Watt linkages and quadrilaterals

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1. Introduction

We define a Watt quadrilateral to be a quadrilateral with a pair of opposite sides of equal length. See Figure 1.

![Figure 1: A Watt quadrilateral: \( |AD| = |BC| \).]

2. Linkages

The Watt linkage (Figure 2) has equal-length cranks \( AD \) and \( BC \), \( A \) and \( B \) fixed, and coupler bar \( CD \). It was devised by James Watt about 1784 to constrain the steam-engine piston rod, connected at \( E \), the midpoint of \( CD \), to approximate straight-line motion over a limited range. James Watt studied the actual path traced out by point \( E \). In Cartesian coordinates it is a closed sextic curve known as Watt's curve. We mention Watt's curve only to indicate some of Watt's contributions, and to justify our naming of Watt quadrilaterals. Standard notation in the subject is to write the various lengths \( |AB| = 2a, |BC| = |AD| = b, |CD| = 2c \). In polar coordinates centred at an origin at the midpoint of \( AB \), \( r_E = |OE|, \theta = \angle BOE \), Watt's curve is, as at equation (5) of [1]:

\[
r_E^2 = b^2 - \left( a \sin \theta \pm \sqrt{c^2 - a^2 \cos^2 \theta} \right)^2.
\]

It is shown with dashes in Figure 2. (See also [2, 3, 4].) The linkage itself is shown in a general position with solid lines.

![Figure 2: Watt's linkage and Watt's curve](https://www.cambridge.org/core)

Starting around 1990, several computer packages have been used in geometry teaching in schools: these packages are able to provide animations. This has popularised linkages as a possible topic in school
geometry. Packages in this genre currently available include

- Cinderella, http://www.cinderella.de

We suggest that all the results treated in this paper can be easily demonstrated with such software. No computer software is needed to prove the results: it would have been possible for Euclid to have proved them — though we have found it easier to establish many of them first by coordinate geometry. We offer our results on Watt quadrilaterals because of their elegance, rather than because of any presumed originality.

The present authors have an earlier paper [1] on a novel application of the Watt linkage. This paper is a natural sequel to [1] in two ways.

1. Property 1(b) of this paper, which is known in the linkage literature, was used in [1]. It was natural to consider Euclidean geometry proofs of Property 1(b). From this, further Euclidean geometry suggested itself.

2. The detailed calculations in our earlier applied-mechanics work [1] were facilitated by using computer algebra packages to handle the algebra from a coordinate geometry description of the Watt linkage. The first proofs of the results of this paper, similarly, involved the use of coordinate geometry, with the Maple computer algebra package handling the routine algebra. The coordinate geometry proofs are at the URL (web address) given at reference [1].

3. Theorem 1 and its proof

3.1 Watt quadrilaterals

We will see the Varignon parallelogram EFOG of the (crossed) quadrilateral ACDB — defined in Property 1(a) — repeatedly in this paper. Sometimes it will appear through triangles similar to FOG.

**Property 1**

(a) Let O be the midpoint of side AB, E be the midpoint of side CD. Let F and G be the midpoints of the diagonals AC and BD respectively. (See Figure 3.) Then EFOG is a rhombus.

(b) Denote the values of the angles BAD, CBA and BOE by \( \alpha \), \( \pi - \beta \) and \( \theta \) respectively. Then \( \theta = \frac{1}{2}(\alpha + \beta) \).

**Proof.** OF is parallel to BC, so \( \angle EOF = \beta - \theta \). Also EF is parallel to DA, so \( \angle FEO = \theta - \alpha \). Both parts of Property 1 follow on observing that the triangle FOE is isosceles.

There are many equivalent ways to state the result. With

\[ 2\phi = \beta - \alpha \] we have \( \angle GOF = 2\phi = \angle FEG \). (1)

Garfunkel [5] defines an *equilic quadrilateral* as a Watt quadrilateral in which \( |\beta - \alpha| = \pi / 3 \). See also [4].
Next we present a result generalising Garfunkel's Theorem 3 from equilic to general Watt quadrilaterals.

**Property 2**
(a) Let \( Q \) be the point of intersection of the perpendicular bisectors of \( AB \) and \( CD \). (See Figure 4.) Then the isosceles triangles \( QDC \) and \( QAB \) are both similar to triangle \( EFG \).

(b) Let \( Q' \) be the point of intersection of the perpendicular bisectors of \( AC \) and \( BD \). Then the isosceles triangles \( Q'BD \) and \( Q'CA \) are similar to triangle \( FEO \).

**Proof.** (a) Triangle \( AQD \) is congruent to triangle \( BQC \) (since corresponding sides are equal). Hence the sum of the angles at the base of triangle \( ABQ \) is \( \alpha + (\pi - \beta) \). Therefore

\[
\angle AQB = 2\phi. \tag{2}
\]

Hence triangle \( QAB \) is similar to triangle \( EFG \). Also

\[
\angle DQC = 2\phi, \tag{3}
\]

showing that triangle \( QDC \) is similar to triangle \( EFG \).
(b) The proof is similar to (a). Triangle $BCQ'$ is congruent to $DAQ'$. Therefore

$$\angle CQ'A = \pi - 2\phi = \angle BQ'D.$$  \hspace{1cm} (4)

**Property 3**

Let $S$ be the point of intersection of lines $AD$ and $BC$.

