REMARKS ON SQUARE FUNCTIONS IN THE
LITTLEWOOD-PALEY THEORY

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We prove that certain square function operators in the Littlewood-Paley theory defined by the kernels without any regularity are bounded on $L^p_w$, $1 < p < \infty$, $w \in A_p$ (the weights of Muckenhoupt). Then, we give some applications to the Carleson measures on the upper half space.

1. INTRODUCTION

In this note we shall prove weighted $L^p$-estimates for the Littlewood-Paley type square functions arising from kernels satisfying only size and cancellation conditions. Suppose that $\psi \in L^1(\mathbb{R}^n)$ satisfies

(1.1) \( \int_{\mathbb{R}^n} \psi(x) \, dx = 0. \)

We consider a square function of Littlewood-Paley type

\[ S(f)(x) = S\psi(f)(x) = \left( \int_0^\infty \left| \psi_t * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \]

where $\psi_t(x) = t^{-n} \psi(t^{-1}x)$.

If $\psi$ satisfies, in addition to (1.1),

(1.2) \( |\psi(x)| \leq c(1 + |x|)^{-n-\varepsilon} \) for some $\varepsilon > 0$

(1.3) \( \int_{\mathbb{R}^n} |\psi(x - y) - \psi(x)| \, dx \leq c |y|^{\varepsilon} \) for some $\varepsilon > 0$,

then it is known that the operator $S$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ (see Benedek, Calderón and Panzone [1]). Well-known examples are as follows.

EXAMPLE 1: Let $P_t(x)$ be the Poisson kernel for the upper half space $\mathbb{R}^n \times (0, \infty)$:

\[ P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}. \]
Put
\[ \psi(x) = \left( \frac{\partial}{\partial t} F_1(x) \right)_{t=1} . \]
Then \( S_\psi(f) \) is the Littlewood-Paley g function.

**EXAMPLE 2:** Consider the Haar function \( \psi \) on \( \mathbb{R} \):
\[ \psi(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x), \]
where \( \chi_E \) denotes the characteristic function of a set \( E \). Then, \( S_\psi(f) \) is the Marcinkiewicz integral
\[ \mu(f)(x) = \left( \int_0^\infty \left| F(x + t) + F(x - t) - 2F(x) \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}, \]
where \( F(x) = \int_0^x f(y) dy \).

In this note, we shall prove that the \( L^p \)-boundedness of \( S \) still holds without the assumption (1.3); the conditions (1.1) and (1.2) only are sufficient. This is already known for the \( L^2 \)-case (see Coifman and Meyer [3, p. 148], and also Journe [7, pp. 81–82] for a proof).

To state our result more precisely, we consider the least non-increasing radial majorant of \( \psi \)
\[ h_\psi(|x|) = \sup_{|y| \geq |x|} |\psi(y)|. \]
We also need to consider two seminorms
\[ B_\varepsilon(\psi) = \int_{|x| > 1} \left| \psi(x) \right| |x|^\varepsilon \, dx \quad \text{for} \quad \varepsilon > 0, \]
\[ D_u(\psi) = \left( \int_{|x| < 1} \left| \psi(x) \right|^u \, dx \right)^{\frac{1}{u}} \quad \text{for} \quad u > 1. \]

We shall prove the following result.

**THEOREM 1.** Put \( H_\psi(x) = h_\psi(|x|) \). If \( \psi \in L^1(\mathbb{R}^n) \) satisfies (1.1) and
\begin{align*}
(1) \quad & B_\varepsilon(\psi) < \infty \quad \text{for some} \ \varepsilon > 0; \\
(2) \quad & D_u(\psi) < \infty \quad \text{for some} \ u > 1; \\
(3) \quad & H_\psi \in L^1(\mathbb{R}^n); 
\end{align*}
then the operator \( S_\psi \) is bounded on \( L^p_w \):
\[ \|S_\psi(f)\|_{L^p_w} \leq C_{p,w} \|f\|_{L^p_w}. \]
for all \( p \in (1, \infty) \) and \( w \in A_p \), where \( A_p \) denotes the weight class of Muckenhoupt (see [6, 7]), and

\[
\|f\|_{L^p_w} = \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}
\]

In fact, we shall prove a more general result.

