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On Mutually m-permutable Products of Smooth Groups

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Abstract. Let G be a finite group and H, K two subgroups of G. A group G is said to be a mutually m-permutable product of H and K if $G = HK$ and every maximal subgroup of H permutes with K and every maximal subgroup of K permutes with H. In this paper, we investigate the structure of a finite group that is a mutually m-permutable product of two subgroups under the assumption that its maximal subgroups are totally smooth.

1 Introduction

Finite groups will be considered in this paper. We use the standard notions and notations as in Schmidt [6]. In addition, n will denote the maximal length of the subgroup lattice $L(G)$, and the set of all distinct primes dividing $|G|$ will be denoted by $\pi(G)$.

Two subgroups H and K of a group G are said to permute if $HK = KH$. It is easily seen that H and K permute if the set HK is a subgroup of G. A subgroup of G is said to be permutable in G if it permutes with every subgroup of G. A subgroup H of a group G is called modular in G if it is modular in the subgroup lattice, $L(G)$, of G. It is well known that the subgroup H of a finite group G is permutable if and only if H is modular and subnormal in G (see [6, p. 201, Theorem 5.1.1]). The permutable subgroups have been studied by several authors. For example, Ore [4] showed that every permutable subgroup of a group is subnormal. Following Ballester-Bolinches et al. [2], a group G is said to be a mutually m-permutable product of the subgroups H and K if $G = HK$ and H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H. By using the mutually m-permutable concept, solvability and supersolvability have been studied by several authors such as Ballester-Bolinches et al. [2] and Asaad [1].

A maximal chain $0 = a_0 < a_1 < a_2 < \cdots < a_n = I$ in a subgroup lattice L with least element 0 and greatest element I is called smooth if $[a_{i+j}/a_j] \cong [a_i/0]$ for all $i, j \in N$ such that $i + j \leq n$. A group G is called smooth if its subgroup lattice $L(G)$ has a smooth chain. Finite smooth groups have been studied by Schmidt [5]. A subgroup lattice L is called totally smooth if all maximal chains of elements of L are smooth. A group G is said to be totally smooth if its subgroup lattice $L(G)$ is totally smooth. Finite totally smooth groups have been studied in [3]. A group G is a P-group if it is either elementary abelian of order p^m or a semidirect product of an

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elementary abelian P of order p^{m-1} by a group Q of prime order $q \neq p$ that induces a nontrivial power automorphism on P, $m \geq 2$ (see [6, p. 49]).

The purpose of this paper is to restrict our attention to the structure of a finite group that is a mutually m-permutable product of two subgroups under the assumption that its maximal subgroups are totally smooth. More precisely, we prove the following result.

Theorem 1.1 (Main Theorem) *Assume that G is a mutually m-permutable product of its proper subgroups H and K with* $|\pi(G)|geq2$. *Suppose further that all maximal subgroups of G are totally smooth. Then one of the following holds:*

(i) *G is a nonabelian P-group;*
(ii) *G is cyclic of square free order;*
(iii) $n = 3$ *and* $|G| = pqr$, *where p, q, and r are distinct primes in* $\pi(G)$*;*
(iv) $n = 3$ *and* $|G| = p^2q$, *where p and q are distinct primes in* $\pi(G)$.

Since every subgroup lattice of length at most 2 is totally smooth, it follows that the structure of groups with this property is well known. So we will usually assume that $n \geq 3$.

2 The Proof of the Main Theorem

We need the following lemma.

Lemma 2.1 *A group G is totally smooth if and only if one of the following holds:*

(i) *G is cyclic of prime power order;*
(ii) *G is a P-group;*
(iii) *G is cyclic of square free order. (See* [3, Theorem 1]*.)*

The proof of the Main Theorem will be included in the following theorems.

Theorem 2.2 *Assume that G is a mutually m-permutable product of its proper subgroups H and K with* $|\pi(G)| = 2$. *Suppose further that all maximal subgroups of G are totally smooth. Then one of the following holds:*

(i) *G is a nonabelian P-group;*
(ii) $n = 3$ *and* $|G| = p^2q$, *where p and q are distinct primes in* $\pi(G)$.

Proof Let H^* and K^* be maximal subgroups of H and K, respectively. By hypothesis, H^* and K^* are totally smooth. Lemma 2.1 implies that the maximal subgroups of G are cyclic of prime power order, P-group, or cyclic of square free order. As $|\pi(G)| = 2$, we have the following cases.

Case 1: Both H and K are of prime power orders.

