LAURENT EXPANSION OF DIRICHLET SERIES

U. Balakrishnan

Let \( (a_n) \) be an increasing sequence of real numbers and \( (b_n) \) a sequence of positive real numbers. We deal here with the Dirichlet series \( f(s) = \sum b_n a_n^{-s} \) and its Laurent expansion at the abscissa of convergence, \( \lambda \), say. When \( a_n \) and \( b_n \) behave like
\[
\sum_{a_n \leq N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log N) + C_k + O(N^{-\varepsilon} \log^k N),
\]
as \( N \to \infty \), where \( P_2(x) \) is a certain polynomial, we obtain the Laurent expansion of \( f(s) \) at \( s = \lambda \), namely
\[
f(s) = P_1(s-\lambda) + \sum_{k=0}^{\infty} k!^{-1} C_k (\lambda-s)^k,
\]
where \( P_1(x) \) is a polynomial connected with \( P_2(x) \) above. Also, the connection between \( P_1 \) and \( P_2 \) is made intuitively transparent in the proof.

Suppose the Dirichlet series \( f(s) = \sum b_n a_n^{-s} \), \( s = \sigma + it \), is convergent for \( \sigma > \lambda \) \((>0)\) and has a pole of order \( d \geq 0 \) at \( s = \lambda \) and also suppose that \( f(s) \) has an analytic continuation to \( \sigma > \sigma_0(< \lambda) \). Then we know that \( f(s) \) has the Laurent expansion
\[
f(s) = P_1(s-\lambda) + \sum_{k=0}^{\infty} k!^{-1} C_k (\lambda-s)^k
\]
where

\[ P_1(x) = (d-1)! \Delta_d x^{-d} + (d-2)! \Delta_{d-1} x^{1-d} + \ldots + \Delta_1 x^{-1 \ldots} \]

at \( s = \lambda \), with some constants \( a_k \) and \( \Delta_i \). It is conjectured that the \( a_k \)'s are given by

\[ a_k = \lim_{N \to \infty} \left( \sum_{n \leq N} b_n a_n^{-\lambda} \log^k a_n - P_2(\log N) \right) \]

where

\[ P_2(x) = \Delta_{d} (k+d) x^{k+d} + \Delta_{d-1} (k+d-1) x^{k+d-1} + \ldots + \Delta_1 (k+1) x^{k+1} \]

In special cases of the function \( f(s) \) this is known to be true (see [3], [4]). What we deal here is a conditional converse of this. We have the following Tauberian theorem:

**THEOREM 1.** Let \( 0 < a_1 \leq a_2 \leq a_3 \ldots \) be an increasing sequence of real numbers and \( 0 \leq b_n \), \( n = 1,2,3, \ldots \) be arbitrary positive real numbers satisfying

\[ \sum_{n \leq N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log N) + c_k + O(N^{-\varepsilon} \log^k N) \]

for \( k = 0,1,2, \ldots, \lfloor \frac{1}{2} \log N \rfloor \), for all \( N \geq N_0 \) with the 0-constant absolute and with a fixed \( \varepsilon \), \( 0 < \varepsilon < \frac{1}{4} \), where \( P_2(x) \) is given by (2). Then the Dirichlet series \( f(s) = \sum b_n a_n^{-s} \) is convergent for \( s > \lambda \) and has the Laurent expansion (1) at \( s = \lambda \), with \( P_1(x) \) as in (1*), provided that \( c_k \ll (2k/\varepsilon)^k \), for \( k \geq \lfloor \frac{1}{2} \varepsilon \log N \rfloor \).

**REMARKS.**

1. It follows that the order of the pole of \( f(s) \) at \( s = \lambda \) is exactly the largest power of \( \log N \) appearing in \( \sum b_n a_n^{-s} \).

2. The condition in Theorem 1 could be altered to

\[ \sum_{n \leq N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log M) + c_k + O(M^{-\varepsilon} \log^k M) \]

for \( k = 0,1,2, \ldots, \lfloor \frac{1}{2} \log M \rfloor \) for all \( N \geq N_0 \) where \( M = M(N) \to \infty \) as \( N \to \infty \) and we restrict \( a_n \) by \( a_n \leq (M(n))^{100} \) for \( n \geq N_0 \).
3. The proof of the Theorem reveals explicitly how the powers of $\log N$ in $\sum b_n a_n^{-\lambda}$ are transformed to powers of $(s-\lambda)^{-1}$ in the Laurent expansion of $f(s)$.