(a) The circumcircles of triangles $QAB$, $QDC$ both pass through point $S$.

(b) The circumcircles of triangles $Q'CA$, $Q'BD$ both pass through point $S$.

**Proof.** (a) First note that

$$\angle ASB = 2\phi.$$  \hspace{1cm} (5)

Equations (2) and (5) show that $ABQS$ is a cyclic quadrilateral. Similarly one finds that $DCQS$ is a cyclic quadrilateral.

(b) The proof of (b) is similar to that of (a).

**Property 4**

(a) $SQ'$ is the internal bisector of angle $ASB$. $SQ$ is the external bisector of angle $ASB$.

(b) $SQ$ is perpendicular to $SQ'$.

(c) $SQ$ is perpendicular to $EO$. $SQ'$ is parallel to $EO$ and $SQ$ is parallel to $FG$.

![Figure 5](https://www.cambridge.org/core/core/terms.https://doi.org/10.1017/S0025557200176107)

**Proof.** (a) See Figure 5. By Property 3(b), $S$, $A$, $Q'$ and $C$ are on the same circle. Hence

$$\angle ASQ' = \angle ACQ',$$  \hspace{1cm} (6)

$$\angle Q'SC = \angle Q'AC.$$  \hspace{1cm} (7)
Since triangle $Q'CA$ is isosceles, $\angle ACQ' = \angle Q'AC$. The last two sentences — equations (6, 7), with either of (4) or (5) — give

$$\angle ASQ' = \angle Q'SC = \phi,$$

which establishes that $SQ'$ bisects the angle $ASB$.

The result concerning $SQ$ being the external bisector of angle $ASB$ is proved similarly.

(b) This is an example of the fact that the internal and external bisectors of an angle are perpendicular.

(c) Since $EF$ is parallel to $SD$, and $EG$ is parallel to $SC$, the diagonal $OE$ of the rhombus $EFOG$ is parallel to the internal bisector of angle $DSC$. Hence $SQ$ is perpendicular to $EO$ as required. The remaining results follow readily.

**Property 5**

Let $P$ be the point where the circle through $A$, $Q'$, $C$ and $S$ intersects again the line $QS$. Similarly let $R$ be the point where the circle through $D$, $Q'$, $B$ and $S$ intersects again the line $SQ$. Then we have the following.

(a) $PQ'$ is the perpendicular bisector of $AC$, a diameter of the circle $PAC$.

$b)$ $RQ'$ is the perpendicular bisector of $BD$, a diameter of the circle $RDB$.

(b) The isosceles triangles $PAC$ and $RDB$ are similar to triangle $EFG$.

**Proof.** (a) See Figures 5 and 6. The circle through $A$, $Q'$, $C$ and $S$ is that of Property 3(b). Since $CAPS$ is a cyclic quadrilateral

$$\angle CAP = \angle CSQ = \angle PSA$$

and since, by Property 4(a), $SQ$ is the external bisector of $\angle ASC$,

$$\angle PSA = \angle PCA.$$ 

Hence triangle $PAC$ is isosceles, and so $P$ lies on the perpendicular bisector of $AC$. By definition, $Q'$ lies on the perpendicular bisector of $AC$. Hence
\(PQ'\) is the perpendicular bisector of the chord \(AC\), and is therefore a diameter of circle \(PAC\).

The result concerning \(RQ'\) is established similarly. By considering angles subtended by line segments \(AQ'\) and \(CQ'\) respectively:
\begin{itemize}
  \item \(\angle APC = \angle ASC = 2\phi\). Hence the isosceles triangles \(PAC\) and \(EFG\) are similar. In like manner, the isosceles triangles \(RDB\) and \(EFG\) are similar.
\end{itemize}

We can now bring together the most important and interesting parts of Properties 2, 3 and 5, to form Theorem 1 below. Note that the definition of \(P\) (and that of \(R\)) in Property 5 differs for the definition of \(\tilde{P}\) (and that of \(\tilde{R}\)) in Theorem 1 below. The definitions given in Theorem 1 enable its statement – though not its proof – to be read and understood immediately after the definition of Watt quadrilaterals, without reference to any of the Properties we have proved.

However, it is a consequence of Property 5 that \(P\) is actually the same point as \(\tilde{P}\) (and \(R\) is the same as \(\tilde{R}\)).

