgroup of* $GL_2(\mathbf{k})$ *formed by matrices of the form* $\left(\begin{smallmatrix} a & 0 \\ b & a^2 \end{smallmatrix}\right)$, *where* $a \in \mathbf{k}^*$ *and* $b \in \mathbf{k}$.

(iii) $\operatorname{Aut}(\mathbf{A}_4(\alpha)) \cong C_2$ *if* $\alpha = 0$ *and* $\operatorname{char}\mathbf{k} \neq 2$; $\operatorname{Aut}(\mathbf{A}_4(\alpha))$ *is trivial if either* $\alpha \in \mathbf{k}^*$ *or* $\alpha = 0$ *and* $\operatorname{char}\mathbf{k} = 2$.

(iv) $\operatorname{Aut}(\mathbf{B}_1(\alpha))$ *is trivial;* $\operatorname{Aut}(\mathbf{B}_2(\alpha)) \cong \mathbf{k}^*$.

(v) $\operatorname{Aut}(\mathbf{B}_3)$ *is isomorphic to the subgroup of* $\mathrm{GL}_2(\mathbf{k})$ *formed by matrices of the form* $\left(\begin{smallmatrix} 1 & 0 \\ b & a \end{smallmatrix}\right)$, *where* $a \in \mathbf{k}^*$ *and* $b \in \mathbf{k}$.

(vi) $\operatorname{Aut}(\mathbf{C}(\alpha, \beta)) \cong C_2$ *if* $\beta = 0$ *and* $\operatorname{char}\mathbf{k} \neq 2$; $\operatorname{Aut}(\mathbf{C}(\alpha, \beta))$ *is trivial if either* $\beta \in \mathbf{k}^*$ *or* $\beta = 0$ *and* $\operatorname{char}\mathbf{k} = 2$.

(vii) $\operatorname{Aut}(\mathbf{D}_1(\alpha, \beta)) \cong C_2$ *if* $\beta = 2\alpha - 1$ *and* $\operatorname{Aut}(\mathbf{C}(\alpha, \beta))$ *is trivial if* $\beta \neq 2\alpha - 1$.

(viii) $\operatorname{Aut}(\mathbf{D}_2(\alpha, \beta)) \cong \mathbf{k}^*$ *and* $\operatorname{Aut}(\mathbf{D}_3(\alpha, \beta))$ *is trivial if* $\alpha + \beta \neq 1$.

In particular,

$$\begin{aligned}\dim \mathrm{Aut}(\mathbf{A}_4(\alpha)) &= \dim \mathrm{Aut}(\mathbf{B}_1(\alpha)) = \dim \mathrm{Aut}(\mathbf{C}_1(\alpha,\beta)) \\ &= \dim \mathrm{Aut}(\mathbf{D}_1(\alpha,\beta)) = \dim \mathrm{Aut}(\mathbf{D}_3(\alpha,\beta)) = 0, \\ \dim \mathrm{Aut}(\mathbf{A}_1(\alpha)) &= \dim \mathrm{Aut}(\mathbf{A}_2) = \dim \mathrm{Aut}(\mathbf{B}_2(\alpha)) = \dim \mathrm{Aut}(\mathbf{D}_2(\alpha,\beta)) = 1, \\ \dim \mathrm{Aut}(\mathbf{A}_3) &= \dim \mathrm{Aut}(\mathbf{B}_3) = 2.\end{aligned}$$

Proof (i)–(iii). Any structure of the class $\mathbf{A}$ has a unique subspace of 2-nil elements generated by e_2. Thus, any automorphism of such an algebra sends e_1 and e_2 to $ae_1 + be_2$ and ce_2, respectively, where $a, c \in \mathbf{k}^*$ and $b \in \mathbf{k}$. It is easy to check that $a = c = 1$ for $\mathbf{A}_1(\alpha)$ and $\mathbf{A}_2$; $c = a^2$ for $\mathbf{A}_3$; $a = \pm 1$, $b = 0$, and $c = 1$ for $\mathbf{A}_4(0)$; and $a = c = 1$, $b = 0$ for $\mathbf{A}_4(\alpha)$ if $\alpha \neq 0$.

(iv) It follows from the proof of Lemma 3.5 that any automorphism of the algebra $\mathbf{B}_i(\alpha)$, where $i \in \{1,2\}$, sends e_1 and e_2 to ae_1 and e_2, respectively, for some $a \in \mathbf{k}^*$. It is easy to see that $a = 1$ for $i = 1$ and a can be arbitrary for $i = 2$.

(v) Since $\mathbf{B}_3(V, V)$ is generated by e_2, any automorphism of $\mathbf{B}_3$ sends e_1 and e_2 to $ae_1 + be_2$ and ce_2, respectively, where $a, c \in \mathbf{k}^*$ and $b \in \mathbf{k}$. It is easy to see that such a map is an automorphism if and only if $a = 1$.

(vi) It follows from the proof of Lemma 3.6 that any automorphism of the algebra $\mathbf{C}(\alpha,\beta)$ sends e_1 and e_2 to $\pm e_1$ and e_2, respectively. It is easy to see that the map that sends e_1 and e_2 to $-e_1$ and e_2 respectively is an automorphism if and only if $\beta = 0$ or $\operatorname{char} \mathbf{k} = 2$.

(vii) It follows from the proof of Lemma 3.7 that any automorphism of the algebra $\mathbf{D}_1(\alpha,\beta)$ sends e_1 to e_1 and sends e_2 either to e_2 or $e_1 - e_2$. It is easy to see that the map that sends e_1 and e_2 to e_1 and $e_1 - e_2$, respectively, is an automorphism if and only if $\beta = 2\alpha - 1$.

(viii) It follows from the proof of Lemma 3.7 that any automorphism of the algebra $\mathbf{D}_2(\alpha,\beta)$ sends e_1 and e_2 to e_1 and ae_2 respectively for some $a \in \mathbf{k}^*$. It follows from the same proof that any automorphism of the algebra $\mathbf{D}_3(\alpha,\beta)$ is trivial. ∎

We will finish the proof of Theorem 3.3 in the next section devoted to the algebras of the class $\mathbf{E}$.

4 Algebras of the Class E

In this section we consider the algebras of the class $\mathbf{E}$. It is clear that such an algebra is isomorphic to $\mathbf{E}_1(\Gamma)$ for some $\Gamma \in \mathbf{k}^4$. First, we describe isomorphisms inside this set and, thus, finish the proof of Theorem 3.3.

Lemma 4.1 $\mathbf{E}_1(\Gamma_1) \cong \mathbf{E}_1(\Gamma_2)$ *if and only if one of the following conditions holds.*

(i) $\Gamma_1 = \Gamma_2$;
(ii) $\mathcal{C}_1(\Gamma_1) = \mathcal{C}_2(\Gamma_2)$ *and* $\mathcal{C}_2(\Gamma_1) = \mathcal{C}_1(\Gamma_2)$;
(iii) $\mathcal{C}_1(\Gamma_1), \mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_1), \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, $\mathcal{C}_1(\Gamma_1) \neq \mathcal{C}_2(\Gamma_1)$, $\mathcal{C}_1(\Gamma_2) \neq \mathcal{C}_2(\Gamma_2)$;
(iv) $\mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_2) \notin \mathcal{T}$, $\mathcal{D}(\Gamma_2) \neq 0$, *and there is some* $\sigma \in S_3$ *such that* $\mathcal{C}_i(\Gamma_1) = \mathcal{C}_{\sigma(i)}(\Gamma_2)$ *for* $i \in \{1,2,3\}$.

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Proof Suppose that $g \in \mathrm{GL}(V)$ is such that $g * \mathbf{E}_1(\Gamma_1) = \mathbf{E}_1(\Gamma_2)$. Then ge_1 and ge_2 are two linearly independent idempotents of $\mathbf{E}_1(\Gamma_2)$. Let us describe all nonzero idempotents of this algebra. Let $\Gamma_2 = (\alpha, \beta, \gamma, \delta)$ and $u = xe_1 + ye_2$ be some element of V. Then $\mathbf{E}_1(\Gamma_2)(u,u) = u$ if and only if

$$x = x^2 + (\alpha + \gamma)xy, \quad y = (\beta + \delta)xy + y^2.$$

The solutions $(x,y) = (0,0), (1,0), (0,1)$ give the obvious idempotents 0, e_1 and e_2. All the other pairs (x,y) satisfying the obtained equations are the solutions of the system of linear equations

$$\begin{cases} (\beta + \delta)x + y = 1, \\ x + (\alpha + \gamma)y = 1 \end{cases}$$

with the additional conditions $x \neq 0$ and $y \neq 0$. Let us consider the following cases.