**Theorem 2.** Suppose that \( \psi \in L^1(\mathbb{R}^n) \) satisfies (1.1) and

1. \( B_{\varepsilon}(\psi) < \infty \) for some \( \varepsilon > 0 \);
2. \( D_u(\psi) < \infty \) for some \( u > 1 \);
3. \( |\psi(x)| \leq h(|x|) \Omega(x') \quad (x' = |x|^{-1} x) \) for some non-negative functions \( h \) and \( \Omega \) such that
   - \( h(\tau) \) is non-increasing for \( \tau \in (0, \infty) \);
   - \( h \) and \( \Omega \) are in \( L^q(S^{n-1}) \) for some \( q \), \( 2 \leq q \leq \infty \).

Then, the operator \( S_\psi \) is bounded on \( L^p_w \) for \( p > q' \) and \( w \in A_p/q' \), where \( q' \) denotes the conjugate exponent of \( q \).

When \( \psi \) is compactly supported, we have another formulation, which is not included in Theorem 2.

**Theorem 3.** Suppose that \( \psi \in L^1(\mathbb{R}^n) \) satisfies (1.1) and

1. \( \psi \) is compactly supported;
2. \( \psi \in L^q(\mathbb{R}^n) \) for some \( q \geq 2 \).

Then \( S_\psi : L^p_w \to L^p_w \) for \( p > q' \) and \( w \in A_p/q' \).

These results will be derived from more abstract ones. Let \( \psi \in L^1(\mathbb{R}^n) \) satisfy (1.1). We also assume the following:

1. There exists \( \varepsilon \in (0, 1) \) such that
   \[
   \int_1^2 |\hat{\psi}(t\xi)|^2 \, dt \leq c \min(|\xi|^\varepsilon, |\xi|^{-\varepsilon}) \quad \text{for all} \quad \xi \in \mathbb{R}^n,
   \]
   where \( \hat{\psi} \) denotes the Fourier transform

   \[
   \hat{\psi}(\xi) = \int \psi(x) e^{-2\pi i x \cdot \xi} \, dx,
   \]
   \( \langle x, \xi \rangle = \sum_{j=1}^n x_j \xi_j \) (the inner product in \( \mathbb{R}^n \)).

2. Let \( 1 \leq s \leq 2 \). For all \( w \in A_s \), we have
   \[
   \sup_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_1^2 |\psi_{2^k} \ast f(x)|^2 \, dt \, w(x) \, dx \leq C_w \|f\|_{L^2_w}^2 \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n),
   \]
   where \( \mathbb{Z} \) denotes the integer group and \( \mathcal{S}(\mathbb{R}^n) \) the Schwartz space.
Under these assumptions the following holds.

**Proposition 1.** For $p > 2/s$ and $w \in A_{ps/2}$, the operator $S_\psi$ is bounded on $L^p_w$.

This will be used to prove the next result.

**Proposition 2.** Put

$$J_\epsilon(\psi) = \sup_{|\xi|=1} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x)\psi(y)\||\langle \xi, x-y \rangle|| \log |\langle \xi, x-y \rangle| \, dx \, dy \quad \text{for} \quad \epsilon \in (0, 1].$$

Let $\psi \in L^1$ satisfy (1.1) and (1.5). Then if $B_\epsilon(\psi) < \infty$ and $J_\epsilon(\psi) < \infty$ for some $\epsilon \in (0, 1]$, the operator $S_\psi$ is bounded on $L^p_w$ for $p > 2/s$ and $w \in A_{ps/2}$.

In Section 2, we shall prove Proposition 1 by the method of the proof of Duoandikoetxea and Rubio de Francia [5, Corollary 4.2] and then Proposition 2 by using Proposition 1. Proposition 2 will be applied to prove Theorems 2 and 3 in Section 3. Finally, in Section 4, we shall give some applications of Theorem 1 to generalised Marcinkiewicz integrals and the Carleson measures on the upper half space $\mathbb{R}^n \times (0, \infty)$.

