A CLASS OF PRIME RINGS

Kwangil Koh and A. C. Mewborn

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1. Introduction. If R is a ring and I is a right ideal of R then I is called faithful if $R - I$ is a faithful right R-module, i. e. if $\{r \in R: Rr \subseteq I\} = (0)$. I is called irreducible [1] provided that if J_1 and J_2 are right ideals such that $J_1 \cap J_2 = I$, then $J_1 = I$ or $J_2 = I$. Let $N(I) = \{r \in R: rI \subseteq I\}$ and $[I: a] = \{r \in R: ar \in I\}$ for $a \in R$. We write $(a)^r$ for $[(0): a]$.

DEFINITION 1.1 If I is a proper right ideal of R, then I is almost maximal provided that I is irreducible and

i) if $a \in R$ and $[I: a] \supset I$, then $a \in I$,

ii) if J is a right ideal of R, $J \supset I$, then $N(I) \cap J \supset I$, and if $a \in R$ such that $[J: a] \supseteq I$, then $[J: a] \supset I$.

In a ring with unity a maximal proper right ideal of R is almost maximal. However, an almost maximal right ideal of R need not be maximal; for example, the zero ideal is almost maximal in the ring of integers.

DEFINITION 1.2 Let V be a (left) vector space over a division ring D and let R be a ring of linear transformations of V. Then R is weakly transitive provided there is a right order K in D and a (K, R)-submodule M of V such that M is uniform [1] as R-module, $DM = V$, and such that if $\{m_i\}_{i=1}^{n}$ is a finite D-linearly independent subset of M and if $\{y_i\}_{i=1}^{n}$ is a sequence from M, then there exists $r \in R$, $k \in K$, $k \neq 0$, such that $m_i r = ky_i$, $1 \leq i \leq n$.

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Each primitive ring is isomorphic to a weakly transitive ring since a transitive ring is obviously weakly transitive. In [6] it was proved that a prime ring with zero (right) singular ideal [3] and containing a uniform right ideal is isomorphic to a weakly transitive ring. A prime ring with zero (right) singular ideal and uniform right ideal has a maximal annihilator right ideal which is faithful and almost maximal. The main result of this paper is that a ring is isomorphic to a weakly transitive ring if and only if it has a faithful, almost maximal right ideal.

2. Let C_α be the class of rings R such that R has a faithful, almost maximal right ideal.

THEOREM 2.1 If $R \in C_\alpha$ and I is a faithful, almost maximal right ideal of R, then I is a prime right ideal [4] of R.

Proof. Let A, B be right ideals of R with $B \neq (0)$ such that $AB \subseteq I$. Suppose $A \not\subseteq I$. Since I is almost maximal, by 1.1 (ii), we can choose $x \in N(I) \cap (A + I)$ such that $x \notin I$. $x = a + i$ for some $a \in A$, $i \in I$, $a \notin I$. Since I is faithful, $RB \not\subseteq I$. Thus $RB + I \supset I$. $a(RB + I) \subseteq AB + I \subseteq I$, since $aI = (i - x)I \subseteq I$. This contradicts 1.1 (i).

COROLLARY. If $R \in C_\alpha$ then R is a prime ring.

Proof. This follows from Theorem 2.1 and [4: p. 800].

THEOREM 2.2 If R is a semi-prime ring such that the (right) singular ideal R_r^Δ of R is zero and R contains a uniform right ideal U then for each $u \in U$ such that $uU \neq 0$, $(u)^r$ is almost maximal.

Proof. Let A be a complement of $(u)^r$. Then A is a uniform right ideal of R and $A \oplus (u)^r$ is a large right ideal of R. By [7: 4.2 p.8], a complement of A containing $(u)^r$ is $(u)^r$. Let $a \in R$ such that $(a)^r \supset (u)^r$. Then $(a)^r \cap A \neq (0)$ and $(a)^r \supseteq (u)^r \oplus A$. Hence a is a singular

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element of R. Since $a \in R_r^{\Delta} = (0)$, $a = 0$. Let J be a right ideal of R such that $J \supset (u)^r$. If $x \in R$ such that $uxJ = (0)$, then $(ux)^r \supset (u)^r$. Hence, $ux = 0$. Notice that $(uJ)(uJ) \neq (0)$ since R is semi-prime. Choose $j \in J$ such that $uju \neq 0$. Then $(ju)^r = (u)^r$ and $ju \in N((u)^r) \cap J$. Thus $N((u)^r) \cap J \supset (u)^r$. Now if J is a right ideal such that $J \supset (u)^r$, then J is a large right ideal. Hence, if $[J: a] \supseteq (u)^r$ for some $a \in R$ then $[J: a] \supset (u)^r$ since $[J: a]$ is large and $(u)^r$ is not large.