**Theorem 1.** Let \(ABCD\) be a Watt quadrilateral, \(AD\) and \(BC\) being the equal sides.

- Let \(S\) be the point of intersection of the lines \(AD\) and \(BC\).
- Let \(Q\) be the point of intersection of the perpendicular bisectors of \(AB\) and \(CD\).
- Let \(\tilde{P}\) be the point of intersection of the perpendicular bisector of \(AC\) with the line \(SQ\).
- Let \(\tilde{R}\) be the point of intersection of the perpendicular bisector of \(BD\) with the line \(SQ\).

Then the four isosceles triangles \(QAB, QDC, \tilde{PAC}, \tilde{RDB}\) are similar. The circumcircles of these four triangles intersect at \(S\).

We now establish some more properties relating to the points defined in Theorem 1.

**Property 6**

Let \(J\) and \(K\) be the midpoints of \(AD\) and \(BC\). Then we have the following.
\begin{itemize}
  \item \(M\) is the midpoint of the segment \(JK\) is the centre of the rhombus \(EFOG\).
  \item The triangle \(QJK\) is similar to triangle \(EFG\). \(MQ\) is the perpendicular bisector of \(JK\).
  \item The circle with diameter \(QQ'\), which (by Property 4(b)) passes through \(S\), also passes through \(J, K\).
\end{itemize}

The lines \(QQ'\) and \(JK\) are perpendicular, and their point of intersection is \(M\).
Proof. (a) We have seen in Property 1 that $EFOG$ is a rhombus, so that $EO$ and $FG$ have a common midpoint $M$. Similarly $JFKG$ is a parallelogram, so $M$ is the midpoint of $JK$ also.

(b) Now both $ABKJ$ and $JKCD$ are Watt quadrilaterals, so our previous results can be applied to them. Consider, for example, the Watt quadrilateral $JKCD$. The apex angle of the rhombus sides parallel to $AD$ and $BC$ with vertices at $M$ and $E$ once again has vertex angle, angle at $E$, equal to $2\phi$. Let $Q_{JK}$ be the point of intersection of the perpendicular bisectors of $DC$ and of $JK$. Now $Q_{JK}DC$ is similar to triangle $EFG$. However, so is $DQC$. Thus $Q = Q_{JK}$. Thus $QJK$ is isosceles, which establishes part (b). It follows readily that $MQ$ is the perpendicular bisector of $JK$.

(c) $ACKJ$ is also a Watt quadrilateral (in crossed configuration). Thus triangle $Q'KJ$ is isosceles and similar to triangle $FOE$. Also $MQ'$ is perpendicular to $KJ$. Combining the last sentence here with the last sentence of our proof of part (b), we see that $QQ'$ is the diameter of circle $Q'KQSJ$.

Property 7

$Q$ is the midpoint of $PR$.

Proof. $PR$, called $SQ$ in Property 4(c), is parallel to $FG$. Hence triangle $Q'GF$ is similar to triangle $Q'RP$. Now consider the line $QMQ'$, the similar triangles $Q'MF$ and $Q'QP$, and the similar triangles $Q'GM$ and $Q'RQ$. Since $M$ is the midpoint of $FG$ we find $Q$ is the midpoint of $PR$.

Property 7 provides further information about the points occurring in Theorem 1. (The results of Properties 6 and 7 can be used in a Euclidean geometry proof of Theorem 2, and indeed were so used before we established our more general Theorem 2g.)

If one looks through geometry books with an awareness of Watt quadrilaterals, isolated examples can be found. For example, in the case when $SDB$ is isosceles, some of our results appear to be contained in published results: [6, p. 28, 43], [7, result 77, p. 51]. Other results, not used in our proofs, follow.

The first of these is related to a special case of a theorem of Menelaus.

Property (Example 110 of [8]).

Let $U$ denote the intersection of the diagonals $AC$ and $BD$. Then, for the Watt quadrilateral $ABCD$, we have the following equation involving lengths: $|AS| |DU| = |CS| |BU|$. 

Property (Examples from [9]).

Let $ABCD$ be a Watt quadrilateral and $O$, $E$ as above. Let $C_E$ be the point of intersection of the lines $BC$ and $OE$. Let $D_E$ be the point of intersection of the lines $AD$ and $OE$. See Figure 7. Then we have the following.
(Example 6.225.) $|DD_E| = |BC_E| = |AD_E| = |CC_E|.$

(Example 6.226.) $\angle AD_E O = \angle OC_E B = \phi$. (Hence the isosceles triangle $SC_E D_E$ is similar to triangle $FOE$. Also, $B, Q, C_E, O$ are concyclic with diameter $BQ$; and $A, Q, D_E, O$ are concyclic with diameter $AQ$.)

(Example 6.227.) $|DD_E| = |CC_E|$. (Hence $ABC_E D_E$ and $DCC_E D_E$ are also Watt quadrilaterals.)

Property (Example 4.23 from [10]).

(a) Let $ABCD$ be a Watt quadrilateral with $|AD| = |BC|$. Circles are described with diameters as the sides of the crossed quadrilateral $ACDB$. The four common chords of pairs of circles on adjacent sides form a rhombus.