To conclude this section, we state a result for the $L^2$-case, from which the result of Coifman-Meyer mentioned above immediately follows, and the idea of the proof will be applied later too (see the proof of Lemma 2).

**Proposition 3.** Suppose that $\psi \in L^1$ satisfies (1.1). Let

$$L(\psi) = \sup_{|\xi|=1} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x)\psi(y)\||\langle \xi, x-y \rangle|| \log |\langle \xi, x-y \rangle| \, dx \, dy.$$ 

Then, if $L(\psi) < \infty$, the operator $S_\psi$ is bounded on $L^2$.

**Proof:** It is sufficient to show that

$$\sup_{|\xi|=1} \int_0^\infty \left| \hat{\psi}(t\xi) \right|^2 \frac{dt}{t} < \infty.$$

We write

$$\left| \hat{\psi}(t\xi) \right|^2 = \hat{\psi}(t\xi)\overline{\hat{\psi}(t\xi)} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi(x)\overline{\psi(y)}e^{-2\pi i t\langle \xi, x-y \rangle} \, dx \, dy,$$

and so

$$\int_0^\infty \left| \hat{\psi}(t\xi) \right|^2 \frac{dt}{t} = \lim_{N \to \infty, \epsilon \to 0} \int_0^N \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi(x)\overline{\psi(y)} \left( \int_\epsilon^N e^{-2\pi i t\langle \xi, x-y \rangle} \frac{dt}{t} \right) \, dx \, dy.$$
Note that
\[ \int_{\epsilon}^{N_e} \left( e^{-2\pi i t (\xi, x-y)} - \cos(2\pi t) \right) \frac{dt}{t} \to -\log|\langle\xi, x-y\rangle| - i\frac{\pi}{2} \text{sgn}(\xi, x-y) \]
as \( N \to \infty \) and \( \epsilon \to 0 \), and the integral is bounded, uniformly in \( \epsilon \) and \( N \), by
\[ c \left( 1 + |\log|\langle\xi, x-y\rangle|\right). \]

Thus, using (1.1) and the dominated convergence theorem, we get
\[ \int_0^\infty \left| \tilde{\psi}(t\xi) \right|^2 \frac{dt}{t} = \iint \left( -\log|\langle\xi, x-y\rangle| - i\frac{\pi}{2} \text{sgn}(\xi, x-y) \right) dx \, dy. \]
This immediately implies the conclusion.

**REMARK.** In the one-dimensional case, it is easy to see that if
\[ \int |\psi(x)| \log(2 + |x|) \, dx < \infty \quad \text{and} \quad \int |\psi(x)| \log(2 + |\psi(x)|) \, dx < \infty, \]
then \( L(\psi) < \infty \), and so \( S^\psi : L^2 \to L^2 \).

### 2. Proofs of Propositions 1 and 2

We use a Littlewood-Paley decomposition. Let \( f \in \mathcal{S}(\mathbb{R}^n) \), and define
\[ \hat{\Delta_j(f)}(\xi) = \Psi(2^j \xi) \hat{f}(\xi) \quad \text{for} \quad j \in \mathbb{Z}, \]
where \( \Psi \in C^\infty \) is supported in \( \{1/2 \leq |\xi| \leq 2\} \) and satisfies
\[ \sum_{j \in \mathbb{Z}} \Psi(2^j \xi) = 1 \quad \text{for} \quad \xi \neq 0. \]

