Note on the Inequality Theorem that

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1) \text{ unless when } 0 < m < 1,$$

$$\text{when } mx^{m-1}(x-1) < x^m - 1 < m(x-1),$$

where x is any positive quantity other than unity.

By H. S. Carslaw.

The proof given in *Chrystal's Algebra*, II., pp. 42–5, of this very important theorem is deduced from elementary algebraical principles: and, though somewhat involved, is of great value, as it establishes what must be considered a fundamental theorem in the Calculus.

In this note I give a proof, in a few lines, which depends upon the First Theorem of Mean Value in Definite Integrals.

Consider the integral

$$\int_1^x x^{m-1}dx$$

where x is any positive quantity.

Then

$$\xi^{m-1}(x-1) = \frac{(x^m-1)}{m},$$

where ξ is some value of x in the range of integration.

Now when we take the curve

$$y = x^{m-1} \qquad (x>0)$$

and note that

$$y' = (m-1)x^{m-2},$$

we see that as x increases the curve continually ascends or descends, according as $m \gtrless 1$.

Therefore, when $m > 1$,

$$x^{m-1} \gtrless \xi^{m-1} \gtrless 1,$$

according as $x \gtrless 1$.

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Hence, when $m>1$,

$$mx^{m-1}(x-1)>x^m-1>m(x-1)$$

in each case, since the negative multiplier $(x-1)$ reverses the signs of inequality which occur for $x<1$.

When $m<1$, the curve descends as x increases.

Hence

$$x^{m-1} \lesseqgtr \xi^{m-1} \lesseqgtr 1,$$

according as $x \gtreqless 1$.

Therefore the former result is reversed when m is positive, *i.e.*, when $0<m<1$; and it remains as before when m is negative, as we now multiply by a negative quantity when $x>1$, and a positive when $x<1$.