ON $\mathfrak{Z}$-HYPEREXCENTRIC MODULES FOR LIE ALGEBRAS

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Abstract

Let $\mathfrak{Z}$ be a saturated formation of soluble Lie algebras over the field $F$, and let $L \in \mathfrak{Z}$. Let $V$ and $W$ be $\mathfrak{Z}$-hypercentral and $\mathfrak{Z}$-hyperexcentric $L$-modules respectively. Then $V \otimes_F W$ and $\text{Hom}_F(V, W)$ are $\mathfrak{Z}$-hyperexcentric and $H^n(L, W) = 0$ for all $n$.


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1. Introduction

Let $\mathfrak{Z}$ be a saturated formation of finite-dimensional soluble Lie algebras over the field $F$. Let $L \in \mathfrak{Z}$ and let $W$ be an $\mathfrak{Z}$-excentric irreducible $L$-module. Results in Barnes and Gastineau-Hills [4] imply that $H^n(L, W) = 0$ for $n \leq 2$, and $H^n(L, W) = 0$ for all $n$ was proved for some special cases, suggesting that this might be true in general. This was proved in Barnes [3] for fields $F$ of characteristic 0. The proof involved a description of the saturated formations over an arbitrary field of characteristic 0. Over a field of non-zero characteristic, the saturated formations are much more complicated and no useful description is available. In this paper, we give a proof independent of the characteristic of the field. All algebras and modules considered are assumed finite-dimensional over $F$.

An irreducible $L$-module $V$ is called $\mathfrak{Z}$-central if the split extension of $V$ by $L/\mathfrak{Z}_L(V)$ is in $\mathfrak{Z}$ and $\mathfrak{Z}$-excentric otherwise. An $L$-module $V$ is called $\mathfrak{Z}$-hypercentral if every composition factor of $V$ is $\mathfrak{Z}$-central and is called $\mathfrak{Z}$-hyperexcentric if ev-
very composition factor of $V$ is $\mathcal{F}$-excentric. We need the following theorems from Barnes [2].

**Theorem 1.1** ([2, Theorem 4.4]). Suppose $L \in \mathcal{F}$ and let $V$ be an $L$-module. Then $V$ is the direct sum of an $\mathcal{F}$-hypercentral $L$-module and an $\mathcal{F}$-hyperexcentric $L$-module.

**Theorem 1.2** ([2, Theorem 2.1]). Let $V$ and $W$ be $\mathcal{F}$-hypercentral $L$-modules. Then $V \otimes_F W$ and $\text{Hom}_F(V, W)$ are $\mathcal{F}$-hypercentral.

Results showing that $H^n(L, V) = 0$ for $\mathcal{F}$-excentric irreducible $L$-modules $V$ are easily extended to $\mathcal{F}$-hyperexcentric modules by using the cohomology exact sequence and induction over the composition length of the module.

## 2. $\mathcal{F}$-hyperexcentric modules

In this section, we obtain a cohomological characterisation of $\mathcal{F}$-hyperexcentric $L$-modules. The characterisation needs to use other algebras besides the algebra $L$ from which we start.

**Definition 2.1.** Suppose $L \in \mathcal{F}$. The cone of $L$ in $\mathcal{F}$ is the class $(\mathcal{F}/L)$ of all pairs $(M, \epsilon)$ where $M \in \mathcal{F}$ and $\epsilon : M \to L$ is an epimorphism. We usually omit $\epsilon$ from the notation, writing simply $M \in (\mathcal{F}/L)$.

Any $L$-module $V$ is an $M$-module via $\epsilon$ for any $M \in (\mathcal{F}/L)$. Then $V$ is $\mathcal{F}$-hypercentral as $M$-module if and only if it is $\mathcal{F}$-hypercentral as $L$-module. It follows that if $V$ is an $\mathcal{F}$-hyperexcentric $L$-module, then $H^n(M, V) = 0$ for all $M \in (\mathcal{F}/L)$ and $n \leq 2$. We would like a converse to this.

**Theorem 2.2.** Let $\mathcal{F}$ be a saturated formation and let $L \in \mathcal{F}$. Suppose $V$ is an $L$-module such that for all $M \in (\mathcal{F}/L)$, $H^1(M, V) = 0$. Then $V$ is $\mathcal{F}$-hyperexcentric.

**Proof.** $V$ is the direct sum of an $\mathcal{F}$-hypercentral module and an $\mathcal{F}$-hyperexcentric module. Thus we may suppose without loss of generality, that $V$ is $\mathcal{F}$-hypercentral, and we then have to prove $V = 0$. Suppose $V \neq 0$ and let $W$ be a minimal submodule of $V$. We form the direct sum $A$ of sufficiently many copies of $W$ to ensure that $\dim \text{Hom}_L(A, V) > \dim H^2(L, V)$, and construct the split extension $M$ of $A$ by $L$. As $W$ is $\mathcal{F}$-central, $M \in (\mathcal{F}/L)$. We use the Hochschild-Serre spectral sequence to calculate $H^1(M, V)$. We have

$$E_2^{20} = H^2(M/A, V^A) = H^2(L, V)$$
and
\[ E_2^{01} = H^0(M/A, H^1(A, V)) = \text{Hom}_F(A, V)^L = \text{Hom}_L(A, V). \]
Thus \( \dim d_2^{01}(E_2^{01}) < \dim H^2(L, V) < \dim E_2^{01} \), so \( E_3^{01} = \ker d_2^{01} \neq 0 \) and so \( H^1(M, V) \neq 0 \) contrary to assumption.