Decompose
\[ f \ast \psi_t(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \Delta_{j+k} (f \ast \psi_k)(x) \chi_{[2^k, 2^{k+1}]}(t) = \sum_{j \in \mathbb{Z}} F_j(x, t), \]
say, and define
\[ T_j(f)(x) = \left( \int_0^\infty \left| F_j(x, t) \right|^2 \frac{dt}{t} \right)^{1/2}. \]
Then
\[ S(f)(x) \leq \sum_{j \in \mathbb{Z}} T_j(f)(x). \]
Put \( E_j = \{ 2^{-1-j} \leq |\xi| \leq 2^{1-j} \} \). Then by the Plancherel theorem and (1.4) we have

\[
\left\| T_j(f) \right\|_2^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |\Delta_{j+k}(f * \psi_t)(x)|^2 \frac{dt}{t} \, dx \\
\leq \sum_{k \in \mathbb{Z}} c \int_{E_{j+k}} \left( \int_{2^k}^{2^{k+1}} \left| \hat{\psi}(t \xi) \right|^2 \frac{dt}{t} \right) \left| \hat{f}(\xi) \right|^2 \, d\xi \\
\leq \sum_{k \in \mathbb{Z}} c \int_{E_{j+k}} \min \left( |2^k \xi|^e, |2^k \xi|^{-e} \right) \left| \hat{f}(\xi) \right|^2 \, d\xi \\
\leq c 2^{-e|j|} \sum_{k \in \mathbb{Z}} \int_{E_{j+k}} \left| \hat{f}(\xi) \right|^2 \, d\xi \\
\leq c 2^{-e|j|} \|f\|_2^2,
\]

where the last inequality holds since the sets \( E_j \) are finitely overlapping. (We denote by \( \| \cdot \|_p \) the ordinary \( L^p \)-norm.)

On the other hand, for \( w \in A_s \) by (1.5) we see that

\[
\left\| T_j(f) \right\|_{L^2_w}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |\Delta_{j+k}(f * \psi_t)(x)|^2 \frac{dt}{t} \, w(x) \, dx \\
\leq \sum_{k \in \mathbb{Z}} c \int_{\mathbb{R}^n} |\Delta_{j+k}(f)(x)|^2 w(x) \, dx \\
\leq c \|f\|_{L^2_w}^2,
\]

where the last inequality follows from a well-known Littlewood-Paley inequality for \( L^2_w \) since \( A_s \subset A_2 \).

Interpolating with change of measures between the two estimates above, we get

\[
\left\| T_j(f) \right\|_{L^2_{(w^u)}} \leq c 2^{-e(1-u)|j|/2} \|f\|_{L^2_{(w^u)}}
\]

for \( u \in (0,1) \). If we choose \( u \) (close to 1) so that \( w^{1/u} \in A_s \), then from this inequality we get

\[
\left\| T_j(f) \right\|_{L^2_w} \leq c 2^{-e(1-u)|j|/2} \|f\|_{L^2_w},
\]

and so

\[
\|S(f)\|_{L^2_w} \leq \sum_{j \in \mathbb{Z}} \|T_j(f)\|_{L^2_w} \leq c \|f\|_{L^2_w}.
\]

Thus the extrapolation theorem of Rubio de Francia [8] implies the conclusion.

To derive Proposition 2 from Proposition 1 we need the following lemmas.
LEMMA 1. If \( \psi \in L^1(\mathbb{R}^n) \) satisfies (1.1) and \( B_\varepsilon(\psi) < \infty \) for \( \varepsilon \in (0, 1] \), then
\[
|\hat{\psi}(\xi)| \leq c |\xi|^\varepsilon \quad \text{for all } \xi \in \mathbb{R}^n.
\]

PROOF: Since \( a \leq a^\varepsilon \) for \( a, \varepsilon \in (0, 1] \), we see that
\[
|\hat{\psi}(\xi)| = \left| \int \psi(x) \left( e^{-2\pi i (x, \xi)} - 1 \right) dx \right| \leq c \int |\psi(x)| \min\left(1, |(x, \xi)|\right) dx
\leq c |\xi|^\varepsilon \int |\psi(x)| |x|^\varepsilon dx.
\]
This completes the proof.