As $|\pi(G)| = 2$, H would be of order p^α with $\alpha \geq 1$ and K would be of order q^β with $\beta \geq 1$ and $q \neq p$. If both H and K are cyclic groups and since every maximal subgroup of H permutes with K, it follows by hypothesis that H^*K is totally smooth, and by Lemma 2.1, H^*K would be a nonabelian P-group or cyclic of square free order. Since H and K are cyclic groups, $|H^*| = p$ and $|K| = q$. Hence $|H| = p^2$, and

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so $|G| = p^2q$. So assume that H is cyclic and K is elementary abelian with $\beta > 1$. If H^*K is cyclic, we get a contradiction, since $|K| = q$. Thus H^*K is a nonabelian P-group. since H^* is a permutable Sylow p-subgroup of H^*K, H^* normal in H^*K and hence $p > q$. We get $|K| = q$, a contradiction. Thus $n \geq 4$.

If H centralizes a proper subgroup K_1 of K, we get $n = 3$, a contradiction. Thus H does not centralize any subgroup of K, $|H| = p$, and $p < q$. This implies that every subgroup containing H is a P-group. Then every subgroup of K is normal in G. Then H induces a universal power automorphism on K and G is a nonabelian P-group.

At the end of this case, assume that both H and K are elementary abelian with $\alpha > 1$ and $\beta > 1$. Let $p > q$. Since every maximal subgroup of H permutes with K, H^*K is a maximal subgroup of G. Hence H^*K would be a nonabelian P-group and so $|K| = q$, a contradiction. Similar, if $p < q$, we get a contradiction.

Case 2: H is cyclic of prime power order and $|\pi(K)| = 2$.

Suppose, first, that K is a nonabelian P-group of order $p^\beta q$ with $\beta \geq 1$ $(p > q)$. If $\beta = 1$, then $n = 3$ and either $|G| = p^2q$ or $|G| = pq^2$, and (ii) holds. So assume that $\beta > 1$. Hence K has a maximal nonabelian P-group K^*. If $|H| = q^\alpha$, HK^* is a maximal subgroup of G which is a nonabelian P-group. Then $G = K$, a contradiction. Thus $|H| = p^\alpha$. Since H is cyclic, we get $|H| = p$. If Q is a Sylow q-subgroup of G, we get that Q normalizes every p-subgroup of G and does not centralize any subgroup of G, which implies that G is a nonabelian P-group, and we are done. So assume that K is cyclic of order pq and let $|H| = p^\alpha$. Then there exists a maximal subgroup M containing H with $q \big| |M|$ by the hypothesis, and hence M is totally smooth. By Lemma 2.1, M is cyclic of square order or nonabelian p-group. Since H is cyclic, H would be of order p. Then $|G| = p^2q$ and $n = 3$.

Case 3: H is elementary abelian and $|\pi(K)| = 2$.

Suppose first, that K is a nonabelian P-group of order $p^\beta q$, $\beta \geq 1$. If $|H| = q^\alpha$ with $\alpha > 1$, there exists a maximal subgroup M of G containing H with $p \big| |M|$, and hence $[M/1]$ is not smooth which contradicts our assumption that all maximal subgroups of G are totally smooth. Thus $|H| = p^\alpha$. Let Q be a Sylow q-subgroup of G and let P be a Sylow p-subgroup of G. Hence P is cyclic or elementary abelian by Lemma 2.1. Since $|H| = p^\alpha$ with $\alpha > 1$, P would be elementary abelian. Let M_1 be a maximal subgroup of G containing H and Q. Clearly, M_1 is totally smooth. Lemma 2.1 implies that M_1 is a nonabelian P-group as $\alpha > 1$. Hence all maximal subgroups of G are P-groups. Therefore, Q induces a power automorphism on P, which is nontrivial. Then G is a nonabelian P-group.

Now consider the case where K is cyclic of order pq. Let H be elementary abelian with $\alpha > 1$. By hypothesis, H^*K is a maximal subgroup of G that is totally smooth. Since K cyclic, it follows by Lemma 2.1 that H^*K would be cyclic of square free order. Then $|H| = p^2$ and hence $|G| = p^2q$.

Case 4: H is a nonabelian P-group of order $p^\alpha q$.

Assume first that K is a nonabelian P-group. Since $|\pi(G)| = 2$, we can assume that $|K| = p^\beta q$. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. We argue that $|Q| = q$.

Let M be a maximal subgroup of G containing Q with $p \big| |M|$. Since M is totally smooth, Lemma 2.1 shows that M is a nonabelian P-group. Hence $|Q| = q$, and so $P \lhd G$, where P is a Sylow p-subgroup of G. Since P is totally smooth, it follows by Lemma 2.1 that P is cyclic or elementary abelian. If P is cyclic, then $n = 3$ and $|G| = p^2q$. So assume that P is elementary abelian. Since every proper subgroup of G is totally smooth, it follows that every proper subgroup containing Q of G is a nonabelian P-group. Then Q does not centralize any p-subgroup of G, and hence G is a nonabelian P-group of order $p^{n-1}q$.