- $\mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, *i.e.*, $\alpha + \gamma = \beta + \delta = 1$. In this case $\mathbf{E}_1(\Gamma_2)(u,u) = u$ if and only if either $u = 0$ or $x + y = 1$. Thus, $ge_1 = ae_1 + (1-a)e_2$ and $ge_2 = be_1 + (1-b)e_2$ for two different $a, b \in \mathbf{k}$. One can check that in this case
$$\Gamma_1 = \big((1-b)\alpha + b\delta, a\beta + (1-a)\gamma, b\beta + (1-b)\gamma, (1-a)\alpha + a\delta \big).$$
If $\mathcal{C}_1(\Gamma_2) = \mathcal{C}_2(\Gamma_2)$, *i.e.*, $(\beta,\delta) = (\gamma,\alpha)$, then we get $\Gamma_1 = \Gamma_2$. If $\mathcal{C}_1(\Gamma_2) \neq \mathcal{C}_2(\Gamma_2)$, then the formula above gives all the possible Γ_1 with $\mathcal{C}_1(\Gamma_1), \mathcal{C}_2(\Gamma_1) \in \mathcal{T}$, and $\mathcal{C}_1(\Gamma_1) \neq \mathcal{C}_2(\Gamma_1)$.
- One of the following three conditions holds.
  - $\mathcal{C}_1(\Gamma_2) \in \mathcal{T}, \mathcal{C}_2(\Gamma_2) \notin \mathcal{T}$, *i.e.*, $\beta + \delta = 1, \alpha + \gamma \neq 1$.
  - $\mathcal{C}_1(\Gamma_2) \notin \mathcal{T}, \mathcal{C}_2(\Gamma_2) \in \mathcal{T}$, *i.e.*, $\beta + \delta \neq 1, \alpha + \gamma = 1$.
  - $\mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_2) \notin \mathcal{T}$, $\mathcal{D}(\Gamma_2) = 0$, *i.e.*, $\beta + \delta, \alpha + \gamma \neq 1$ and our system has zero determinant.

  It is easy to see that in all of these cases our system of linear equations does not have solutions satisfying the additional conditions, *i.e.*, e_1 and e_2 are all the nonzero idempotents of $\mathbf{E}_1(\Gamma_2)$. Thus, either $ge_1 = e_1$, $ge_2 = e_2$, and $\Gamma_1 = \Gamma_2$ or $ge_1 = e_2$, $ge_2 = e_1$, $\mathcal{C}_1(\Gamma_1) = \mathcal{C}_2(\Gamma_2)$, and $\mathcal{C}_2(\Gamma_1) = \mathcal{C}_1(\Gamma_2)$.
- $\mathcal{C}_1(\Gamma_2), \mathcal{C}_2(\Gamma_2) \notin \mathcal{T}$, $\mathcal{D}(\Gamma_2) \neq 0$, *i.e.*, $\beta + \delta \neq 1, \alpha + \gamma \neq 1$ and our system has nonzero determinant. In this case $(x,y) = \big(\frac{\alpha+\gamma-1}{\mathcal{D}(\Gamma_2)}, \frac{\beta+\delta-1}{\mathcal{D}(\Gamma_2)}\big)$ is the unique solution of our system of linear equations. Hence, e_1, e_2, and $e_3 = \frac{\alpha+\gamma-1}{\mathcal{D}(\Gamma_2)}e_1 + \frac{\beta+\delta-1}{\mathcal{D}(\Gamma_2)}e_2$ are all the nonzero idempotents of $\mathbf{E}_1(\Gamma_2)$. Thus, there is $\sigma \in S_3$ such that $ge_i = e_{\sigma(i)}$ for $i \in \{1,2\}$. Then direct calculations show that $\mathcal{C}_i(\Gamma_1) = \mathcal{C}_{\sigma(i)}(\Gamma_2)$ for $i \in \{1,2,3\}$. ∎

Now we can finish the proof of Theorem 3.3.

Proof of Theorem 3.3. By Corollary 3.2, the algebra A belongs to one of the classes **A**–**E** and the class containing A is unique. If A belongs to one of the classes **A**–**D**, then the statement of the theorem follows from Lemmas 3.4–3.7.

Suppose that A belongs to the class **E**. Then A can be represented by $\mathbf{E}_1(\Gamma)$ for some $\Gamma = (\alpha,\beta,\gamma,\delta) \in \mathbf{k}^4$. Now we have the following.

- If $\mathcal{C}_1(\Gamma) = \mathcal{C}_2(\Gamma) \in \mathcal{T}$, then $\mathbf{E}_1(\Gamma) = \mathbf{E}_4(\beta)$.
- If $\mathcal{C}_1(\Gamma)$, $\mathcal{C}_2(\Gamma) \in \mathcal{T}$ and $\mathcal{C}_1(\Gamma) \neq \mathcal{C}_2(\Gamma)$, then $\mathbf{E}_1(\Gamma) \cong \mathbf{E}_1(1,1,0,0) = \mathbf{E}_4$ by Lemma 4.1.

- If $\mathcal{C}_1(\Gamma) \in \mathcal{T}$ and $\mathcal{C}_2(\Gamma) \notin \mathcal{T}$, then $\mathbf{E}_1(\Gamma) \cong \mathbf{E}_1(1-\beta,\gamma,\beta,\alpha) = \mathbf{E}_2(\beta,\gamma,\alpha)$ by Lemma 4.1.
- If $\mathcal{C}_1(\Gamma) \notin \mathcal{T}$ and $\mathcal{C}_2(\Gamma) \in \mathcal{T}$, then $\mathbf{E}_1(\Gamma) = \mathbf{E}_2(\gamma,\beta,\delta)$.
- If $\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma) \notin \mathcal{T}$, $\mathcal{D}(\Gamma) = 0$ and $\alpha+\gamma \in \mathbf{k}^*_{>1}$, then

$$\mathbf{E}_1(\Gamma) = \mathbf{E}_3\big(\gamma(\beta+\delta), \beta(\alpha+\gamma), \alpha+\gamma\big).$$

- If $\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma) \notin \mathcal{T}$, $\mathcal{D}(\Gamma) = 0$, and $\alpha+\gamma \notin \mathbf{k}^*_{>1}$, then, by Lemma 4.1,

$$\mathbf{E}_1(\Gamma) \cong \mathbf{E}_1(\delta,\gamma,\beta,\alpha) = \mathbf{E}_3(\beta(\alpha+\gamma), \gamma(\beta+\delta), \beta+\delta).$$

- If $\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma) \notin \mathcal{T}$ and $\mathcal{D}(\Gamma) \neq 0$, then there is a unique $\sigma \in S_3$ such that ${}^{\sigma^{-1}}(\mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma), \mathcal{C}_3(\Gamma)) \in \tilde{\mathcal{V}}$ and we have $\mathbf{E}_1(\Gamma) \cong \mathbf{E}_1(\Gamma')$ by Lemma 4.1, where $\Gamma' \in \mathcal{V}$ is such that $\mathcal{C}_i(\Gamma') = \mathcal{C}_{\sigma(i)}(\Gamma)$ for $i \in \{1,2,3\}$.

By Lemma 4.1, the structures from the set

$$\Big\{\mathbf{E}_1(\Gamma)\}_{\Gamma\in\mathcal{V}} \cup \{\mathbf{E}_2(\alpha,\beta,\gamma)\}_{(\alpha,\beta,\gamma)\in\mathbf{k}^3\smallsetminus\mathbf{k}\times\mathcal{T}} \\ \cup \{\mathbf{E}_3(\alpha,\beta,\gamma)\}_{(\alpha,\beta,\gamma)\in\mathbf{k}^2\times\mathbf{k}^*_{>1}} \cup \{\mathbf{E}_4\} \cup \{\mathbf{E}_5(\alpha)\Big\}_{\alpha\in\mathbf{k}}$$

are pairwise non-isomorphic. ∎

As a consequence of the proof of Lemma 4.1 we can describe the automorphism groups of algebras of the class $\mathbf{E}$.