LEMMA 2. If \( \psi \in L^1(\mathbb{R}^n) \) and \( J_\varepsilon(\psi) < \infty \) for \( \varepsilon \in (0, 1] \), then
\[
\int_1^2 |\hat{\psi}(t\xi)|^2 dt \leq c |\xi|^{-\varepsilon} \quad \text{for all } \xi \in \mathbb{R}^n.
\]

PROOF: As in the proof of Proposition 3, we see that
\[
\int_1^2 |\hat{\psi}(t\xi)|^2 dt = \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi(x)\overline{\psi(y)} \frac{e^{-4\pi i (\xi, x-y)} - e^{2\pi i (\xi, x-y)}}{-2\pi i (\xi, x-y)} dx dy.
\]
Thus
\[
\int_1^2 |\hat{\psi}(t\xi)|^2 dt \leq c \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x)| |\psi(y)| \min\left(1, |(\xi, x-y)|^{-1}\right) dx dy
\leq c J_\varepsilon(\psi) |\xi|^{-\varepsilon}.
\]
This completes the proof.

Now, we can see that Proposition 1 implies Proposition 2, since the condition (1.4) follows from Lemmas 1 and 2.

3. PROOFS OF THEOREMS 2 AND 3

To get Theorem 2 from Proposition 2 we need Lemmas 3 and 4 below. First, we give a sufficient condition for \( J_\varepsilon(\psi) < \infty \).

LEMMA 3. Let \( h(r), h \geq 0, \) be a non-increasing function for \( r > 0 \) satisfying \( H \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), where \( H(x) = h(|x|) \), and let \( \Omega \in L^v(S^{n-1}), v > 1, \Omega \geq 0. \) Suppose that \( F \) is a non-negative function such that
\[
F(x) \leq h(|x|)\Omega(x') \quad \text{for } |x| > 1
\]
and $D_u(F) < \infty$ for $u > 1$. Then $J_\varepsilon(F) < \infty$ if $\varepsilon < \min(1/u', 1/v')$.

**Proof:** For non-negative functions $f$, $g$ and $\xi \in S^{n-1}$ put

$$L_\varepsilon(f, g; \xi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)g(y)|\xi, x - y|^{-\varepsilon} \, dx \, dy.$$  

Decompose $F$ as $F = E + G$, where $E(x) = F(x)$ if $|x| < 1$ and $E(x) = 0$ otherwise. Then

$$L_\varepsilon(F, F; \xi) = L_\varepsilon(E, E; \xi) + 2L_\varepsilon(E, G; \xi) + L_\varepsilon(G, G; \xi).$$

We show that each of $L_\varepsilon(E, E; \xi)$, $L_\varepsilon(E, G; \xi)$ and $L_\varepsilon(G, G; \xi)$ is bounded by a constant independent of $\xi$ if $\varepsilon < \min(1/u', 1/v')$.

First, by Hölder’s inequality and a change of variables

$$L_\varepsilon(E, E; \xi) \leq \|E\|_u^2 \left( \int_{|x| < 1, |y| < 1} |x_1 - y_1|^{-\varepsilon u'} \, dx \, dy \right)^{1/u'},$$

where we note that $\|E\|_u = D_u(F)$.

Next, by Hölder’s inequality again

$$L_\varepsilon(E, G; \xi) \leq \|E\|_u \left( \int_{|x| < 1} \left( \int_{\mathbb{R}^n} G(y)|x_1 - \langle \xi, y \rangle|^{-\varepsilon} \, dy \right)^{u'} \, dx \right)^{1/u'}.$$  

For $s > 0$, let

$$I_\varepsilon(s) = \int_{S^{n-1}} |x_1 - \langle \xi, s\omega \rangle|^{-\varepsilon} \Omega(\omega) \, d\sigma(\omega)$$

for fixed $x_1$ and $\xi$, where $d\sigma$ denotes the Lebesgue surface measure of $S^{n-1}$ (when $n = 1$, let $\sigma\{1\} = \sigma\{-1\} = 1$). Then by Hölder’s inequality

$$I_\varepsilon(s) \leq (N_\varepsilon(s))^{1/u'} \|\Omega\|_u,$$

where

$$N_\varepsilon(s) = \int_{S^{n-1}} |x_1 - s\omega_1|^{-\varepsilon} \, d\sigma(\omega).$$  