So let K be cyclic of order pq and let P_1 be a Sylow p-subgroup of H. It is clear that P_1K is a maximal subgroup of G. Hence by our assumption, P_1K would be totally smooth. Since K is cyclic and H is a nonabelian P-group, we get $|P_1K| = pq$. Therefore, $n = 3$ and $|G| = pq^2$, and we are done.

Case 5: H and K are cyclic groups of square free orders.

As $|\pi(G)| = 2$, we get H is cyclic of order pq and K is cyclic of order pq. Therefore, $n = 3$ and hence $|G| = p^2q$. This completes our proof. ■

Now we can assume that $|G|$ is divisible by $m \geq 3$ different primes.

Theorem 2.3 *Assume that G is a mutually m-permutable product of its proper subgroups H and K with* $|\pi(G)| \geq 3$. *Suppose further that all maximal subgroups of G are totally smooth. Then one of the following holds:*

(i) *G is cyclic of square free order;*
(ii) $n = 3$ *and* $|G| = pqr$, *where p, q, and r are distinct primes in* $\pi(G)$.

Proof As all maximal subgroups of G are totally smooth, Lemma 2.1 shows that the maximal subgroups of G are cyclic of prime power order, P-group, or cyclic of square free order. Since $|\pi(G)| \geq 3$, we have the following cases.

Case 1: H is cyclic of prime power order and $|\pi(K)| \geq 2$.

Suppose first that K is a nonabelian P-group of order $p_1^{\alpha} p_2$, $p_1 > p_2$. Then $|\pi(G)| = 3$, and so we can assume that $|H| = p^{\beta}$. Let P_i be a Sylow p_i-subgroup of K, $(i = 1, 2)$. Since H permutes with every maximal subgroup of K, HP_1 is a maximal subgroup of G. Hence it is totally smooth and by Lemma 2.1, HP_1 is cyclic of square free order or a nonabelian P-group. If HP_1 is cyclic of square free order, $|H| = p$ and $|P_1| = p_1$. Then $n = 3$ and $|G| = p_1p_2p_3$, and (ii) holds. So suppose HP_i is a nonabelian P-group $(i = 1, 2)$. If p is the largest prime dividing the order of G and since H is cyclic, it follows that $|H| = p$, and hence $|G| = pp_1p_2$ and (ii) holds. Therefore, $p < p_i$ for each $i = 1, 2$. Hence H would be of order p.

So assume, for a contradiction, that $|P_1| > p_1$. Then P_1 has a normal subgroup L of G and hence LHP_2 is a subgroup of G that is totally smooth. Lemma 2.1 shows that LHP_2 is cyclic of square free order. Then P_2 centralizes L, which contradicts our choice of K since $LP_2 < K$ and K is a nonabelian P-group. Thus $|P_1| = p_1$ and $|G| = pp_1p_2$.

Now assume that K is cyclic of order $p_1p_2 \cdots p_m$ with $m > 1$. Let H be of order p^{α} and let P_i be Sylow p_i-subgroups of K, $i = 1, 2, \ldots, m$. If $n = 3$, $|K| = p_1p_2$. By hypothesis and Lemma 2.1, HP_i is cyclic or nonabelian P-group with $i = 1, 2$. If

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HP_i is cyclic for some i, $|G|$ would be of order $p_1p_2p_3$ or cyclic of square free order. So HP_i is a nonabelian P-group for every $i = 1, 2$. If $\alpha > 1$, then H has a normal subgroup L of G. Since LK is totally smooth and $|\pi(LK)| = 3$, it follows by Lemma 2.1 that LK is cyclic, a contradiction. Thus $n \geq 4$.

Hence there exists a maximal subgroup M of G containing H with $|\pi(M)| \geq 3$. Since M is totally smooth, Lemma 2.1 shows that M would be cyclic of square free order. This implies that $|H| = p$ and H centralizes every subgroup of K. Then G is cyclic of square free order.

Case 2: H is elementary abelian and $|\pi(K)| \geq 2$.

Assume that K is a nonabelian P-group of order $p_1^{\alpha}p_2$ with $p_1 > p_2$. Then $|\pi(G)| = 3$. Let P_i be a Sylow p_i-subgroup of K $(i = 1, 2)$. If $|P_1| > p_1$, there exists a totally smooth maximal subgroup HK^* of G with $|\pi(HK^*)| = 3$. Since K is a nonabelian P-group, $[HK^*/1]$ is not smooth, which contradicts our assumption. Thus $|P_1| = p_1$. If H would have a maximal subgroup H^*, then H^*K is totally smooth subgroup of G. As $|\pi(H^*K)| = 3$, Lemma 2.1 shows that H^*K would be cyclic, contradicting the choice of K. Thus H would be of prime order, and hence $|G| = p_1p_2p_3$.