Corollary 4.2 (i) *For* $\Gamma \in \mathcal{V}$,
  (a) $\operatorname{Aut}(\mathbf{E}_1(\Gamma))$ *is trivial if* $\mathcal{C}_1(\Gamma) \neq \mathcal{C}_2(\Gamma)$,
  (b) $\operatorname{Aut}(\mathbf{E}_1(\Gamma)) \cong C_2$ *if* $\mathcal{C}_1(\Gamma) = \mathcal{C}_2(\Gamma) \neq (-1,-1)$,
  (c) $\operatorname{Aut}(\mathbf{E}_1(-1,-1,-1,-1)) \cong S_3$ *if* $\operatorname{char}\mathbf{k} \neq 3$.
(ii) $\operatorname{Aut}(\mathbf{E}_4)$ *and* $\operatorname{Aut}(\mathbf{E}_2(\alpha,\beta,\gamma))$ *are trivial for* $(\alpha,\beta,\gamma) \in \mathbf{k}^3 \smallsetminus \mathbf{k}\times\mathcal{T}$.
(iii) *For* $(\alpha,\beta,\gamma) \in \mathbf{k}^2 \times \mathbf{k}^*_{>1}$, $\operatorname{Aut}(\mathbf{E}_3(\alpha,\beta,\gamma)) \cong C_2$ *if* $\gamma = -1$ *and* $\alpha = \beta$ *and* $\operatorname{Aut}(\mathbf{E}_3(\alpha,\beta,\gamma))$ *is trivial otherwise.*
(iv) $\operatorname{Aut}(\mathbf{E}_5(\alpha))$ *is isomorphic to the subgroup of* $\mathrm{GL}_2(\mathbf{k})$ *formed by matrices of the form* $\left(\begin{smallmatrix} a & b \\ 1-a & 1-b \end{smallmatrix}\right)$, *where* $a, b \in \mathbf{k}$, $a \neq b$.

In particular, we have

$$\begin{aligned} \dim\operatorname{Aut}(\mathbf{E}_1(\Gamma)) &= \dim\operatorname{Aut}(\mathbf{E}_2(\alpha,\beta,\gamma)) = \dim\operatorname{Aut}(\mathbf{E}_3(\alpha,\beta,\gamma)) \\ &= \dim\operatorname{Aut}(\mathbf{E}_4) = 0, \\ \dim\operatorname{Aut}(\mathbf{E}_5(\alpha)) &= 2. \end{aligned}$$

Proof (i) Any automorphism of $\mathbf{E}_1(\Gamma)$ must send e_1 and e_2 to $e_{\sigma(1)}$ and $e_{\sigma(2)}$, respectively, for some $\sigma \in S_3$, where e_3 is defined in the proof of Lemma 4.1. Such a map is an automorphism if and only if $\mathcal{C}_i(\Gamma) = \mathcal{C}_{\sigma(i)}(\Gamma)$ for $i = 1, 2$. If $\mathcal{C}_1(\Gamma) \neq \mathcal{C}_2(\Gamma)$, then we have also $\mathcal{C}_3(\Gamma) \neq \mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma)$ and, hence, only the identical element of S_3 determines an automorphism. If $\mathcal{C}_1(\Gamma) = \mathcal{C}_2(\Gamma) \neq (-1,-1)$, then one can check that $\mathcal{C}_3(\Gamma) \neq \mathcal{C}_1(\Gamma), \mathcal{C}_2(\Gamma)$ and, hence, only the identical element of S_3 and the element that swaps e_1 and e_2 determine automorphisms. If $\mathcal{C}_1(\Gamma) = \mathcal{C}_2(\Gamma) = (-1,-1)$, then any $\sigma \in S_3$ determines an automorphism. Note that $(-1,-1,-1,-1) \in \mathcal{V}$ if and only if $\operatorname{char}\mathbf{k} \neq 3$.

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(ii) Follows directly from the proof of Lemma 4.1.

(iii) It follows from the proof of Lemma 4.1 that an automorphism of $\mathbf{E}_3(\alpha,\beta,\gamma)$ is either trivial or swaps e_1 and e_2. The last mentioned map is an automorphism if and only if $\gamma=-1$ and $\alpha=\beta$.

(iv) It follows directly from the proof of Lemma 4.1 that automorphisms of $\mathbf{E}_5(\alpha)$ are exactly the linear maps that send e_1 and e_2 to $ae_1+(1-a)e_2$ and $be_1+(1-b)e_2$ for two different $a,b\in\mathbf{k}$. ∎

Now we will discuss some facts about degenerations of the form $A\to B$, where A is an algebra of the class $\mathbf{E}$. First of all, let us prove the following lemma.

Lemma 4.3 (i) *For any $\Gamma\in\mathcal{V}$ and $(\beta,\gamma)\in\mathcal{C}(\Gamma)$, there exists a degeneration $\mathbf{E}_1(\Gamma)\to\mathbf{D}_2(\beta,\gamma)$.*

(ii) *For any $(\alpha,\beta,\gamma)\in\mathbf{k}^3\smallsetminus\mathbf{k}\times\mathcal{T}$, there exists a degeneration $\mathbf{E}_2(\alpha,\beta,\gamma)\to\mathbf{D}_2(\beta,\gamma)$.*

(iii) *For any $(\alpha,\delta,\epsilon)\in\mathbf{k}^2\times\mathbf{k}^*_{>1}$ and $(\beta,\gamma)\in\mathcal{C}(\alpha,\delta,\epsilon)$, there exists a degeneration $\mathbf{E}_3(\alpha,\delta,\epsilon)\to\mathbf{D}_2(\beta,\gamma)$.*

Proof The parametrized basis $E_1^t=e_1$, $E_2^t=te_2$ gives the degeneration $\mathbf{E}_1(\Gamma)\to\mathbf{D}_2(\mathcal{C}_1(\Gamma))$ for any $\Gamma\in\mathbf{k}^4$. If $\Gamma\in\mathcal{V}$ and $1\le i\le3$, then, by Lemma 4.1, there exists $\Gamma'\in\mathbf{k}^4$ such that $\mathbf{E}_1(\Gamma)\cong\mathbf{E}_1(\Gamma')$ and $\mathcal{C}_i(\Gamma)=\mathcal{C}_1(\Gamma')$. Hence, $\mathbf{E}_1(\Gamma)\cong\mathbf{E}_1(\Gamma')\to\mathbf{D}_2\big(\mathcal{C}_i(\Gamma)\big)$. We also have $\mathbf{E}_2(\alpha,\beta,\gamma)=\mathbf{E}_1(1-\alpha,\beta,\alpha,\gamma)\to\mathbf{D}_2(\beta,\gamma)$ for $(\alpha,\beta,\gamma)\in\mathbf{k}^3\smallsetminus\mathbf{k}\times\mathcal{T}$ and

$$
\begin{aligned}
\mathbf{E}_3(\alpha,\delta,\epsilon)&=\mathbf{E}_1\Big((1-\alpha)\epsilon,\frac{\delta}{\epsilon},\alpha\epsilon,\frac{1-\delta}{\epsilon}\Big)\\
&\cong\mathbf{E}_1\Big(\frac{1-\delta}{\epsilon},\alpha\epsilon,\frac{\delta}{\epsilon},(1-\alpha)\epsilon\Big)\to\mathbf{D}_2\big(\alpha\epsilon,(1-\alpha)\epsilon\big),\mathbf{D}_2\Big(\frac{\delta}{\epsilon},\frac{1-\delta}{\epsilon}\Big)
\end{aligned}
$$

for $(\alpha,\delta,\epsilon)\in\mathbf{k}^2\times\mathbf{k}^*_{>1}$. ∎

For $\Gamma=(\alpha,\beta,\gamma,\delta)\in\mathbf{k}^4$, let us define the following subset of $\mathcal{A}_2$.

$$
\mathcal{G}(\Gamma)=\left\{\mu\;\middle|\;
\begin{aligned}
&c_{22}^1=0;\ c_{21}^1=\gamma c_{22}^2;\ c_{12}^1=\alpha c_{22}^2;\\
&\big(1-\gamma-\delta(\alpha+\gamma)\big)c_{12}^2-\big(1-\alpha-\beta(\alpha+\gamma)\big)c_{21}^2\\
&\qquad=\big(\beta(1-\gamma)-\delta(1-\alpha)\big)c_{11}^1;\\
&\big(1-\alpha-\beta(\alpha+\gamma)\big)^2c_{11}^2c_{22}^2\\
&\qquad=(\beta c_{11}^1-c_{12}^2)\big(\mathcal{D}(\Gamma)c_{12}^2+((\alpha-1)(\delta-1)-\beta\gamma)c_{11}^1\big);\\
&\big(1-\gamma-\delta(\alpha+\gamma)\big)^2c_{11}^2c_{22}^2\\
&\qquad=(\delta c_{11}^1-c_{21}^2)\big(\mathcal{D}(\Gamma)c_{21}^2+((\beta-1)(\gamma-1)-\alpha\delta)c_{11}^1\big);\\
&\big(1-\alpha-\beta(\alpha+\gamma)\big)\big(1-\gamma-\delta(\alpha+\gamma)\big)c_{11}^2c_{22}^2\\
&\qquad=(\beta c_{11}^1-c_{12}^2)\big(\mathcal{D}(\Gamma)c_{21}^2+((\beta-1)(\gamma-1)-\alpha\delta)c_{11}^1\big);\\
&\big(1-\alpha-\beta(\alpha+\gamma)\big)\big(1-\gamma-\delta(\alpha+\gamma)\big)c_{11}^2c_{22}^2\\
&\qquad=(\delta c_{11}^1-c_{21}^2)\big(\mathcal{D}(\Gamma)c_{12}^2+((\alpha-1)(\delta-1)-\beta\gamma)c_{11}^1\big)
\end{aligned}
\right\}.
$$

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Here and further in a definition of a subset of $\mathcal{A}_2$ we always assume by default that $\mu \in \mathcal{A}_2$ and c_{ij}^k $(i, j, k \in \{1,2\})$ are structure constants of μ. The following lemma will allow us to use $\mathcal{G}(\Gamma)$ as a separating set for some non-degenerations. Its proof is a direct calculation and so it is left to the reader.