(b) The vertices of the rhombus are the orthocentres of the triangles $ABS$, $ACS$, $DBS$, $DCS$.

Item (a) is that in [10, p. 125]; item (b) is a small addition to help relate the Property to our Theorem 2(c) of §4. In particular, we will see that the rhombus of this last property is similar to rhombus $EFOG$.

3.2 General quadrilaterals

After establishing the results for Watt quadrilaterals, we noted the following generalisations.

Theorem 1g: Let $ABCD$ be a quadrilateral. Let $O, E, F, G$ be the midpoints of the segments $AB, CD, AC, BD$, so that $EFOG$ is a parallelogram.

- Let $S$ be the point of intersection of the lines $AD$ and $BC$.
- Let $Q_g$ be the Miquel point of $ABCD$, the point ($\neq S$) of intersection of the circumcircles of $ABS$ and $DCS$. 
• Let $P_g$ be the second point of intersection of the circumcircle of $ACS$ with the line $SQ_g$.
• Let $R_g$ be the second point of intersection of the circumcircle of $DBS$ with the line $SQ_g$.

Then the four triangles $Q_gAB$, $Q_gDC$, $P_gAC$, $R_gDB$ are similar to the triangle $EFG$. The circumcircles of these four triangles intersect at $S$.

For the proof of this and of related properties it is also useful to define $Q'_g$ to be the Miquel point of $ABCD$, the point $(\neq S)$ of intersection of the circumcircles of $ACS$ and $DBS$. Analogues of the later properties are as follows.

6g (a) Let $J$, $K$ denote the midpoints of the sides $AD$, $BC$. The midpoint $M$ of the segment $JK$ coincides with the point of intersection of $OE$ and $FG$.

(b) Triangle $Q_gJK$ is similar to triangle $EFG$.

(c) The circumcircle of $SJK$ passes through both $Q_g$ and $Q'_g$.

7g The point $Q_g$ is midway between $P_g$ and $R_g$.

4. Associated quadruples of circumcentres

The main result of this section is part (a) of Theorem 2, concerning circumcentres. Part (b) is trivial and part (c-i) is a combination of the preceding parts. The rhombus-of-orthocentres result in part (c-ii) is in [10].

Theorem 2. Let $ABCD$ be a Watt quadrilateral.

(a) (i) The centres of the circumcircles of Theorem 1, namely the circumcentres $E_O$, $F_O$, $O_O$, $G_O$ of the triangles $DCS$, $ACS$, $ABS$, $DBS$ respectively, form a rhombus. (ii) This rhombus of circumcentres is similar to $EFOG$ but rotated through a right angle.

(b) (i) The centroids $E_G$, $F_G$, $O_G$, $G_G$ of the triangles $DCS$, $ACS$, $ABS$, $DBS$ respectively, form a rhombus. (ii) This rhombus of centroids is similar to $EFOG$ with corresponding sides parallel.

(c) Form convex combinations of the vectors and so obtain points on the Euler lines (for the appropriate triangle) as indicated:

\[ E_t = tE_G + (1 - t)E_O, \quad F_t = tF_G + (1 - t)F_O, \]
\[ O_t = tO_G + (1 - t)O_O, \quad G_t = tG_G + (1 - t)G_O. \]

(i) $E_tF_tO_tG_t$ forms a rhombus. (ii) When $t = 3$, the points defined by the convex combinations are the orthocentres of the triangle, and the rhombus is again similar to $EFOG$ but rotated through a right angle.

Figure 8 is based on that used by a referee in a Euclidean geometry proof of part (a) following from the Properties of §3.1.
There are many related results. The Euler lines of the triangles \( ABS, ACS, DCS \) and \( DBS \) are relevant to the parallelograms \( E_iF_iO_iG_i \), and the Euler line of \( JKS \) is relevant to the centres of the parallelograms. Concerning these Euler lines, we have the following.

**Property 8**

For a Watt quadrilateral, the Euler lines of triangles \( ABS, DCS, JKS \) and the line \( SQ \) are concurrent, and the Euler lines of triangles \( ACS, BDS, JKS \) and the line \( SQ' \) are concurrent.

The Euler line for \( JSK \) is parallel to the Gauss line \( FG \).

We next, in §4.1, discuss some results about general quadrilaterals, before returning to Watt quadrilaterals in §4.2.

### 4.1 General quadrilaterals

Theorem 2g(a) below, especially part (a-i), seems a beautiful and simple result. It also generalises in several different directions, some of which we report below. Our treatment is abbreviated – omitting most proofs – for three reasons. The first is that a lengthy treatment of general quadrilaterals would distract from our main interest associated with Watt quadrilaterals. The second is that we suspect that Theorem 2g or some generalisation must already be known and it is our hope that readers will be able to clarify the history. (There might be ‘triangle’ variants of Theorem 2g, in which points \( C, E, D \) coincide with \( S \).) The third reason is that, for some items associated with Theorem 2g(c), we have only our Maple-based coordinate geometry proofs.
We remark that parts of Theorem 2g below generalise to $n$ dimensions. One considers ‘$n$ lines through the origin $S = 0$ with linearly independent directions’, upon each of which there are given two distinct non-zero points.