THEOREM 2.3 If R is a prime ring and I is an almost maximal right ideal of R which is not large, then I is a maximal annihilator right ideal of R.

Proof. Since I is not large, there exists a right ideal $A \neq (0)$ in R such that $I \cap A = (0)$. Let $J = I \ominus A$. Let $x \in N(I) \cap J$ such that $x \notin I$. This choice of x is possible since I is almost maximal. $x = i + a$ for some $i \in I$ and $a \in A$, $a \neq 0$. $(i + a)I \subseteq I$. Thus $aI' \in A \cap I = (0)$. Hence $I \subseteq (a)^r$. Suppose there is $b \in R$, $b \neq 0$, such that $(b)^r \supset I$. Then $(b)^r$ is large since I is irreducible. Thus $R_r^{\Delta} \neq (0)$. Let $L = R_r^{\Delta} \cap I^{\ell}$ where $I^{\ell} = \{ r \in R : rI = (0)\}$. Then $L \neq (0)$ since R is prime. If $y \in L$ then $(y)^r \supset I$ for if $(y)^r = I$ then $y \notin R_r^{\Delta}$. Then $y \cdot (y)^r \subseteq I$ implies that $y \in I$; thus $L \subseteq I$. This is impossible since R is prime. Hence $I = (a)^r$ is a maximal annihilator right ideal.

COROLLARY. Let R be a prime ring. R contains a non-large, almost maximal right ideal if and only if $R_r^{\Delta} = (0)$ and R contains a uniform right ideal.

Proof. Sufficiency follows from Theorem 2.2 and necessity follows from Theorem 2.3.

REMARK 2.4 By Theorem 2.1, in a prime ring an

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almost maximal right ideal which is faithful is a prime right ideal. However, a prime right ideal of a ring is not necessarily almost maximal. For example, if R is an integral domain which is not a right Ore-domain, then (0) is a prime right ideal, which is not almost maximal.

3. THEOREM 3.1 If R is a ring and I is an almost maximal right ideal of R then:

i) $N(I)/I$ is a right Ore-domain;

ii) $N(I)/I$ is isomorphic to a subring of the centralizer $\mathrm{Hom}_R(R - I, R - I)$ of the R-module $R - I$;

iii) $\mathrm{Hom}_R(R - I, R - I)$ is a right quotient ring of $N(I)/I$ in the sense of [3] ;

iv) In case R has a unity, $N(I)/I = \mathrm{Hom}_R(R - I, R - I)$.

Proof.

i) Let x_1, x_2 be non-zero elements of $N(I)/I$. Suppose $x_1 \cdot x_2 = 0$. $x_1 = n_1 + I$ and $x_2 = n_2 + I$ for some $n_i \in N(I)$, $n_i \notin I$, $i = 1, 2$. Since $n_1 \cdot n_2 \in I$, $[I: n_1] \supset I$. Hence by 1.1 i), $n_1 \in I$ which is absurd. Now $(n_1 + I)R \cap (n_2 + I)R = M$ is a non-zero submodule of $R - I$ since I is irreducible. Let $M = J - I$ for some right ideal J of R. Then $J \supset I$. Hence there is an element $a \in N(I) \cap J$ such that $a \notin I$. $a + I = n_1 r_1 + I = n_2 r_2 + I$ for some r_1 and r_2 in R. $a - n_i r_i \in I$ for $i = 1, 2$. Hence $n_i r_i \in N(I)$ for $i = 1, 2$. If $r_i I \nsubseteq I$ for $i = 1$ or $i = 2$, then $r_i I + I \supset I$ and $n_i(r_i I + I) \subseteq I$ implies that $n_i \in I$. This is impossible. Thus $r_i I \subseteq I$ for $i = 1, 2$; hence $r_i \in N(I)$. This implies that $(0) \neq (n_1 + I)(r_1 + I) = (n_2 + I)(r_2 + I)$ and $N(I)/I$ is an (right) Ore-domain.