Lemma 4.4 *The set $\mathcal{G}(\Gamma)$ is closed upper invariant and contains $E_1(\Gamma)$ for any $\Gamma \in \mathbf{k}^4$.*

5 Degenerations of Two-dimensional Algebras

In this section we describe all degenerations of two-dimensional algebras. Note that the results are valid for algebras over an algebraically closed field of arbitrary characteristic.

Theorem 5.1 *$\mathcal{A}_2$ has the graph of primary degenerations presented in Figure 1.*

Proof All primary degenerations that do not follow from Lemma 4.3 are presented in Table 2. Table 3 describes separating sets for all required non-degenerations and, thus, finishes the proof of the theorem.

The verification of degenerations is an easy direct calculation in each case. An example clarifying how to do this can be found in the proof of [21, Theorem 3]. The verification of Table 3 is more difficult. To clarify how one can achieve it, let us consider the first row of the table. It is easy to prove that $\mathbf{A}_4(\alpha)$ belongs to the presented separating set that we denote by $\mathcal{R}$. The fact that $\mathcal{R}$ is upper invariant can be checked by a direct calculation. What exactly one has to check is explained in Section 2. Let us prove that the orbits of $\mathbf{B}_2(\gamma)$, $\mathbf{D}_2(\beta,\gamma)$, and $\mathbf{E}_5(\beta)$ do not intersect $\mathcal{R}$. Let λ be one of these structures. Suppose that the structure constants c_{ij}^k $(i, j, k = 1, 2)$ of λ in the basis f_1, f_2 satisfy the defining equations of $\mathcal{R}$. Then $c_{22}^1 = c_{22}^2 = 0$, and hence $\lambda(f_2, f_2) = 0$. Since $\mathcal{R}$ is invariant under the basis rescaling, we may assume that $f_2 \in \{e_1, e_2\}$ if $\lambda = \mathbf{B}_2(\gamma)$, $f_2 = e_2$ if $\lambda = \mathbf{D}_2(\beta,\gamma)$, and $f_2 = e_2 - e_1$ if $\lambda = \mathbf{E}_5(\beta)$. Now in view of the upper invariance of $\mathcal{R}$, we may assume that $(f_1, f_2) \in \{(e_1, e_2), (e_2, e_1)\}$ if $\lambda = \mathbf{B}_2(\beta,\gamma)$, $(f_1, f_2) = (e_1, e_2)$ if $\lambda = \mathbf{D}_2(\beta,\gamma)$, and $(f_1, f_2) = (e_1, e_2 - e_1)$ if $\lambda = \mathbf{E}_5(\beta)$. We have the following.

- $c_{12}^1 + c_{21}^1 = 1 \neq 0$ if $\lambda = \mathbf{B}_2(\beta,\gamma)$, $f_1 = e_1$, $f_2 = e_2$.
- $c_{12}^2 + c_{21}^2 = 1 \neq 0 = c_{11}^1$ if $\lambda = \mathbf{B}_2(\beta,\gamma)$, $f_1 = e_2$, $f_2 = e_1$.
- $\alpha^2 c_{12}^1 c_{11}^2 = 0 \neq (c_{11}^1)^2$ if either $\lambda = \mathbf{D}_2(\beta,\gamma)$, $f_1 = e_1$, $f_2 = e_2$ or $\lambda = \mathbf{E}_5(\beta)$, $f_1 = e_1$, $f_2 = e_2 - e_1$.

Thus, the structure constants of λ in any basis do not satisfy the defining equations of $\mathcal{R}$, *i.e.*, $O(\lambda) \cap \mathcal{R} = \varnothing$. The other nondegenerations can be considered in the same manner. ∎

Let us recall that n-dimensional algebra A has a *level* m if

- there exists a sequence of n-dimensional algebras $A_0, \dots, A_m$ such that $A_0 = \mathbf{k}^n$, $A_m = A$, and, for $0 \le i \le m-1$, one has $A_{i+1} \to A_i$ and $A_{i+1} \not\cong A_i$;
- if $A_0, \dots, A_{m+1}$ is a sequence of algebras such that $A_0 = \mathbf{k}^n$, $A_{m+1} = A$, and $A_{i+1} \to A_i$ for $1 \le i \le m$, then $A_{i+1} \cong A_i$ for some $1 \le i \le m$.

Theorem 5.1 gives us the following partition of $\mathcal{A}_2$ to levels:

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Level	Algebra Structures
0	$\mathbf{k}^2$
1	$\mathbf{A}_3, \mathbf{B}_3, \mathbf{E}_5(\alpha)$
2	$\mathbf{A}_1(\alpha), \mathbf{A}_2, \mathbf{B}_2(\alpha), \mathbf{D}_2(\alpha,\beta), \mathbf{E}_4$
3	$\mathbf{A}_4(\alpha), \mathbf{B}_1(\alpha), \mathbf{C}(\alpha,\beta), \mathbf{D}_1(\alpha,\beta), \mathbf{D}_3(\alpha,\beta), \mathbf{E}_1(\Gamma), \mathbf{E}_2(\alpha,\beta,\gamma)$

Note also that the algebras $\mathbf{k}^2$, $\mathbf{B}_3$, $\mathbf{E}_4$, and $\mathbf{E}_5(\alpha)$ ($\alpha \in \mathbf{k}$) form a closed subset of $\mathcal{A}_2$ that has two interesting descriptions. First of all, these are exactly all two-dimensional algebras that do not degenerate to $\mathbf{A}_3$. Secondly, any one-generated subalgebra of such an algebra is one-dimensional and this property does not hold for other algebras. In fact, these two descriptions define the same set of algebras in a variety of algebras of any dimension. Note also that $\mathbf{E}_4$ is a unique two-dimensional algebra of level two that does not have non-trivial derivations.

6 Closures for Orbits of Infinite Series

In this section we describe closures of orbits for infinite series from our classification. To make this description nicer and more complete, we introduce two additional series and one additional algebra. For $\alpha \in \mathbf{k}$, we introduce $\mathbf{D}_2'(\alpha) = \mathbf{D}_2(\alpha,-\alpha)$ and $\mathbf{D}_3'(\alpha) = \mathbf{D}_3(\alpha,-\alpha)$. Note that $\mathbf{D}_2'(*) \subset \mathbf{D}_2(*)$ and $\mathbf{D}_3'(*) \subset \mathbf{D}_3(*)$. Also we define $\mathbf{A}_4' = \mathbf{A}_4(0) \in \mathbf{A}_4(*)$. Henceforth for a symbol X, we denote by $\mathbf{X}(*)$ the set formed by all $\mathbf{X}(\Gamma)$ that are defined. For example, $\mathbf{D}_2(*) = \{\mathbf{D}_2(\Gamma) \mid \Gamma \in \mathbf{k}^2\}$, $\mathbf{E}_3(*) = \{\mathbf{E}_3(\Gamma) \mid \Gamma \in \mathbf{k}^2 \times \mathbf{k}^*\}$.

Theorem 6.1 *For each row of* Table 4*, the second column contains all isomorphism classes of algebras whose orbits lie in the closure of the orbit of the series of algebras contained in the first column of the same row.*

Proof All required degenerations that do not follow from Theorem 5.1 are proved in Table 5. Table 6 describes separating sets for all required non-degenerations and, thus, finishes the proof of the theorem.