**Theorem 2.3.** Let \( \mathcal{F} \) be a saturated formation and let \( L \in \mathcal{F} \). Suppose \( V \) is an \( \mathcal{F} \)-hypercentral \( L \)-module and let \( W \) be an \( \mathcal{F} \)-hyperexcentric \( L \)-module. Then \( V \otimes_F W \) and \( \text{Hom}_F(V, W) \) are \( \mathcal{F} \)-hyperexcentric.

**Proof.** Let \( M \in (\mathcal{F}/L) \). Then \( V \) and \( W \) are \( \mathcal{F} \)-hypercentral and \( \mathcal{F} \)-hyperexcentric respectively as \( M \)-modules, and every \( M \)-module extension \( X \) of \( W \) by \( V \) splits. Thus \( H^1(M, \text{Hom}_F(V, W)) = 0 \). By Theorem 2.2, \( \text{Hom}_F(V, W) \) is \( \mathcal{F} \)-hypercentral. By Theorem 1.2, the dual module \( V^* = \text{Hom}_F(V, F) \) is \( \mathcal{F} \)-hypercentral. As
\[ V \otimes_F W \simeq V^{**} \otimes_F W \simeq \text{Hom}_F(V^*, W), \]
the result follows. \( \square \)

This suggests that we could have some sort of \( \mathbb{Z}_2 \)-grading on the class of all \( L \)-modules. However, the tensor product of two \( \mathcal{F} \)-hyperexcentric modules need not be \( \mathcal{F} \)-hypercentral. Anything can happen as is shown by the following examples. Here, \( \mathcal{N} \) denotes the saturated formation of all nilpotent algebras.

**Example 2.4.** Suppose the characteristic of \( F \) is not 2. Let \( L = \langle e \rangle \) be the 1-dimensional algebra, and let \( V = \langle v \rangle \) and \( W = \langle w \rangle \) be the modules with action given by \( ev = v \) and \( ew = w \). Then \( V \) and \( W \) are \( \mathcal{N} \)-excentric and \( V \otimes_F W \) is \( \mathcal{N} \)-excentric.

**Example 2.5.** Let \( L = \langle e \rangle \) be the 1-dimensional algebra, and let \( V = \langle v \rangle \) and \( W = \langle w \rangle \) be the modules with action given by \( ev = v \) and \( ew = -w \). Then \( V \) and \( W \) are \( \mathcal{N} \)-excentric and \( V \otimes_F W \) is \( \mathcal{N} \)-central.

**Example 2.6.** Suppose the characteristic of \( F \) is not 2. Let \( i \in \tilde{F} \) have minimum polynomial \( x^2 + 1 \). Let \( L = \langle e \rangle \) be the 1-dimensional algebra, and let \( V \) and \( W \) be 2-dimensional modules with the action given by the matrix \( A = \left( \begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array} \right) \). The eigenvalues of the action of \( e \) on \( V \otimes_F W \) are the sums of the eigenvalues on \( V \) and \( W \), thus \( 2i, 0, 0, -2i \). Thus \( V \otimes_F W \) is the direct sum of a 2-dimensional module on which the action is trivial, and a 2-dimensional module on which the action is given by the matrix \( 2A \). It is thus the sum of an \( \mathcal{N} \)-hypercentral and an \( \mathcal{N} \)-excentric module.
3. Cohomology of \( \mathfrak{F} \)-hyperexcentric modules

We can now prove the desired theorem on the cohomology of \( \mathfrak{F} \)-hyperexcentric modules.

**Theorem 3.1.** Let \( \mathfrak{F} \) be a saturated formation and let \( L \in \mathfrak{F} \). Let \( V \) be an \( \mathfrak{F} \)-hyperexcentric \( L \)-module. Then \( H^n(L, V) = 0 \) for all \( n \).

**Proof.** By the cohomology exact sequence for a submodule \( W \)

\[
\cdots \rightarrow H^n(L, W) \rightarrow H^n(L, V) \rightarrow H^n(L, V/W) \rightarrow \cdots ,
\]

we need only consider the case in which \( V \) is irreducible. We use induction over \( \dim L \). The result holds if \( \dim L = 1 \), so suppose \( \dim L > 1 \). Let \( A \) be a minimal ideal of \( L \). We use the Hochschild-Serre spectral sequence. We have

\[
E_2^{rs} = H^r(L/A, H^s(A, V)).
\]

If \( A \) acts non-trivially on \( V \), then \( V_A = 0 \) and \( H^s(A, V) = 0 \) for all \( s \) by Barnes [1, Theorem 1]. If on the other hand, \( A \) acts trivially on \( V \), then \( H^s(A, V) = \text{Hom}_F(\Lambda^s A, V) \). Now \( \Lambda^s A \) is a submodule of the tensor power of \( A \), so is \( \mathfrak{F} \)-hypercentral by Theorem 1.2. By Theorem 2.3, \( \text{Hom}_F(\Lambda^s A, V) \) is \( \mathfrak{F} \)-hyperexcentric. By induction over \( \dim L \), we have \( H^r(L/A, H^s(A, V)) = 0 \) for all \( r, s \). In either case, we have \( H^r(L/A, H^s(A, V)) = 0 \) for all \( r, s \). By the Hochschild-Serre spectral sequence, \( H^n(L, V) = 0 \) for all \( n \). \( \square \)

### References


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