Thus, using Hölder’s inequality,

$$\int_{\mathbb{R}^n} G(y)|x_1 - \langle \xi, y \rangle|^{-\varepsilon} \, dy \leq \int_0^\infty h(s)s^{n-1} I_\varepsilon(s) \, ds \leq \|\Omega\|_u \int_0^\infty h(s)s^{n-1}(N_\varepsilon(s))^{1/u'} \, ds \leq c \|H\|_1^{1/u'} \|\Omega\|_u \left( \int_0^\infty h(s)s^{n-1}N_\varepsilon(s) \, ds \right)^{1/u'} = c \|H\|_1^{1/u'} \|\Omega\|_u \left( \int_{\mathbb{R}^n} h(|y|)|x_1 - y_1|^{-\varepsilon u'} \, dy \right)^{1/u'}.$$


Therefore, the desired estimate for $L_e(E, G; \xi)$ follows if we show that

\begin{equation}
\sup_{z_1 \in \mathbb{R}^n} \int_{\mathbb{R}^n} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy < \infty.
\end{equation}

To see this, we split the domain of the integration as follows:

\[ \int_{\mathbb{R}^n} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy = \int_{|z_1 - y_1| < 1} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy + \int_{|z_1 - y_1| > 1} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy = I_1 + I_2, \]

Clearly $I_2 \leq \|H\|_1$. To estimate $I_1$ we may assume that $n \geq 2$; the case $n = 1$ can be easily disposed of since $h$ is bounded. We need further splitting of the domain of the integration. We write $y = (y_1, y')$, $y' \in \mathbb{R}^{n-1}$. Then

\[ I_1 = \int_{|z_1 - y_1| < 1} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy + \int_{|z_1 - y_1| < 1} h(|y|) |x_1 - y_1|^{-\epsilon_1} dy = I_3 + I_4, \]

It is easy to see that

\[ I_3 \leq \|H\|_\infty \int_{|y_1| < 2} |y_1|^{-\epsilon_1} dy < \infty. \]

Next, since $h(|y|) \leq h(|y'|)$,

\[ I_4 \leq \int_{|y_1| < 1} |y_1|^{-\epsilon_1} dy_1 \int_{|y'| > 1} h(|y'|) dy' \leq c \int_{|y_1| < 1} |y_1|^{-\epsilon_1} dy_1 \int_{|y'| > 1} h(|y|) dy < \infty. \]

It remains to estimate $L_e(G, G; \xi)$. Note that

\begin{equation}
L_e(G, G; \xi) \leq \int_0^\infty \int_0^\infty h(r) h(s) r^{n-1} s^{n-1} I_e(r, s) \, dr \, ds,
\end{equation}

where

\[ I_e(r, s) = \int_{S^{n-1} \times S^{n-1}} |(\xi, r\theta - s\omega)|^{-\epsilon} \Omega(\theta) \Omega(\omega) \, d\sigma(\theta) \, d\sigma(\omega). \]
By Hölder’s inequality

$$I_{e}(r,s) \leq \left( N_{e,r}(r,s) \right)^{1/v'} \|\Omega\|_{v}^{2},$$

where

$$N_{e}(r,s) = \int_{S^{n-1} \times S^{n-1}} |r \theta_{1} - s \omega_{1}|^{-\varepsilon} \, d\sigma(\theta) \, d\sigma(\omega).$$

Using the estimate (4.3) in (4.2) and then applying Hölder’s inequality, we see that

$$L_{e}(G,G;\xi) \leq c \|H\|_{1}^{2/v} \|\Omega\|_{v}^{2} \left( \int_{0}^{\infty} \int_{0}^{\infty} N_{e,r}(r,s) h(r) h(s) r^{n-1} s^{n-1} \, dr \, ds \right)^{1/v'}$$

$$= c \|H\|_{1}^{2/v} \|\Omega\|_{v}^{2} \left( \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} h(|x|) h(|y|) |x_{1} - y_{1}|^{-e v'} \, dx \, dy \right)^{1/v'}.$$