Note that the structures $\mathbf{B}_1(\alpha)$, $\mathbf{D}_1(\alpha,\beta)$, and $\mathbf{E}_3(\alpha,\beta,\gamma)$ do not lie in the corresponding separating sets, but the structures

$$\begin{pmatrix}0&1\\1&0\end{pmatrix} * \mathbf{B}_1(\alpha), \quad \begin{pmatrix}0&1\\1&0\end{pmatrix} * \mathbf{D}_1(\alpha,\beta), \quad \begin{pmatrix}1&\gamma\\0&1\end{pmatrix} * \mathbf{E}_3(\alpha,\beta,\gamma)$$

satisfy the required conditions. ∎

Theorems 5.1 and 6.1 give a lattice of subsets for $\mathcal{A}_2$. This lattice is presented in Figure 2. In this figure the leftmost set coincides with $\mathcal{A}_2$ and sets placed in one column have the same dimension equal to the number standing above them. Two sets of dimensions i and $i+1$ are connected by an edge if and only if the set of dimension $i+1$ contains the set of dimension i. Moreover, if $X, Y \subset \mathcal{A}_2$ correspond to two vertices of the diagram, then $X \cap Y$ is equal to the union of all $Z \subset \mathcal{A}_2$ corresponding to vertices of the diagram such that there exist paths from X to Z and from

Y to Z going from left to right. For example, $\overline{O(\mathbf{D}_1(*))} \cap \overline{O(\mathbf{C}(*))} = \overline{O(\mathbf{A}_4')}$ and $\overline{O(\mathbf{D}_1(*))} \cap \overline{O(\mathbf{D}_2(*))} = \overline{O(\mathbf{B}_2(*))} \cup \overline{O(\mathbf{D}_2'(*))}$.

7 Subvarieties Defined by Identities

Now we are going to apply the results of previous sections to develop the varieties of two-dimensional flexible and bicommutative algebras. In particular, we will describe the varieties of commutative and anticommutative algebras. Since there exists only one nontrivial two-dimensional anticommutative algebra, the last mentioned problem is not of big interest. Note also that in the same way one can recover the results of [3], where the analogous problems were solved for two-dimensional Novikov and pre-Lie algebras. Since the classifications of flexible and bicommutative algebras depend on the characteristic of the ground field, we assume everywhere in this section that $\operatorname{Chara} \mathbf{k} \neq 2$.

7.1 Flexible Algebras

By definition, an algebra is called *flexible* if it satisfies the identity $(xy)x = x(yx)$. It is clear that all commutative and anticommutative algebras are flexible. Using Theorem 3.3, one can verify that any two-dimensional flexible algebra is either (anti)commutative or $\mathbf{E}_5(\alpha)$. For $\alpha, \beta \in \mathbf{k}$, let us introduce the algebras

$$\mathbf{D}_2^c(\alpha) = \mathbf{D}_2(\alpha, \alpha), \qquad \mathbf{E}_2^c(\alpha) = \mathbf{E}_2(\tfrac{1}{2}, \alpha, \alpha),$$
$$\mathbf{E}_3^c(\alpha) = \mathbf{E}_3(\tfrac{1}{2}, \tfrac{1}{2}, \alpha), \qquad \mathbf{E}_1^c(\alpha, \beta) = \mathbf{E}_1(\alpha, \beta, \alpha, \beta).$$

It follows from our classification that any nontrivial two-dimensional commutative algebra can be represented by a unique structure from the set

$$\Big\{ \mathbf{A}_1(\tfrac{1}{2}), \mathbf{A}_3, \mathbf{B}_2(\tfrac{1}{2}), \mathbf{C}(\tfrac{1}{2}, 0), \mathbf{D}_1(\tfrac{1}{2}, 0), \mathbf{E}_5(\tfrac{1}{2}) \} \cup \{ \mathbf{D}_2^c(\alpha), \mathbf{E}_2^c(\alpha) \}_{\alpha \in \mathbf{k} \setminus \{\frac{1}{2}\}} \cup \{ \mathbf{E}_3^c(\alpha) \}_{\alpha \in \mathbf{k}^*_{>1}} \cup \{ \mathbf{E}_1^c(\alpha, \beta) \Big\}_{(\alpha,\beta,\alpha,\beta) \in \mathcal{V}}$$

and that any noncommutative flexible algebra can be represented by a structure from the set $\{\mathbf{B}_3\} \cup \{\mathbf{E}_\alpha\}_{\alpha \in \mathbf{k} \setminus \{\frac{1}{2}\}}$. Using Theorem 5.1, we get the graph of primary degenerations for the variety of two-dimensional flexible algebras presented in Figure 3.

It is easy to see that the variety of commutative algebras is simply $\mathbf{k}^6$, *i.e.*, irreducible. Then it is clear that the variety of flexible algebras has two irreducible components. The first component is $\overline{\{O(\mathbf{E}_5(\alpha))\}_{\alpha \in \mathbf{k}}} = \{\mathbf{E}_5(\alpha), \mathbf{B}_3, \mathbf{k}^2\}_{\alpha \in \mathbf{k}}$. The second component, formed by all commutative algebras, is equal to the closure of the orbit of the algebra series $\mathbf{E}_1^c(*)$.

This variety of flexible algebras does not have rigid algebras, and has the lattice of subsets presented in Figure 4. The lattice satisfies the same properties as the lattice from the previous section. To prove this it is enough to use Theorem 6.1 and its proof. The only difference is that one must use the parametrized indices $(\frac{1}{2} + \frac{t}{2}, \frac{1}{2} + \frac{t}{2})$, $(\frac{1}{2}, \frac{1}{2} - \frac{t^2}{2}, \frac{1}{2} - \frac{t^2}{2})$, and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{t})$ in the degenerations $\mathbf{D}_2(*) \to \mathbf{A}_1(\frac{1}{2})$,

$\mathbf{E}_2(*) \to \mathbf{C}(\frac{1}{2},0)$, and $\mathbf{E}_3(*) \to \mathbf{D}_1(\frac{1}{2},0)$, respectively, to obtain the degenerations $\mathbf{D}_2^c(*) \to \mathbf{A}_1(\frac{1}{2})$, $\mathbf{E}_2^c(*) \to \mathbf{C}(\frac{1}{2},0)$, and $\mathbf{E}_3^c(*) \to \mathbf{D}_1(\frac{1}{2},0)$.

7.2 Bicommutative Algebras

The variety of bicommutative algebras (see, for example, [10]) is defined by the identities $x(yz) = y(xz)$ and $(xy)z = (xz)y$. It follows from Theorem 3.3 that any nontrivial bicommutative algebra is isomorphic to a unique algebra from the set

$$\{\mathbf{A}_3, \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0), \mathbf{D}_2(1,1), \mathbf{D}_2(0,0), \mathbf{E}_1(0,0,0,0)\}.$$

Using Theorem 5.1, we get the graph of primary degenerations for the variety of two-dimensional bicommutative algebras. This graph is presented in Figure 5.

Thus, the irreducible components in the variety of two-dimensional bicommutative algebras are

$$\overline{O\big(\mathbf{D}_1(0,0)\big)} = \{\mathbf{D}_1(0,0), \mathbf{D}_2(0,0), \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{A}_3, \mathbf{k}^2\},$$
$$\overline{O\big(\mathbf{E}_1(0,0,0,0)\big)} = \{\mathbf{E}_1(0,0,0,0), \mathbf{D}_2(0,0), \mathbf{D}_2(1,1), \mathbf{A}_3, \mathbf{k}^2\}.$$

These components are generated by the rigid bicommutative algebras $\mathbf{D}_1(0,0)$ and $\mathbf{E}_1(0,0,0,0)$ and all have dimension 4.

Appendix A. Tables.