4. We have given in Theorem 2 below a class of sequences satisfying the hypothesis of Theorem 1.

Proof of Theorem 1. We write $a_n^{-s}$ in the form

$$a_n^{-s} = a_n^{-\lambda} \left( 1 + \eta \log a_n + 2^{-1} \eta^2 \log^2 a_n + \ldots + t^{-1} \eta^t \log^t a_n \right) + O(t^{-1} |\eta|^t a_n^{-\lambda} \log^t a_n),$$

where we have denoted $\lambda - s$ by $\eta$ and have used the fact that for $x \in \mathbb{C}$

$$e^x = 1 + x + 2^{-1} x^2 + \ldots + t^{-1} x^t + O(t^{-1} |x|^t e^{|x|}).$$

Now we consider $a_1^{-s}$, $a_2^{-s}$, ..., $a_N^{-s}$ for the above expansion and by columnwise addition we get

$$\sum_{a_n \leq N} b_n a_n^{-s} = \sum_{a_n \leq N} b_n a_n^{-\lambda} + \eta \sum_{a_n \leq N} b_n a_n^{-\lambda} \log a_n + \ldots + t^{-1} \eta^t \sum_{a_n \leq N} b_n a_n^{-\lambda} \log^t a_n + O(t^{-1} |\eta|^t a_n^{-\lambda} \log^t a_n),$$

and using the hypothesis of the theorem we get for

(3) $0 \neq |\eta| \leq 10^{-6} \varepsilon \min(1, \lambda)$ and $t = \lfloor \varepsilon \log N \rfloor$

that

(4) $\sum_{a_n \leq N} b_n a_n^{-s} = \sum_{k=0}^t k^{-1} \eta^k \sum_{r=1}^d \Delta_r (k+r)^{-1} (\log N)^{k+r} + \sum_{k=0}^t k^{-1} \eta^k \sum_{a_n \leq N} b_n a_n^{-\lambda+|\eta|} \log^t a_n) + O(N^{-\varepsilon} \sum_{k=0}^\infty k^{-1} |\eta| \log N |^k + t^{-1} |\eta|^t \sum_{a_n \leq N} b_n a_n^{-\lambda+|\eta|} \log a_n).$

Now using the hypothesis of the theorem again, we get

(5) $t^{-1} |\eta|^t \sum_{a_n \leq N} b_n a_n^{-\lambda+|\eta|} \log^t a_n \ll N |\eta| \log^d N t^{-1} |\eta| \log N |^t$

$$\ll N |\eta|^{-\varepsilon} \log^d N,$$

using the choice of $t$ from (3). Now let
We write for a fixed \( r \)

\[(k+1) \ldots (k+r-1) = A_1 + A_2(k+r) + A_3(k+r)(k+r-1) + \ldots + A_r(k+r)(k+r-1) \ldots (k+2),\]

as an identity in \( k \). It is easy to check that, for \( 1 \leq i \leq r-1 \),

\[(7) \quad i!^{-1}A_1 + (i-1)!^{-1}A_2 + \ldots + A_{i+1} = 0; \quad A_1 = (-1)^{r-1}(r-1)!; \quad A_m = A_m(r).\]

Let us use this expansion in (6) and get

\[(8) \quad Q = \sum_{r=1}^{d} \Delta_r n^{-r} \sum_{i=1}^{r} A_i (k+r-i+1)!^{-1} (n \log N)^{k+r-i+1}
= \sum_{r=1}^{d} \Delta_r n^{-r} \sum_{i=1}^{r} A_i (n \log N)^{i-1} \sum_{k=0}^{t} (k+r-i+1)!^{-1} (n \log N)^{k+r-i+1}
= \sum_{r=1}^{d} \Delta_r n^{-r} \sum_{i=1}^{r} A_i (n \log N)^{i-1} \left\{ n^r - (1 + n \log N + 2^{-1}(n \log N)^2 + \ldots + (r-i)!^{-1}(n \log N)^{r-i}) + O(t!^{-1}n^{t+d}) \right\}
= \sum_{r=1}^{d} \Delta_r n^{-r} \sum_{i=1}^{r} A_i (n \log N)^{i-1} (n^r + O(n^{t-\varepsilon} \log d) N)
= \sum_{r=1}^{d} \Delta_r n^{-r} \left\{ A_1 + (A_1 + A_2)n \log N + \ldots + (n \log N)^{r-1} \sum_{i=1}^{r} (r-i)!^{-1} A_i \right\}.

Using (7), we have all of \( A_1 + A_2, 2!^{-1}A_1 + A_2 + A_3, \ldots , \sum_{i=1}^{r} (r-i)!^{-1} A_i \) are zero. Also by the choice of \( \eta \) and \( t \) as in (3) and reading \( A_1 \) from (7) we get from (8) that

\[(9) \quad Q = -\sum_{r=1}^{d} (r-1)! (-1)^{r-1} \Delta_r n^{-r} + O(N^n \log^d N + N^\varepsilon \log^2 d \ N).\]

It now follows from (5), (6) and (9) that

\[\sum b_n = \sum_{r=1}^{d} (r-1)! \Delta_r (n^{-r} + \sum_{k=0}^{t} t!^{-1} a_n^{k} + O(N^n \log^d N + N^{-\frac{i}{2}}(N))].\]
We are in \( \sigma > \lambda \) and hence \( \Re \eta < 0 \) and the truth of the theorem follows, as we let \( N \to \infty \).