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ii) Let χ be an element of $N(I)/I$. Then $\chi = n + I$ for some $n \in N(I)$. Define $f_\chi(r + I) = n \cdot r + I$ for all $r \in R$. Then f_χ is an R-homomorphism of $R - I$ into $R - I$. It is easy to see $\chi \to f_\chi$ is an isomorphism of $N(I)/I$ into $\mathrm{Hom}_R(R - I, R - I)$.

iii) Let g be a non-zero element of $\mathrm{Hom}_R(R - I, R - I)$. Then $g(R - I) = J - I$ for some right ideal J of R such that $J \supset I$. Let $n \in N(I) \cap J$ such that $n \notin I$. There is $r_o \in R$ such that $g(r_o + I) = n + I$. Let the kernel of $g = K - I$ for some right ideal K of R. If the kernel of g is zero then $r_o i \in I$ for all $i \in I$ since $ni \in I$ for all $i \in I$. Hence $g f_{r_o + I} = f_{n+I} \neq 0$. We claim that the kernel of g is zero. Suppose the kernel of g is not zero. Then $K \supset I$ and $[K: r_o] \supset I$ by 1.1 ii). Let $a \in [K: r_o]$ such that $a \notin I$. Then $g(r_o + I)a = g(r_o a + I) = 0 = na + I$, and $[I: n] \supset I$. By 1.1 i), $n \in I$, which is absurd. Thus the kernel of g must be zero.

iv) In case R has a unity 1, the module $R - I$ is strictly cyclic, and since $(1 + I)^r = I$ and $I = \{k \in N(I) : kR \subseteq (1 + I)^r\}$, by [2: p. 25] $N(I)/I$ is $\mathrm{Hom}_R(R - I, R - I)$.

THEOREM 3.2 If I is an almost maximal right ideal of R then the extended centralizer [3] of the R-module $M = R - I$ exists and is a division ring.

Proof. Let f be a non-zero semi-endomorphism [3] of M and let J', J be right ideals of R such that $J' - I$ is the domain of f and $J - I$ is the kernel of f. Also let K be the right ideal such that $f(J' - I) = K - I$. Then $K \supset I$, because $f \neq 0$. Suppose the kernel of f is non-zero, i.e. $J \supset I$. Let $k \in N(I) \cap K$, $k \notin I$. Let $b \in J'$ such that $f(b + I) = k + I$. Since $k \in N(I)$, $bI \subseteq J$. By 1.1 (ii),

$[J: b] \supset I$. Let $r \in [J: b]$, $r \notin I$, $(0) = f(br + I) = kr + I$ implies $kr \in I$. This contradicts 1.1 (i). Thus the kernel of f is (0). It follows that each semi-endomorphism of $R - I$ has a unique maximal extension. Therefore the extended centralizer exists. The extended centralizer is a division ring because $R - I$ is a uniform module, [see 3].

In the sequel, we let I be an almost maximal right ideal of a ring R, $M = R - I$, and D the extended centralizer of M.

LEMMA 3.3 There exists a (D, R)-module V which is a quasi-injective [5] extension of M as R-module and such that $DM = V$.

Proof. Let $\hat{M}$ be the minimal injective extension of M. Let $Q = \mathrm{Hom}_R(\hat{M}, \hat{M})$, and $V = QM$. If $f \in Q$, then $M_f = \{m \in M: f(m) \in M\}$ is a large submodule of M and the contraction of f to M_f is a semi-endomorphism. If $\ker f \neq (0)$, then $\ker f \cap M_f \neq (0)$. In the proof of Theorem 3.2 it was shown that a non-zero semi-endomorphism has zero kernel. Therefore either $\ker f = (0)$ or $\ker f \supseteq M$.