Table 1

$\mathbf{A}_1(\alpha)$, $\alpha \in \mathbf{k}$	$e_1e_1 = e_1 + e_2$,	$e_1e_2 = \alpha e_2$,	$e_2e_1 = (1-\alpha)e_2$,	$e_2e_2 = 0$
$\mathbf{A}_2$	$e_1e_1 = e_2$,	$e_1e_2 = e_2$,	$e_2e_1 = -e_2$,	$e_2e_2 = 0$
$\mathbf{A}_3$	$e_1e_1 = e_2$,	$e_1e_2 = 0$,	$e_2e_1 = 0$,	$e_2e_2 = 0$
$\mathbf{A}_4(\alpha)$, $\alpha \in \mathbf{k}_{\geq 0}$	$e_1e_1 = \alpha e_1 + e_2$,	$e_1e_2 = e_1 + \alpha e_2$,	$e_2e_1 = -e_1$,	$e_2e_2 = 0$
$\mathbf{B}_1(\alpha)$, $\alpha \in \mathbf{k}$	$e_1e_1 = 0$,	$e_1e_2 = (1-\alpha)e_1 + e_2$,	$e_2e_1 = \alpha e_1 - e_2$,	$e_2e_2 = 0$
$\mathbf{B}_2(\alpha)$, $\alpha \in \mathbf{k}$	$e_1e_1 = 0$,	$e_1e_2 = (1-\alpha)e_1$,	$e_2e_1 = \alpha e_1$,	$e_2e_2 = 0$
$\mathbf{B}_3$	$e_1e_1 = 0$,	$e_1e_2 = e_2$,	$e_2e_1 = -e_2$,	$e_2e_2 = 0$
$\mathbf{C}(\alpha,\beta)$, $(\alpha,\beta) \in \mathbf{k} \times \mathbf{k}_{\geq 0}$	$e_1e_1 = e_2$,	$e_1e_2 = (1-\alpha)e_1 + \beta e_2$,	$e_2e_1 = \alpha e_1 - \beta e_2$,	$e_2e_2 = e_2$
$\mathbf{D}_1(\alpha,\beta)$, $(\alpha,\beta) \in \mathcal{U}$	$e_1e_1 = e_1$,	$e_1e_2 = (1-\alpha)e_1 + \beta e_2$,	$e_2e_1 = \alpha e_1 - \beta e_2$,	$e_2e_2 = 0$
$\mathbf{D}_2(\alpha,\beta)$, $(\alpha,\beta) \in \mathbf{k}^2 \setminus \mathcal{T}$	$e_1e_1 = e_1$,	$e_1e_2 = \alpha e_2$,	$e_2e_1 = \beta e_2$,	$e_2e_2 = 0$
$\mathbf{D}_3(\alpha,\beta)$, $(\alpha,\beta) \in \mathbf{k}^2 \setminus \mathcal{T}$	$e_1e_1 = e_1$,	$e_1e_2 = e_1 + \alpha e_2$,	$e_2e_1 = -e_1 + \beta e_2$,	$e_2e_2 = 0$
$\mathbf{E}_1(\alpha,\beta,\gamma,\delta)$, $(\alpha,\beta,\gamma,\delta) \in \mathcal{V}$	$e_1e_1 = e_1$,	$e_1e_2 = \alpha e_1 + \beta e_2$,	$e_2e_1 = \gamma e_1 + \delta e_2$,	$e_2e_2 = e_2$
$\mathbf{E}_2(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma) \in \mathbf{k}^3 \setminus \mathbf{k} \times \mathcal{T}$	$e_1e_1 = e_1$,	$e_1e_2 = (1-\alpha)e_1 + \beta e_2$,	$e_2e_1 = \alpha e_1 + \gamma e_2$,	$e_2e_2 = e_2$
$\mathbf{E}_3(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma) \in \mathbf{k}^2 \times \mathbf{k}^*_{>1}$	$e_1e_1 = e_1$,	$e_1e_2 = (1-\alpha)\gamma e_1 + \frac{\beta}{\gamma} e_2$,	$e_2e_1 = \alpha\gamma e_1 + \frac{1-\beta}{\gamma} e_2$,	$e_2e_2 = e_2$
$\mathbf{E}_4$	$e_1e_1 = e_1$,	$e_1e_2 = e_1 + e_2$,	$e_2e_1 = 0$,	$e_2e_2 = e_2$
$\mathbf{E}_5(\alpha)$, $\alpha \in \mathbf{k}$	$e_1e_1 = e_1$,	$e_1e_2 = (1-\alpha)e_1 + \alpha e_2$,	$e_2e_1 = \alpha e_1 + (1-\alpha)e_2$,	$e_2e_2 = e_2$

Table 2		
degenerations	parametrized bases	
$\mathbf{A}_1(\alpha) \to \mathbf{A}_3$	$E_1^t = te_1$	$E_2^t = t^2 e_2$
$\mathbf{A}_1(\alpha) \to \mathbf{E}_5(\alpha)$	$E_1^t = e_1$	$E_2^t = e_1 + t^{-1} e_2$
$\mathbf{A}_2 \to \mathbf{A}_3$	$E_1^t = te_1$	$E_2^t = t^2 e_2$
$\mathbf{A}_2 \to \mathbf{B}_3$	$E_1^t = e_1$	$E_2^t = t^{-1} e_2$
$\mathbf{A}_4(\alpha) \to \mathbf{A}_2$	$E_1^t = te_1 - e_2$	$E_2^t = t^2 e_2$
$\mathbf{B}_1(\gamma) \to \mathbf{A}_2$	$E_1^t = e_1 + te_2$	$E_2^t = -t^2 e_2$
$\mathbf{B}_1(\gamma) \to \mathbf{B}_2(\gamma)$	$E_1^t = te_1$	$E_2^t = e_2$
$\mathbf{B}_2(\gamma) \to \mathbf{A}_3$	$E_1^t = e_1 + te_2$	$E_2^t = te_1$
$\mathbf{C}(\alpha, \beta) \to \mathbf{A}_1(\alpha)$	$E_1^t = te_1 + e_2$	$E_2^t = t^2 e_2$
$\mathbf{D}_1(\alpha, \beta) \to \mathbf{B}_2(\alpha)$	$E_1^t = te_1$	$E_2^t = e_2$
$\mathbf{D}_1(\alpha, \beta) \to \mathbf{B}_2(1 - \alpha + \beta)$	$E_1^t = te_2$	$E_2^t = e_1 - e_2$
$\mathbf{D}_1(\alpha, \beta) \to \mathbf{D}_2(\beta, -\beta)$	$E_1^t = e_1$	$E_2^t = te_2$
$\mathbf{D}_2(\beta, \gamma) \to \mathbf{A}_3$	$E_1^t = te_1 + te_2$	$E_2^t = t^2 e_1 + (\beta + \gamma) t^2 e_2$
$\mathbf{D}_3(\beta, \gamma) \to \mathbf{A}_2$	$E_1^t = \frac{t}{1-\beta-\gamma} e_1 - e_2$	$E_2^t = te_2$
$\mathbf{D}_3(\beta, \gamma) \to \mathbf{D}_2(\beta, \gamma)$	$E_1^t = e_1$	$E_2^t = te_2$
$\mathbf{E}_2(\alpha, \beta, \gamma) \to \mathbf{A}_1(\alpha)$	$E_1^t = te_1 + e_2$	$E_2^t = (1 - \beta - \gamma) t^2 e_1$
$\mathbf{E}_3(\alpha, \delta, \epsilon) \to \mathbf{B}_2\left(\frac{1-\delta-(1-\alpha)\epsilon}{1-\epsilon}\right)$	$E_1^t = te_1$	$E_2^t = \frac{\epsilon e_1 - e_2}{\epsilon - 1}$
$\mathbf{E}_4 \to \mathbf{B}_3$	$E_1^t = e_1 - e_2$	$E_2^t = te_2$
$\mathbf{E}_4 \to \mathbf{E}_5(\alpha)$	$E_1^t = \alpha e_1 + (1 - \alpha) e_2$	$E_2^t = (\alpha - t) e_1 + (1 - \alpha + t) e_2$