Therefore, the desired estimates follows again from (4.1). This completes the proof. \qed

For a non-negative function $\Omega$ on $S^{n-1}$ we define a non-isotropic Hardy-Littlewood maximal function

$$M_{\Omega}(f)(x) = \sup_{r>0} r^{-n} \int_{|y|<r} |f(x-y)| \Omega(|y|^{-1} y) \, dy.$$

To prove Theorem 2 we also need the following (see Duoandikoetxea [4]).

**Lemma 4.** If $\Omega \in L^{q}(S^{n-1})$, $q \geq 2$, and $w \in A_{2/q'}$, then $M_{\Omega}$ is bounded on $L^{2}_{w}$.

Now we can prove Theorem 2. As in Stein [10, pp.63-64], we can show that

$$\sup_{t>0} |\psi_{t} * f(x)| \leq c M_{\Omega}(f).$$

So, by Lemma 4 we see that the condition (1.5) holds for $\psi$ of Theorem 2 with $s = 2/q'$.

Next, applying Lemma 3, we see that $J_{e}(\psi) < \infty$ for $\varepsilon < \min(1/u', 1/q')$ (note that $h(r)$ of Theorem 2 (3) is bounded for $r \geq 1$). Combining these facts with the assumption in Theorem 2 (1), we can apply Proposition 2 to reach the conclusion.

Finally, we give the proof of Theorem 3. Clearly $B_{1}(\psi) < \infty$, and $J_{1/(2q')}(\psi) < \infty$ by applying Lemma 3 suitably. Therefore, the conclusion follows from Proposition 2 if we show that the condition (1.5) holds with $s = 2/q'$. But, for $q > 2$ this is a consequence of the inequality

$$\sup_{t>0} |\psi_{t} * f(x)| \leq c M \left( |f|^{q'} \right)^{1/q'},$$
where $M$ denotes the Hardy-Littlewood maximal operator. (This inequality is easily proved from Hölder’s inequality.)

To prove condition (1.5) when $q = 2$ and $w \in A_1$, we may assume that $\psi$ is supported in $\{|x| < 1\}$. Then by Schwarz’s inequality

$$|\psi_t * f(x)|^2 \leq t^{-n} \|\psi\|_{2}^2 \int_{|y| < t} |f(x - y)|^2 \, dy.$$ Integrating with the measure $w(x) \, dx$ and using a property of the $A_1$-weight function, we get

$$\int |\psi_t * f(x)|^2 w(x) \, dx \leq \|\psi\|_{2}^2 \int |f(y)|^2 t^{-n} \int_{|x - y| < t} w(x) \, dx \, dy$$

$$\leq C_w \|\psi\|_{2}^2 \int |f(y)|^2 w(y) \, dy$$
uniformly in $t$. From this the desired inequality follows.

4. APPLICATIONS

It is to be noted that Theorem 1 can be applied to study the $L_p^w$-boundedness of generalised Marcinkiewicz integrals.

**Corollary 1.** For $\varepsilon > 0$, let

$$\psi(x) = |x|^{-n+\varepsilon} \Omega(x') \chi_{(0,1)}(|x|),$$

where $\Omega \in L^\infty(S^{n-1})$ and $\int \Omega(x') \, d\sigma(x') = 0$. Define a Marcinkiewicz integral

$$\mu(f)(x) = \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$ Then, the operator $\mu$ is bounded on $L_p^w$ for all $p \in (1, \infty)$ and $w \in A_p$:

$$\|\mu(f)\|_{L_p^w} \leq C_{p,w} \|f\|_{L_p^w}.$$ This result, in particular, removes the Lipschitz condition assumed for $\Omega$ in Stein [9, Theorem 1 (2)].

Next, we consider applications to Carleson measures on the upper half spaces.