So assume that K is cyclic of order $p_1p_2 \cdots p_m$ with $m > 1$. Let H be of order p^{α}. Consider $|\pi(G)| > 3$. Let K^* be a maximal subgroup of K. Then HK^* is a subgroup of G by the hypothesis. Hence HK^* is totally smooth, which would be cyclic of square free order by Lemma 2.1. Then $|H| = p$ and H centralizes every subgroup of K, since K^* is any subgroup of K. Therefore, G is cyclic of square free order and (i) holds. So assume that $|\pi(G)| = 3$. We argue that $|H| = p$. If not, p would be the largest prime dividing $|G|$. Hence $H \lhd G$ and it has a proper subgroup L that is normal in G. Then LK is totally smooth by the hypothesis. This implies that LK is cyclic of square free order, since $|\pi(LK)| = 3$ and by Lemma 2.1. Hence $|H| = p^2$.

Let P_i be a Sylow p_i-subgroup of K for some $i = 1, 2$. Then HP_i is a totally smooth subgroup of G. Since L centralizes P_i, it follows by Lemma 2.1 that HP_i would be cyclic of square free order. Hence $|H| = p$, a contradiction since $|H| = p^2$. Thus H would be of prime order and so $|G| = pqr$.

Case 3: H and K are nonabelian P-groups.

Since H and K are nonabelian P-groups, $|\pi(G)|$ would be at most 4. Let $|\pi(G)| = 4$ and let K^* be a maximal subgroup of K. Then HK^* is a totally smooth subgroup of G by the hypothesis. Since $|\pi(HK^*)| \geq 3$, it is clear by Lemma 2.1 that HK^* is cyclic. As H is a nonabelian P-group, we get a contradiction. Thus $|\pi(G)| = 3$. Let P_i be a Sylow p_i-subgroups of G, $i = 1, 2, 3$.

If $|G| = p_1p_2p_3$, we get that (ii) holds and we are done. So suppose, for a contradiction, that $|P_i| > p_i$ for some i; $i = 1, 2, 3$.

As both H and K are nonabelian P-groups, we get $|P_1| > p_1$, where p_1 is the largest prime in $\pi(G)$. Then G has a normal p_1-subgroup N of P_1. Hence we get by the hypothesis that NP_2P_3 is a totally smooth subgroup of G that is cyclic of square free order by Lemma 2.1, a contradiction, since both H and K are nonabelian P-group. Thus $|P_1| = p_1$, and this completes the proof of this case.

Case 4: H is a nonabelian P-group and K is cyclic of order $p_1p_2 \cdots p_m$ with $m > 1$.

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Let H^* be a maximal subgroup of H with $|\pi(H^*)| = 2$. Hence H^*K is a totally smooth subgroup of G. We get by Lemma 2.1 that H^*K is cyclic of square free order that contradicts our choice of H. Thus H^* would be of prime order, which implies that $|H| = pq$. If $H \cap K = 1$ and since $|\pi(K)| \geq 2$, it follows that $|\pi(HK^*)| \geq 3$. Then by Lemma 2.1, HK^* would be cyclic, a contradiction, since H is a nonabelian P-group. Thus $H \cap K \neq 1$. Let q be the smallest prime in $\pi(G)$ and let Q be a Sylow q-subgroup of G. Then G has a normal q-complement N, say. It follows by the hypothesis and Lemma 2.1 that N is a nonabelian P-group or cyclic. If N is a nonabelian P-group, $|\pi(G)| = 3$ and N has a proper normal subgroup L of G.

Suppose, for a contradiction, that $p_j^2 \big| |N|$ for some prime $p_j \in \pi(G)$. It follows that G has a maximal subgroup M containing both L and Q with $|\pi(M)| \geq 3$. By the hypothesis and Lemma 2.1, M is cyclic, a contradiction since N is a nonabelian P-group. Thus $|N| = p_1p_2$, and hence $|G| = p_1p_2p_3$. Thus N is cyclic of square free order. We argue that $|\pi(N)| = 2$. If not, then there is a maximal subgroup M of G containing H with $|\pi(M)| \geq 3$. Since H is a nonabelian P-group, $[M/1]$ is not smooth, which contradicts the hypothesis. Thus $|\pi(N)| = 2$. Once again, $|G| = p_1p_2p_3$.

Case 5: H and K are cyclic groups of square free orders.

If K^* is a maximal subgroup of K, it follows by the hypothesis that HK^* is a maximal subgroup of G, and hence it is totally smooth. Then HK^* would be cyclic of square free order as H cyclic by Lemma 2.1. Hence every maximal subgroup of G containing H or K is cyclic of square free order, which implies that every Sylow subgroup of G is of prime order and would be normal in every maximal subgroup containing it. Therefore, G is cyclic of square free order. This final case completes the proof of the Main Theorem. ■

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