Table 3	
non-degenerations	separating sets
$\mathbf{A}_4(\alpha) \not\to \mathbf{B}_2(\gamma), \mathbf{D}_2(\beta, \gamma), \mathbf{E}_5(\beta)$	$\left\{ \mu \;\middle|\; \begin{array}{l} c_{22}^1 = c_{22}^2 = c_{12}^1 + c_{21}^1 = 0, c_{12}^2 + c_{21}^2 = c_{11}^1, \\ \alpha^2 c_{12}^1 c_{11}^2 = (c_{11}^1)^2 \end{array} \right\}$
$\mathbf{B}_1(\gamma) \not\to \mathbf{B}_2(\beta)\ (\beta \neq \gamma), \mathbf{D}_2(\beta, \delta), \mathbf{E}_5(\alpha),$	$\left\{ \mu \;\middle|\; \begin{array}{l} c_{22}^1 = c_{22}^2 = 0, c_{12}^2 + c_{21}^2 = -c_{11}^1, \\ c_{11}^2(c_{21}^1 + c_{12}^1) = -(c_{11}^1)^2, \gamma c_{12}^1 = (1 - \gamma) c_{21}^1 \end{array} \right\}$
$\mathbf{C}(\alpha, \beta) \not\to \mathbf{B}_2(\gamma), \mathbf{B}_3, \mathbf{D}_2(\gamma, \delta), \mathbf{E}_5(\gamma)\ (\gamma \neq \alpha)$	$\left\{ \mu \;\middle|\; \begin{array}{l} c_{22}^1 = 0, c_{21}^2 + c_{12}^2 = c_{11}^1, \\ c_{21}^1 = \alpha c_{22}^2, c_{12}^1 = (1 - \alpha) c_{22}^2, \\ (\alpha c_{21}^2 - (1 - \alpha) c_{12}^2)^2 = \beta^2 c_{11}^2 c_{22}^2 \end{array} \right\}$
$\mathbf{D}_1(\alpha, \beta) \not\to \begin{array}{l} \mathbf{B}_2(\gamma)\ (\gamma \notin \{\alpha, 1 - \alpha + \beta\}), \mathbf{B}_3, \\ \mathbf{D}_2(\gamma, \delta)\ \left((\gamma, \delta) \neq (\beta, -\beta) \right), \mathbf{E}_5(\gamma) \end{array}$	$\left\{ \mu \;\middle|\; \begin{array}{l} c_{22}^1 = c_{22}^2 = 0, \alpha c_{12}^1 = (1 - \alpha) c_{21}^1, \\ (\alpha - \beta) c_{12}^2 - (1 - \alpha + \beta) c_{21}^2 = \beta c_{11}^1, \\ c_{11}^1(c_{12}^2 + c_{21}^2) = c_{11}^2(c_{12}^1 + c_{21}^1), \end{array} \right\}$
$\mathbf{D}_3(\beta, \gamma) \not\to \mathbf{B}_2(\delta), \mathbf{D}_2(\delta, \epsilon)\ \left((\delta, \epsilon) \neq (\beta, \gamma) \right), \mathbf{E}_5(\alpha)$	$\left\{ \mu \;\middle|\; \begin{array}{l} c_{22}^1 = c_{22}^2 = c_{12}^1 + c_{21}^1 = 0, \\ c_{12}^2 + c_{21}^2 = (\beta + \gamma) c_{11}^1, \\ (1 - \beta - \gamma)(c_{12}^2 - \beta c_{11}^1) c_{11}^1 = c_{11}^2 c_{12}^1 \end{array} \right\}$
$\mathbf{E}_1(\Gamma) \not\to \mathbf{B}_2(\gamma), \mathbf{B}_3, \mathbf{D}_2(\beta, \gamma)\ \left((\beta, \gamma) \notin \mathcal{C}(\Gamma) \right), \mathbf{E}_5(\alpha)$	$\mathcal{G}(\Gamma)$
$\mathbf{E}_2(\alpha, \beta, \gamma) \not\to \begin{array}{l} \mathbf{B}_2(\delta), \mathbf{B}_3, \mathbf{D}_2(\delta, \epsilon)\ \left((\delta, \epsilon) \neq (\beta, \gamma) \right), \\ \mathbf{E}_5(\delta)\ (\delta \neq \alpha) \end{array}$	$\mathcal{G}(1 - \alpha, \beta, \alpha, \gamma)$
$\mathbf{E}_3(\alpha, \delta, \epsilon) \not\to \begin{array}{l} \mathbf{B}_2(\gamma)\ \left(\gamma \neq \frac{1-\delta-(1-\alpha)\epsilon}{1-\epsilon} \right), \mathbf{B}_3, \\ \mathbf{D}_2(\beta, \gamma)\ \left((\beta, \gamma) \notin \mathcal{C}(\alpha, \delta, \epsilon) \right), \mathbf{E}_5(\gamma) \end{array}$	$\mathcal{G}\left((1 - \alpha)\epsilon, \frac{\delta}{\epsilon}, \alpha\epsilon, \frac{1-\delta}{\epsilon} \right)$
$\mathbf{E}_4 \not\to \mathbf{A}_3$	$\left\{ \mu \;\middle|\; c_{22}^1 = c_{11}^2 = 0, c_{12}^1 + c_{21}^1 = c_{22}^2, c_{12}^2 + c_{21}^2 = c_{11}^1 \right\}$

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Table 4	
$\mathbf{A}_1(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_3, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{A}_4(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4(*), \mathbf{B}_3, \mathbf{E}_4, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{B}_1(*)$	$\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}'_4, \mathbf{B}_1(*), \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{k}^2$
$\mathbf{B}_2(*)$	$\mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{k}^2$
$\mathbf{C}(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4(*), \mathbf{B}_3, \mathbf{C}(*), \mathbf{E}_4, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{D}_1(*)$	$\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}'_4, \mathbf{B}_1(*), \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{D}_1(*), \mathbf{D}'_2(*), \mathbf{D}'_3(*), \mathbf{k}^2$
$\mathbf{D}_2(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{D}_2(*), \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{D}'_2(*)$	$\mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_3, \mathbf{D}'_2(*), \mathbf{k}^2$
$\mathbf{D}_3(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4(*), \mathbf{B}_1(*), \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{D}_2(*), \mathbf{D}_3(*), \mathbf{E}_4, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{D}'_3(*)$	$\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}'_4, \mathbf{B}_3, \mathbf{D}'_2(*), \mathbf{D}'_3(*), \mathbf{k}^2$
$\mathbf{E}_1(*)$	$\mathcal{A}_2$
$\mathbf{E}_2(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4(*), \mathbf{B}_1(*), \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{C}(*), \mathbf{D}_2(*), \mathbf{D}_3(*), \mathbf{E}_2(*), \mathbf{E}_4, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{E}_3(*)$	$\mathbf{A}_1(*), \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4(*), \mathbf{B}_1(*), \mathbf{B}_2(*), \mathbf{B}_3, \mathbf{D}_1(*), \mathbf{D}_2(*), \mathbf{D}_3(*), \mathbf{E}_3(*), \mathbf{E}_4, \mathbf{E}_5(*), \mathbf{k}^2$
$\mathbf{E}_5(*)$	$\mathbf{B}_3, \mathbf{E}_5(*), \mathbf{k}^2$

Table 5		
degenerations	parametrized bases	parametrized indices
$\mathbf{A}_1(*) \to \mathbf{A}_2$	$E_1^t = te_1, E_2^t = t^2e_2$	$\epsilon(t) = \frac{1}{t}$
$\mathbf{A}_4(*) \to \mathbf{A}_1(\alpha)$	$E_1^t = te_1 + (1-\alpha)e_2, E_2^t = t^2e_2$	$\epsilon(t) = \frac{1}{t}$
$\mathbf{A}_4(*) \to \mathbf{E}_4$	$E_1^t = te_1, E_2^t = te_1 + e_2$	$\epsilon(t) = \frac{1}{t}$
$\mathbf{B}_1(*) \to \mathbf{A}'_4$	$E_1^t = -t^{-1}e_1 + te_2, E_2^t = -t^2e_2$	$\epsilon(t) = \frac{1}{t^2}$
$\mathbf{B}_2(*) \to \mathbf{A}_2$	$E_1^t = e_1 + te_2, E_2^t = -t^2e_2$	$\epsilon(t) = \frac{1}{t}$
$\mathbf{C}(*) \to \mathbf{A}_4(\alpha)$	$E_1^t = te_1 + \alpha e_2, E_2^t = t^2e_2$	$\epsilon(t) = \left(-\frac{1}{t^2}, \frac{\alpha(1+t^2)}{t^3}\right)$
$\mathbf{D}_1(*) \to \mathbf{B}_1(\alpha)$	$E_1^t = te_1, E_2^t = e_2$	$\epsilon(t) = \left(\alpha, \frac{1}{t}\right)$
$\mathbf{D}_1(*) \to \mathbf{D}'_3(\alpha)$	$E_1^t = e_1, E_2^t = te_2$	$\epsilon(t) = \left(-\frac{1}{t}, \alpha\right)$
$\mathbf{D}_2(*) \to \mathbf{A}_1(\alpha)$	$E_1^t = e_1 + e_2, E_2^t = te_2$	$\epsilon(t) = (\alpha + t, 1-\alpha)$
$\mathbf{D}_2(*) \to \mathbf{B}_2(\alpha)$	$E_1^t = e_2, E_2^t = te_1$	$\epsilon(t) = \left(\frac{\alpha}{t}, \frac{1-\alpha}{t}\right)$
$\mathbf{D}'_2(*) \to \mathbf{A}_2$	$E_1^t = te_1 - e_2, E_2^t = te_2$	$\epsilon(t) = \frac{1}{t}$
$\mathbf{D}_3(*) \to \mathbf{A}_4(\alpha)$	$E_1^t = \alpha e_1 + \frac{1}{\alpha t}e_2, E_2^t = e_2$	$\epsilon(t) = \left(1 + t + \frac{1}{\alpha^2 t}, -\frac{1}{\alpha^2 t}\right)$
$\mathbf{D}_3(*) \to \mathbf{B}_1(\alpha)$	$E_1^t = -e_2, E_2^t = te_1$	$\epsilon(t) = \left(\frac{\alpha}{t}, \frac{1-\alpha}{t}\right)$
$\mathbf{D}'_3(*) \to \mathbf{A}'_4$	$E_1^t = te_1 - \frac{1}{t}e_2, E_2^t = e_2$	$\epsilon(t) = -\frac{1}{t^2}$
$\mathbf{E}_2(*) \to \mathbf{C}(\alpha, \beta)$	$E_1^t = t^{-1}e_1 - t^{-1}e_2, E_2^t = e_2$	$\epsilon(t) = \left(\alpha, \alpha + \beta t, 1 - \alpha - \beta t - t^2\right)$
$\mathbf{E}_2(*) \to \mathbf{D}_3(\alpha, \beta)$	$E_1^t = e_1, E_2^t = te_2$	$\epsilon(t) = \left(-\frac{1}{t}, \alpha, \beta\right)$
$\mathbf{E}_3(*) \to \mathbf{D}_1(\alpha, \beta)$	$E_1^t = e_1, E_2^t = te_2$	$\epsilon(t) = \left(\alpha, \frac{\beta}{t}, \frac{1}{t}\right)$
$\mathbf{E}_3(*) \to \mathbf{D}_3(\alpha, \beta)$	$E_1^t = e_1, E_2^t = -te_2$	$\epsilon(t) = \left(\frac{\alpha+\beta}{t}, \frac{\alpha}{\alpha+\beta}, \frac{1}{\alpha+\beta}\right)$
$\mathbf{E}_5(*) \to \mathbf{B}_3$	$E_1^t = te_1, E_2^t = e_2 - e_1$	$\epsilon(t) = \frac{1}{t}$