(a) The centres of the $2^n$ spheres, each passing through $S = 0$ and precisely one of the given points on each of the lines, lie on a parallelepiped.

(b) Given $n$ points — identified with vectors — define their centroid to be the vector which is the average of the $n$ points. Using the same sets of points as in (a), form the $2^n$ centroids. That there is a parallelepiped of centroids can be established by coordinate geometry.

We return now to $n = 2$.

**Theorem 2g.** Let $ABCD$ be the quadrilateral and $S, E, F, O, G$ the associated points defined in Theorem 1g.

(a) (i) Let $E_0, F_0, O_0, G_0$ be the circumcentres respectively of triangles $DCS, ACS, ABS, DBS$. These circumcentres are the vertices of a parallelogram. (ii) This parallelogram of circumcentres is similar to the parallelogram $EFOG$ of Theorem 1g. (iii) Furthermore, the direction of the longer sides of the parallelogram of circumcentres is perpendicular to that of the shorter sides of $EFOG$.

(b) (i) The centroids $E_G, F_G, O_G, G_G$ of the triangles $DCS, ACS, ABS$ and $DBS$, form a parallelogram. (ii) This parallelogram of centroids is similar to $EFOG$ with corresponding sides parallel.

(c) Form convex combinations of the vectors and so obtain points on the Euler lines (for the appropriate triangle) as indicated:

- $E_t = tE_G + (1 - t)E_0,$
- $F_t = tF_G + (1 - t)F_0,$
- $O_t = tO_G + (1 - t)O_0,$
- $G_t = tG_G + (1 - t)G_0.$

(i) $E_tF_tO_tG_t$ forms a parallelogram and $|E_tF_t|/|EF| = |E_GG_t|/|EG|$. Also define the corresponding convex-combination triangle point $M_t$ for triangle $JSK$. $M_t$ is the midpoint of the parallelogram $E_tF_tO_tG_t$. (ii) When $t = 3$, the points defined by the convex combinations are the orthocentres of the triangle, and the parallelogram is again similar to $EFOG$. Furthermore the direction of the longer (resp. shorter) diagonal of the parallelogram of orthocentres is perpendicular to that of the longer (resp. shorter) diagonal of $EFOG$: the direction of the longer (resp. shorter) sides of the parallelogram of orthocentres is perpendicular to that of the longer (resp. shorter) sides of $EFOG$.

We do not yet have Euclidean geometry proofs for all parts of the above. Part (b), however, is genuinely elementary. A Euclidean geometry proof of part (a-i) is simple, while that for the later parts of (a) (e.g. that $OF$ is perpendicular to $O_0G_0$ and $OG$ is perpendicular to $O_0F_0$) are harder, and part (c) is more general than (a).

**Proof.** We use coordinate geometry. Part (b) is elementary, in any form of proof. Our account below deals with parts (a) and (c). To shorten the
formulæ, it is convenient, for this proof, to translate the origin of coordinates to \( S \). Then, if the point \( A \) has complex coordinates \( z_A = x_A + iy_A \), and \( B \) has complex coordinates \( z_B \), for real numbers \( s_1 \) and \( s_2 \) we have \( z_D = s_1 z_A \) and \( z_C = s_2 z_B \).

Various 'triangle points' occur. Considering the triangle with vertices \( S = 0, z_1 \) and \( z_2 \) all our triangle points \( c(z_1, z_2) \) satisfy:

\[
\begin{align*}
  c(z_2, z_1) &= c(z_1, z_2) \quad \text{(symmetry)} \\
  c(r z_1, r z_2) &= r c(z_1, z_2) \quad \text{(scaling)} \\
  c(z_1, \bar{z}_2) &= \bar{c}(z_1, z_2) \quad \text{(reflection)} \\
  c(z_1, \bar{z}_1) \text{ is real} \\
  c(e^{i\pi/3}, e^{-i\pi/3}) &= \frac{1}{2\sqrt{3}}.
\end{align*}
\]

Our two main examples are \( c_0 \) giving the circumcentres of (a), and \( c_G \) giving the centroids of (b). Now, for any points \( z_1, z_2 \), the centre of the circle through \( 0, z_1, z_2 \) is at

\[
c_0(z_1, z_2) = \frac{z_1 z_2 (\bar{z}_2 - \bar{z}_1)}{z_1 \bar{z}_2 - \overline{z_1 z_2}},
\]

and the centroid of this triangle is at

\[
c_G(z_1, z_2) = \frac{1}{3}(z_1 + z_2).
\]

Also define

\[
c_t(z_1, z_2) = (1 - t) c_0(z_1, z_2) + t c_G(z_1, z_2).
\]

(\( t = 3 \) gives the orthocentre: \( t = 3/2 \) gives the nine-point-circle centre.)