Now suppose $g \in \mathrm{Hom}_R(V, V)$, $g \neq 0$. We wish to show that $\ker g = (0)$. Let $f \in Q$, $m \in M$, such that $g(f(m)) \neq (0)$. Since $\ker gf \not\supseteq M$, $\ker gf = (0)$. Now $f(M) \cap M \neq (0)$; hence $M \not\subseteq \ker g$. Then $\ker g = (0)$. It follows that each element d of D has a unique extension to an element $\bar{d}$ of $\mathrm{Hom}_R(V, V)$. The mapping $d \to \bar{d}$ is an isomorphism of D onto $\mathrm{Hom}_R(V, V)$. Each semi-endomorphism α of V has an extension $\bar{\alpha}$ in $\mathrm{Hom}_R(\hat{M}, \hat{M})$ and the contraction of $\bar{\alpha}$ to V is an extension of α in $\mathrm{Hom}_R(V, V)$. Thus V is quasi-injective. Clearly $DM = QM = V$.

LEMMA 3.4 Let $K = N(I)/I$. Let J be any non-zero right ideal of R. If $x, y \in M$ such that $x \cdot J \neq (0)$ and $K \cdot y \neq (0)$, then $x \cdot J \cap K \cdot y \neq (0)$.

Proof. Let $x = r(x) + I$, $y = r(y) + I$ for some $r(x)$,

$r(y) \in R$. Since $x \cdot J \neq 0$, $r(x) \cdot J \nsubseteq I$. Thus $(r(x) \cdot J + I) \cap N(I) \supset I$. Let $a \in (r(x) \cdot J + I) \cap N(I)$ such that $a \notin I$. Then $a = r(x) \cdot j + i$ for some $j \in J$, $i \in I$ such that $r(x) \cdot j \in N(I)$ and $r(x) \cdot j \notin I$. Let $k \in K$, $k \neq 0$. Then $k = b + I$ for some $b \in N(I)$ and $b \notin I$. $k(r(y) + I) = br(y) + I \neq (0)$. $(r(x)j + I)(br(y) + I) = r(x)j \cdot br(y) + I \neq (0)$. $r(x)j \cdot br(y) + I \in x\ J$ and since $(r(x) \cdot j \cdot b + I)(r(y) + I) = (r(x) \cdot j \cdot b \cdot r(y) + I)$ and $r(x) \cdot j \cdot b + I \in K$, $r(x) \cdot j \cdot b \cdot r(y) + I \in K \cdot y$. Thus $x \cdot J \cap K \cdot y \neq (0)$.

LEMMA 3.5 If $\{x_i\}_{i=1}^{n}$ is a finite linearly independent subset of V contained in M and y is an element of M, then y is a D-linear combination of $\{x_i\}_{i=1}^{n}$ if and only if

$$\bigcap_{i=1}^{n} (x_i)^r \subseteq (y)^r .$$

Proof. This is an immediate consequence of [5: 2.2].

LEMMA 3.6 Let $K = N(I)/I$. Then K is a right order in D.

Proof. Let $d \in D$, $d \neq 0$. Let $U = \{m \in M : dm \in M\}$. Then $U \neq (0)$, and $U = J - I$ for some right ideal $J \supset I$. Since I is almost maximal, $N(I) \cap J \supset I$. Choose $j_o \in N(I) \cap J$ such that $j_o \notin I$. Then $j_o + I \in K$, and $(j_o + I)m \in U$ for all $m \in M$. Thus $0 \neq d[(j_o + I)] \in K$.

THEOREM 3.7 A ring R has a faithful almost maximal right ideal if and only if R is weakly transitive.

Proof. Let I be a faithful almost maximal right ideal of R and let $M = R - I$. Let D be the extended centralizer of M. Then D is a division ring by Theorem 3.2. By Corollary to Lemma 3.3, $V = D \cdot M$ is a left vector space over D. Let $K = N(I)/I$. Then K is a right order in D by Lemma 3.6 and $KM \subseteq M$. Let $\{x_i\}_{i=1}^{n}$ be a finite linearly independent subset of V contained in M. Let $\{y_i\}_{i=1}^{n}$ be a finite sequence in M.

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Let $I_j = \bigcap_{i=1 \atop i \neq j}^{n} (x_i)^r$. Then $x_j I_j \neq (0)$, by Lemma 3.5, for any $1 \leq j \leq n$. $x_j I_j \cap Ky_j \neq (0)$ for any $y_j \neq 0$, $1 \leq j \leq n$ by Lemma 3.4. Hence $x_j a_j = k_j y_j \neq 0$ for some $a_j \in I_j$, $y_j \neq 0$, $1 \leq j \leq n$. If $y_j = 0$, for some $1 \leq j \leq n$, then we let $a_j = 0$.