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Table 6	
non-degenerations	separating sets
$\mathbf{B}_1(*) \not\to \mathbf{D}_2'(\alpha)$	$\left\{\mu \,\middle|\, c_{22}^1 = c_{22}^2 = c_{12}^1 + c_{21}^1 = c_{11}^1 = 0\right\}$
$\mathbf{C}(*) \not\to \mathbf{B}_2(\alpha), \mathbf{D}_2(\alpha,\beta)\ (\alpha+\beta \neq 1)$	$\left\{\mu \,\middle|\, c_{22}^1 = 0, c_{12}^1 + c_{21}^1 = c_{22}^2, c_{12}^2 + c_{21}^2 = c_{11}^1\right\}$
$\mathbf{D}_1(*) \not\to \mathbf{A}_4(\alpha)\ (\alpha \neq 0), \mathbf{D}_2(\alpha,\beta)\ (\alpha+\beta \neq 0), \mathbf{E}_5(\alpha)$	$\left\{\mu \,\middle|\, c_{22}^1 = c_{12}^1 + c_{21}^1 = c_{11}^1 = 0\right\}$
$\mathbf{D}_3'(*) \not\to \mathbf{B}_2(\alpha)$	$\left\{\mu \,\middle|\, c_{22}^1 = c_{22}^2 = c_{12}^1 + c_{21}^1 = c_{12}^2 + c_{21}^2 = 0\right\}$
$\mathbf{E}_2(*) \not\to \mathbf{D}_1(\alpha,\beta), \mathbf{E}_1(\Gamma)\ (\Gamma \in \mathcal{V}), \mathbf{E}_3(\alpha,\beta,\gamma)\ (\gamma \neq 1)$	$\left\{\mu \,\middle|\, c_{22}^1 = 0, c_{12}^1 + c_{21}^1 = c_{22}^2\right\}$
$\mathbf{E}_3(*) \not\to \mathbf{C}(\alpha,\beta), \mathbf{E}_1(\Gamma)\ (\Gamma \in \mathcal{V}), \mathbf{E}_2(\alpha,\beta,\gamma)\ (\beta+\gamma \neq 1)$	$\left\{\mu \,\middle|\, c_{22}^1 = c_{22}^2 = 0\right\}$

Appendix B. Figures.

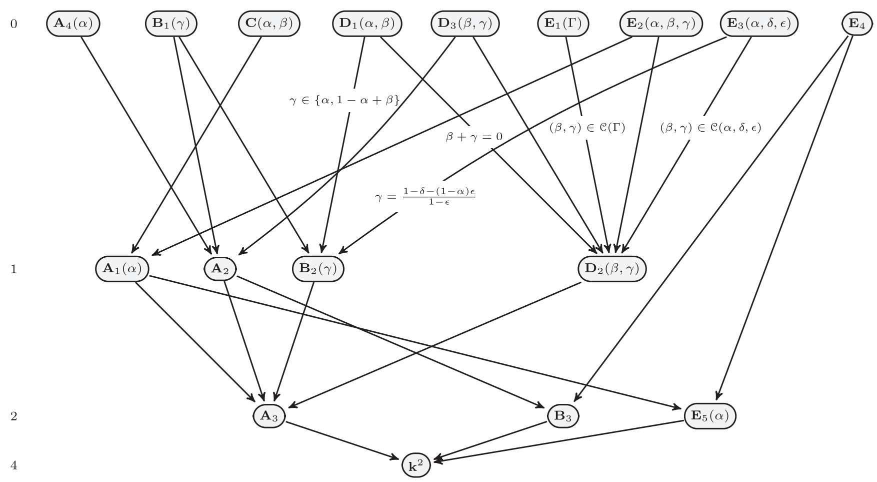

Figure 1

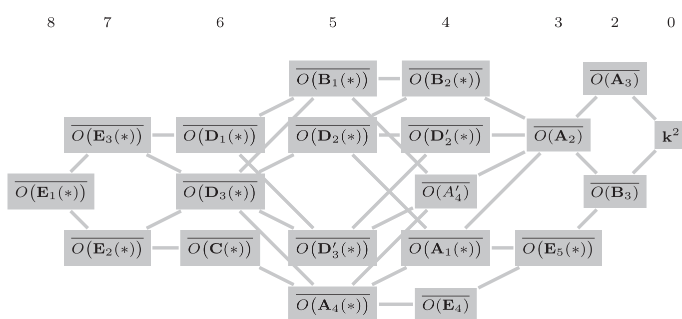

Figure 2

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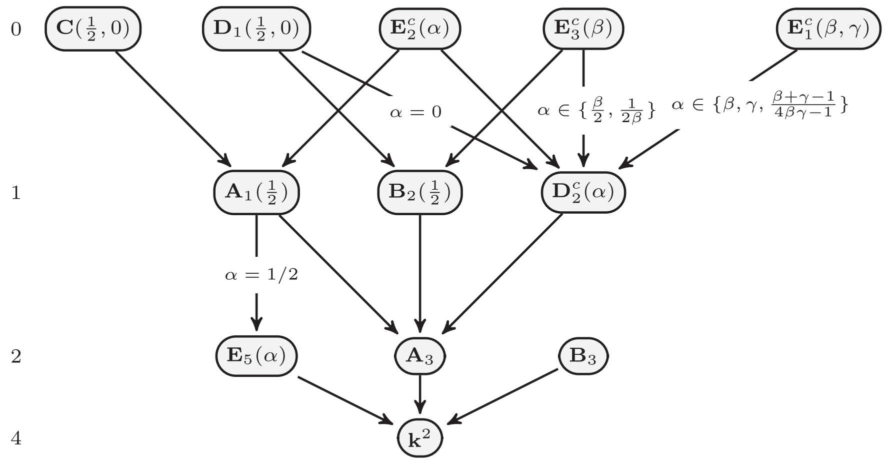

Figure 3

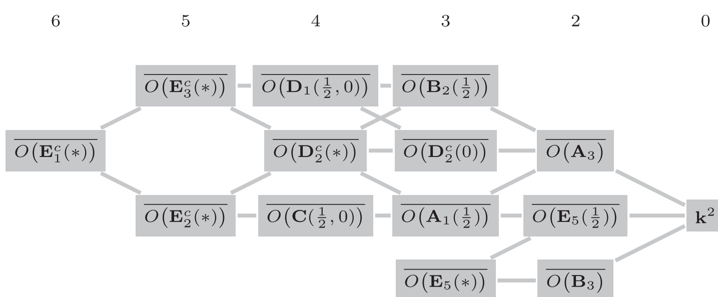

Figure 4

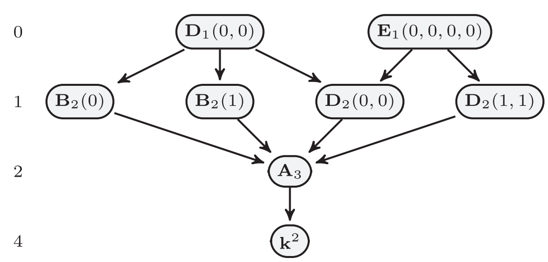

Figure 5

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