The result that the four circumcentres form the vertices of a parallelogram (obvious when \( t = 0 \) and \( t = 1 \) from considerations of Euclidean geometry) is also verified from the identity:

\[
c_t(z_A, z_B) - c_t(s_1 z_A, z_B) = c_t(z_A, s_2 z_B) - c_t(s_1 z_A, s_2 z_B).
\]

Also set

\[
\begin{align*}
  z_O &= \frac{1}{2}(z_A + z_B) = \frac{1}{2} c_G(z_A, z_B), \\
  z_G &= \frac{1}{2}(s_1 z_A + z_B) = \frac{1}{2} c_G(s_1 z_A, z_B), \\
  z_E &= \frac{1}{2}(s_1 z_A + s_2 z_B) = \frac{1}{2} c_G(s_1 z_A, s_2 z_B), \\
  z_F &= \frac{1}{2}(z_A + s_2 z_B) = \frac{1}{2} c_G(z_A, s_2 z_B).
\end{align*}
\]

It is straightforward to calculate lengths, and thereby to check that \( F_0 O_0 E_O \) and \( E_G F_G O_G \) are both similar to triangle \( EFG \).

Another consequence from calculating lengths is that

\[
\left| \frac{c_t(z_A, z_B) - c_t(s_1 z_A, z_B)}{c_t(z_A, z_B) - c_t(s_1 z_A, s_2 z_B)} \right| = \left| \frac{c_t(z_A, z_B) - c_t(z_A, s_2 z_B)}{c_1(z_A, z_B) - c_1(z_A, s_2 z_B)} \right|
\]
i.e. ratios of lengths of corresponding sides of the parallelograms are equal. Of course, this doesn't ensure that the two parallelograms are similar, as corresponding angles might differ, and generally do.

It remains to prove (a-iii) and (c-ii). Consider directions \([x, y]\) and \([\xi, \eta]\), and write \(z = x + iy, \zeta = \xi + i\eta\). These directions are orthogonal if and only if the real part of \(z\zeta\) is zero. The result of 2g(a-iii) is established by showing

\[
\text{RealPart} \left( (c_O(z_A, z_B) - c_O(s_1z_A, z_B))\bar{z}_B \right) = 0,
\]

\[
\text{RealPart} \left( (c_O(z_A, z_B) - c_O(z_A, s_2z_B))\bar{z}_A \right) = 0.
\]

The corresponding calculation involving \(c_3\) to establish part (c-ii) is similar.

The circumcentre case corresponds to \(t = 0\), the centroid case corresponds to \(t = 1\), the orthocentre case corresponds to \(t = 3\), and for each of these and also for \(t = t_s = 3/(1 + 2\cos 2\phi)\) the parallelogram of triangle centres is similar to \(EFOG\). \(t = 3/2\) is the nine-point-circle centre case, and the parallelogram of these centres is not, in general, similar to \(EFOG\).

The parallelogram of circumcentres is larger than \(EFOG\), and for the ratio of the linear dimensions we find \(|OF|/|O_FO| = |OG|/|O_OG| = \sin 2\phi\). The parallelogram of orthocentres is similar to \(EFOG\) and for the ratio of the linear dimensions \(|OF|/|O_HF_H| = |OG|/|O_HG_H| = 2 \tan 2\phi\).

Consider the triangles \(ABS\) and \(DCS\). The line joining the centroids of these triangles is parallel to \(OE\). The line joining the circumcentres of these triangles is perpendicular to \(SQ'\). The line joining the orthocentres of these triangles is perpendicular to \(FG\). (For a Watt quadrilateral, each of these three lines, and more generally, any \(O_iE_i\) is parallel to \(OE\) and also to \(SQ'\).)

It is easy with computer support to calculate dot products and thereby to find lengths, and to evaluate expressions such as \(\cos^2 \angle G_iO_iF_i\). Concerning lengths, we find

\[
|O_iE_i|^2 = \frac{(t-1)(t-3)}{3} \frac{FG^2}{\sin^2 2\phi} + \frac{t(t-3)}{9} \frac{OE^2}{\cos^2 2\phi} + \frac{t(t-1)}{6} \frac{FG^2}{\cos^2 2\phi}
\]

\[
|O_iF_i|^2 = |O_iG_i|^2 = \frac{(t-1)(t-3)}{3} \frac{1}{\sin^2 2\phi} + \frac{t(t-3)}{9} \frac{4}{\cos^2 2\phi}
\]

\[
|F_iG_i|^2 = \frac{(t-1)(t-3)}{3} \frac{OE^2}{\sin^2 2\phi} + \frac{t(t-3)}{9} \frac{4|FG|^2}{\cos^2 2\phi} + \frac{t(t-1)}{6} \frac{OE^2}{\cos^2 2\phi}
\]