$\bigcap_{j=1}^{m} k_j y_j R = A$ is a non-zero submodule of M since M is uniform, where $y_j \neq 0$, $1 \leq j \leq m \leq n$. Let $k_j = n_j + I$, where $n_j \in N(I)$ and $y_j = r(y_j) + I$ for some $r(y_j) \in R$, $1 \leq j \leq m$. Let $A = E - I$ for some right ideal E of R. Let $b_o \in (E + I) \cap N(I)$ such that $b_o \notin I$. Then $b_o - n_j r(y_j) \cdot r_j \in I$ for some $n_j r(y_j) \cdot r_j + I \in k_j y_j \cdot R$. Let $x_j = r(x_j) + I$ for some $r(x_j) \in R$, for each $1 \leq j \leq n$. Let $r = \sum_{j=1}^{n} a_j r_j r(y_j)$ and $k = b_o + I$. Then $x_j r = ky_j$ for all $1 \leq j \leq n$, since

$$\begin{aligned} x_j \cdot r &= x_j a_j r_j \cdot r(y_j) = k_j y_j r_j r(y_j) = n_j r(y_j) r_j \cdot r(y_j) + I \\ &= (n_j r(y_j) r_j + I) \cdot (r(y_j) + I) \\ &= k \cdot y_j, \quad 1 \leq j \leq n . \end{aligned}$$

Conversely, assume V, M, D and K are given as in Definition 1.2. Let $0 \neq v_o \in M$ and let $I = (v_o)^r$. First we prove that I is faithful. Suppose S is a two sided ideal of R which is contained in I. If $v \in M$ then there exist $r \in R$ and $k \in K$, $k \neq 0$, such that $v_o r = kv$. If $S \neq (0)$ then there is $s_1 \in S$, $s_1 \neq 0$ such that $0 = v_o r s_1 = kvs_1$. Since $v_o R s_1 = (0)$, and $kvs_1 = 0$ for any $k \in K$ implies that $vs_1 = 0$, $M \cdot s_1 = (0)$. Since $V = D \cdot M$, $V \cdot s_1 = (0)$ and $s_1 = 0$. To prove I is irreducible, we note that $v_o R \cong R - I$ and $v_o R$ is a uniform R-module. Hence I is irreducible.

Let J be a right ideal of R such that $J \supset I$ and let $a \in R$ such that $aJ \subseteq I$. Then $(v_o a)^r \supset I$. If $v_o a \neq 0$, then $\{v_o, v_o a\}$ is a linearly independent set; for otherwise $dv_o = v_o a$ and $v_o = d^{-1} v_o a$ for some $d \in D$, and $(v_o)^r = (v_o a)^r$. Since $\{v_o, v_o a\}$ is a linearly independent subset of M, there is $r \in R$, $r \neq o$, such that $v_o r = 0$ and $(v_o a)r \neq 0$. This implies that $I \supseteq (v_o a)^r$. Thus $v_o a = 0$.

Let J be a right ideal of R such that $J \supset I$. Then $v_o j \neq 0$ for some $j \in J$. There exist r in R, k in K, $k \neq 0$, such that $v_o jr = kv_o \neq 0$. Hence $v_o(jr)I = 0$ and $jr \in N(I) \cap J$ and $jr \notin I$. Now let a be an element of R such that $a \cdot I \subseteq J$. If $a \in I$ then $[J: a] = R$. Suppose $a \notin I$. If $\{v_o a, v_o\}$ is a linearly dependent set then $(v_o a)^r = (v_o)^r$ and $a \in N(I)$. Hence $J \cap (a \cdot J + I) \cap N(I) \supset I$. Pick $x \in J \cap (a \cdot J + I) \cap N(I)$ such that $x \notin I$. Then $x = aj + i$ for some $j \in J$, $j \notin I$ and $i \in I$. $aj = i - x \notin I$ and $aj \in J$. Thus $[J: a] \supset I$ since $j \notin I$. If $\{v_o a, v_o\}$ is a linearly independent set, then by the weak transitivity property we may find $r \in R$, $k \in K$, $k \neq 0$ such that $v_o ar = 0$ and $v_o r = kv_o \neq 0$. Thus, $[J: a] \supset I$.

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North Carolina State University at Raleigh
and the
University of North Carolina at Chapel Hill.

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