Now we verify the hypotheses of theorem 1 for the case \( f(\sigma) = \zeta(\sigma,a) \), the Hurwitz zeta function. We have

\[
\zeta(\sigma,a) = a^{-\sigma} + (1+a)^{-\sigma} + (2+a)^{-\sigma} + \ldots, \quad \sigma > 1, \quad 0 < a \leq 1.
\]

We consider the sum

\[
\sum_{n=0}^{N} (n+a)^{-1} \log^n (n+a) = \sum_{n=1}^{\infty} \left\{ (n+a)^{-1} \log^n (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right\}
\]

\[
+ a^{-1} \log^n a + \int_1^{N} u^{-1} \log^k u \, du + O(N^{-1} \log^k N)
\]

\[
+ \left( \sum_{n=N}^{\infty} (n+a)^{-1} \log^n (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right)
\]

Now

\[
\left| (n+a)^{-1} \log^n (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right| \leq 2(n^{-1} \log^n (n+1) - (n+1)^{-1} \log^n n)
\]

\[
\leq 2n^{-1}(\log^n (n+1) - \log^n n) + 2n^{-2} \log^{k-1}(n+1)
\]

\[
\leq 2n^{-2}(k(\log^n (n+1))^{k-1} + \log^k (n+1))
\]

so the first sum on the right side of (10) is absolutely convergent and further, for \( k \leq \frac{\epsilon}{2} \log N \),

\[
\sum_{n=N}^{\infty} \left| (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right| \leq 6 \sum_{n=N}^{\infty} (n+1)^{-2} \log^k (n+1)
\]

\[
\leq 12N^{-1} \log^k N.
\]

Evaluating the integral we can write (10) as

\[
\sum_{n=0}^{N} (n+a)^{-1} \log^k (n+a) = (k+1)^{-1}(\log^k N)^{k+1} + c_k(a) + O(N^{-1} \log^k N),
\]

and so the hypotheses of the Theorem 1 are satisfied, with \( A_1 = 1, \quad d = 1 \) and even \( \epsilon = 1 \). Also observe using (11) that
Now Theorem 1 gives us the Laurent expansion of $\zeta(s,a)$ as

$$\zeta(s,a) = (s-1)^{-1} + b_0(a) + (1-s) b_1(a) + (1-s)^2 b_2(a) + \ldots$$

with

$$b_k(a) = k!^{-1} \lim_{N \to \infty} \left\{ \sum_{n=0}^{N} (n+a)^{-1} \log^k (n+a) - \frac{(k+1)^{-1}}{k+1} \log^k N \right\}.$$ 

Of course, for $a = 1$ we get the Laurent expansion of $\zeta(s)$ at $s = 1$. This expression for $b_k(a)$ is already present in [1] and [2]. We can easily see from the above estimates that $b_k(a) = k!^{-1} c_k(a) \ll 1$, which implies the validity of the Laurent expansion of $\zeta(s,a)$ in $|1-s| < 1$. A better estimation of $b_k(a)$ is given in [1].

Below we include a theorem, without proof, which gives a good degree of freedom in choosing a sequence $a_n$ satisfying the hypotheses of Theorem 1. We restrict ourselves to the special case $d = 1 = \lambda$ and $b_n = 1$ for all $n$. If $S_N$ denote the number of $a_n$'s in the sequence with $a_n \leq N$ we would expect $S_N$ to behave as $S_N = \Delta_1 N + O(N^{1-\epsilon})$.

This is in fact true provided we choose the $\tilde{\Delta}_1 N$ numbers as described below.

**Theorem 2.** Let integer $G \geq 1$, positive real numbers $A, B, T$ and $0 < \epsilon \leq \frac{1}{2}$ be fixed. Suppose for each $n \geq n_0$, we choose $G$ real numbers from the interval $[T_n - An^{1-\epsilon}, T_n + Bn^{1+\epsilon}]$ (the same real number may be chosen for different $n$'s, provided that we pick them from the prescribed interval) and insert into the sequence thus formed any number of positive real numbers subject to the condition that $S_N = T^{-1} G N + O(N^{1-\epsilon})$, for $N \geq N_0$. Then the sequence thus constructed satisfies the hypotheses of Theorem 1 with $\Delta_1 = T^{-1} G$ and $\lambda = 1 = d$. 

References

[1] U. Balakrishnan, "On the Laurent expansion of $\zeta(s,a)$ at $s = 1$", 


     functions defined by Dirichlet series", Illinois J. Math., 5 
     (1961), 43-44.


School of Mathematics, 
Tata Institute of Fundamental Research, 
Homi Bhabha Road, 
Bombay - 5, 
India.