Concerning angles, we have

\[
\frac{O_iF_i.O_iG_i}{OF.OG} = \frac{(t-1)(t-3)}{3} \frac{1}{\sin^2 2\phi} + \frac{t(t-3)}{9} \frac{4}{\cos^2 2\phi} - \frac{t(t-1)}{6} \frac{4\cos^2 2\phi}{\sin^2 2\phi}
\]

\[
\gamma(t) = \cos^2 \angle G_iO_iF_i = \frac{(O_iF_i.O_iG_i)^2}{|O_iF_i|^2 |O_iG_i|^2}
\]
We have already noted the four values of $t$ for which $\cos^2 \angle G_i O_i F_i = \cos^2 2\phi$. We also find the following:

(i) $E_i F_i O_i G_i$ becomes a line segment,

$$\gamma(t) = \cos^2 \angle G_i O_i F_i = 1, \text{ when } t = \frac{3}{1 \pm 2 \cos 2\phi}. \quad (11)$$

(In the numerical cases we have checked, these two lines intersect orthogonally at $S$.)

(ii) $E_i F_i O_i G_i$ is a rectangle,

$$\gamma(t) = \cos^2 \angle G_i O_i F_i = 0, \text{ when } t = \frac{3}{3 \pm 2 \sin 2\phi}. \quad (12)$$

(In the numerical cases we have checked, these two rectangles are similar but the one is rotated through a right angle from the other.)

There are many further results for special types of quadrilaterals. For example, for a cyclic quadrilateral, the Euler lines of the four triangles $ABS$, $DBS$, $DCS$, $ACS$ are concurrent. (In general, this point of concurrency cannot be found by setting a given value of $t$ into the formulae for $O_i, G_i, E_i, F_i$.)

For any quadrilateral where at some value of $t$ we have $E_i = F_i = O_i = G_i$, all the parallelograms will be similar to $EFOG$.

The referee suggested that we consider a quadrilateral in orthogonal configuration to be one where $\angle ASB = 2\phi$ is a right angle. For such quadrilaterals, $EFOG$ from Theorem 2g is a rectangle. Also the orthocentres of the four triangles coincide with $S$. (Hence, the Euler lines of the four triangles pass through $S$.) For any quadrilateral in orthogonal configuration, the $E_i F_i O_i G_i$ are rectangles similar to $EFOG$, with $E_0 F_0 O_0 G_0 = EFOG$.

It may be that the only time that the four Euler lines are concurrent at the same value of $t$ is when $ABCD$ is a quadrilateral in orthogonal configuration.

The earlier part of our paper, §3, and our main ‘possibly new’ contribution – Theorem 2(a), 2g(a) – concerns circumcentres, the orthocentres result in Theorem 2(c) being in [10]. Accordingly, we return to considerations of circumcentres in order to place our result in context with those that are well known. Amongst the better known results in the general area is the following. (See for example: [4]; [11, Chapter II, p. 71, Examples 3, 5]; [8, Examples 308, 309]; [7, p. 119]; [12, p. 254]; [13].)
Wallace-Miquel Theorem

Let the notation be as in Theorems 1g and 2g. Additionally, define \( T \) as the intersection of the lines \( AB \) and \( DC \). Consider the circumcentres of \( SAB, SDC, TAD, TBC \).
(a) The four circumcircles intersect at a common point which we denote \( Q_g \).
(b) The four circumcentres are concyclic.
(c) The point \( Q_g \) lies on this circle of circumcentres.

The Wallace-Miquel Theorem is best derived as a consequence of Miquel’s Theorem (1838) for triangles. (See, for example, [11].) Coordinate geometry and complex coordinates yield proofs too: a useful fact is that, given a set of four complex numbers, one tests whether they are concyclic (or collinear in special cases) by showing their cross-ratio is real. (In [14] there is some description of a third approach to proofs of both Miquel’s Theorem and the Wallace-Miquel Theorem. Both can be interpreted by projecting relatively simple 3-dimensional configurations to the plane. See [14, p. 133 and p. 110 respectively]. The geometrical graphics software mentioned in §2 – or Maple – provide excellent tools for visualisation here.)

Theorem 2g(a) seems to be a separate result from the Wallace-Miquel Theorem. For Watt quadrilaterals, and the geometry of the Wallace-Miquel theorem, the now-familiar isosceles triangle appears again. For a Watt quadrilateral, the triangle with vertex \( Q \) and base vertices the circumcentres of \( TAD \) and of \( TBC \) is isosceles and similar to triangle \( FEG \).

Some information on the quadruples of circumcentres is summarised in Table 1. Consider the quadrilateral \( ABCD \) and the points \( S, T, U \). It is natural to consider the 12 circumcircles, each passing through one of \( S, T \) or \( U \) and an appropriate pair of vertices of \( ABCD \). The entries in each column of the table are a set of four triangles whose four circumcentres form a parallelogram. The entries in each row of the table are a set of four triangles whose four circumcentres are concyclic, and whose Miquel point is denoted at the left.