**Corollary 2.** Suppose $\psi \in L^1$ satisfies (1.1) and

$$|\psi(x)| \leq c(1 + |x|)^{-n-\varepsilon} \quad \text{for some} \quad \varepsilon > 0.$$
Take $b \in BMO$ and $w \in A_2$. Then the measure
\[ d\nu(x,t) = \left| \psi_t * b(x) \right|^2 \frac{dt}{t} w(x) \, dx \]
on the upper half space $\mathbb{R}^n \times (0, \infty)$ is a Carleson measure with respect to the measure $w(x) \, dx$, that is,
\[ \nu(S(Q)) \leq C_w \|b\|_{BMO}^2 \int_Q w(x) \, dx \]
for all cubes $Q$ in $\mathbb{R}^n$, where
\[ S(Q) = \{(x,t) \in \mathbb{R}^n \times (0, \infty) : x \in Q, \, 0 < t \leq \ell(Q) \}, \]
with $\ell(Q)$ denoting the sidelength of $Q$.

This can be proved by using $L^2_w$-boundedness of the operator $S_\psi$ (see Theorem 1) as in Journé [7, Chapter 6 III, pp.85–87]. In [7], a similar result has been proved with an additional assumption on the gradient of $\psi$.

Arguing as in [7, Chapter 6 III, p.87], by Corollary 2 we can get the following.

**COROLLARY 3.** Let $\psi$ and $b$ be as in Corollary 2. Suppose $\varphi$ satisfies
\[ |\varphi(x)| \leq c(1 + |x|)^{-n-\delta} \]
for $\delta > 0$. Then, the sublinear operator
\[ T_b(f)(x) = \left( \int_0^\infty \left| \psi_t * b(x) \right|^2 |\varphi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \]
is bounded on $L^p_w$ for all $p \in (1, \infty)$ and $w \in A_p$:
\[ \|T_b(f)\|_{L^p_w} \leq C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}. \]

Here again we don’t need the assumption on the gradient of $\psi$. See Coifman and Meyer [3, p.149] for the $L^2$-case.

**COROLLARY 4.** Suppose $\eta \in L^1(\mathbb{R}^n)$ satisfies the assumptions of Theorem 1 for $\psi$. Let $\psi$, $\varphi$ and $b$ be as in Corollary 3, and define a paraproduct
\[ \pi_b(f)(x) = \int_0^\infty \eta_t * ((\psi_t * b)(\varphi_t * f))(x) \frac{dt}{t}. \]
Then, the operator $\pi_b$ is bounded on $L^p_w$ for all $p \in (1, \infty)$ and $w \in A_p$:
\[ \|\pi_b(f)\|_{L^p_w} \leq C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}. \]
PROOF: Let \( g \in L^2(w^{-1}), w \in A_2 \). Then, since \( w^{-1} \in A_2 \), by Schwarz's inequality, Theorem 1 and Corollary 3, for \( 0 < u < v \), we see that

\[
\left| \int_u^v \eta_t \ast \left( (\psi_t \ast b)(\varphi_t \ast f) \right)(x) \frac{dt}{t} g(x) \, dx \right| \\
\leq \left( \int_u^v \left| \eta_t \ast g(x) \right|^2 \frac{dt}{t} w^{-1}(x) \, dx \right)^{1/2} \| T_b(f) \|_{L^2(w)} \\
\leq C_w \| b \|_{BMO} \| g \|_{L^2(w^{-1})} \| f \|_{L^2(w)},
\]

where \( \eta(x) = \eta(-x) \). From this estimate we can see that \( \pi_b(f) \) is well-defined (see Christ [2, III, Section 3]). Taking the supremum over \( g \) with \( \| g \|_{L^2(w^{-1})} \leq 1 \), we get the \( L^2_w \)-boundedness, and so the extrapolation theorem of Rubio de Francia implies the conclusion. This completes the proof. \( \square \)

See Coifman and Meyer [3, p.149, Proposition 1] for a similar result in the \( L^2 \)-case.

REFERENCES