<table>
<thead>
<tr>
<th>( c^\prime )</th>
<th>( P )</th>
<th>( S )</th>
<th>( T )</th>
<th>( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>SAB, SCD</td>
<td>TAD, TBC</td>
<td>UAD, UBC</td>
<td></td>
</tr>
<tr>
<td>( Q' )</td>
<td>SAC, SBD</td>
<td>TAC, TBD</td>
<td>UAB, UCD</td>
<td></td>
</tr>
</tbody>
</table>

| \( Q'' \) | SAB, SCD | TAD, TBC | UAD, UBC |

TABLE 1: Parallelograms and concyclic quadrilaterals of circumcentres
(There are also other results concerning the other special points of the triangles, e.g. orthocentres. See for example [12].)
4.2 Watt quadrilaterals and other special cases

Proof of Theorem 2.

From Property 1(a) \( EFOG \) is a rhombus. Combining this with Theorem 2g establishes the result.

More generally, the ratio of the adjacent sides of the parallelogram \( EFOG \) is determined by the ratio of the lengths of \( AD \) and \( BC \), so would be invariant as a four-bar linkage moves through its states. The angle at \( E \) is \( 2\phi \), and varies as one imagines the linkage moved. The ratio and the angle of the last two sentences completely determine the shape of \( EFOG \).

For a Watt quadrilateral, the diagonals \( OE \) and \( FG \) make equal angles at the intersection point \( M: EFOG \) is (a rhombus so) inscribable. For a quadrilateral in orthogonal configuration, the lengths of \( OE \) and \( FG \) are equal: \( EFOG \) is (a rectangle so) circumscribable.

Proof of Property 8.

So far, our only proof of Property 8 is computer-aided. The concurrence of the triples of Euler lines is closely related to the parallelograms \( E_iF_iO_iG_i \), collapsing to line segments at the values of \( t \) given in equation (11).

There are many formulae which are simpler in the Watt quadrilateral case than for general quadrilaterals. For example, for a Watt quadrilateral

\[
\frac{|ME|}{|NE_O|} = \frac{b^2}{8},
\]

while for a quadrilateral in an orthogonal configuration

\[
\frac{|ME|}{|NE_O|} = \frac{1}{16} \left( |AD|^2 + |BC|^2 \right).
\]

For a Watt quadrilateral, any line \( O_iE_i \) is parallel to \( OE \) and also to \( SQ' \). The lines \( OO_0 \), \( EE_1 \), and \( SQ \) are concurrent and \( OO_1 \cap EE_1 = S \), and for Watt: \( OO_0 \cap EE_1 = Q \).

The lines \( FF_1 \), \( GG_1 \), and \( SQ' \) are concurrent and \( FF_1 \cap GG_1 = S \) also true in general, and for Watt: \( FF_0 \cap GG_0 = Q' \).

A Watt quadrilateral in orthogonal configuration is a Watt quadrilateral with \( 2\phi = \pi/2 \). For these we have (with \( |AD| = b = |BC| \)) the following properties:

\[
|JK|^2 - \frac{1}{2} (|AB|^2 + |CD|^2 - b^2) = 0,
\]

\[
M = N, \quad E = E_0, \quad F = F_0, \quad G = G_0, \quad O = O_0.
\]

The two coincident rhombuses are squares. \( |PR| = 2 |FG| = 2 |EO| \). For a Watt quadrilateral in orthogonal configuration, each Euler line is a line through \( S \) and one of the points \( E, F, O, G \), and, for all \( t \), \( E_iF_iO_iG_i \) is a square.

We expect that there are many more properties to be discovered, and that many of the properties of Watt quadrilaterals will have analogues in corresponding properties of other special types of quadrilaterals.
Acknowledgements

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References

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Irrational behaviour

Although confused by magic, the conviction of Pythagoras that numbers were the key to the universe, and indeed to all scientific knowledge, was a big step in man's thinking. His discovery of irrational numbers, that were both odd and even at the same time, caused the collapse of the Brotherhood.

Graham Howlett, of Godalming, found this in The History of Clocks and Watches, by Eric Bruton. It does not say whether the Brotherhood collapsed in grief or laughter.

Touché

... the character of $\pi + e, \pi e, \pi e$ is unknown. (March 2004)
It is well-known that $\pi, e, \pi e$... are all irrational. (July 2004)

Nick Lord was surprised to see these conflicting claims about $\pi e$ in successive issues of the Gazette and was sceptical that a proof had emerged sometime in between.

With a name like that....

Family celebration: Todd Hamilton with his wife Jaque and children Tyler, Kaylee and Drake.

Nick Lord saw this caption to a photograph of the new Open Golf champion in the Daily Telegraph of 20th June 2004, and speculated on the likelihood of Todd's daughter becoming a mathematician.

Tailor expansion?

It is only Dior who can transform nature, at least as far as the female body is concerned. He can choose any shape of Euclidean geometry and finish the job within a fortnight.

Michel Bataille, of Rouen, found this in George Mikes' book Little Cabbages, and asked how long Dior would take over a non-Euclidean dress.