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On groups of exponent four with generators of order two

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The largest group of exponent 4 generated by n elements of order 2 has nilpotency class $n + 1$ when $n \geq 3$.

Introduction

Let $G(n) = \left\langle a_1, \ldots, a_n; a_i^2, w^4 \right\rangle$ $(n \geq 2)$ be the freest group of exponent 4 on n generators of order 2. It is almost immediate that

$$G(2) = \left\langle a_1, a_2; a_1^2, a_2^2, w^4 \right\rangle = \left\langle a_1, a_2; a_1^2, a_2^2, [a_1, a_2]^2 \right\rangle =$$
$$= \left\langle a_1, a_2; a_1^2, a_2^2, [a_1, a_2, a_2], [a_1, a_2, a_1] \right\rangle ,$$

so that $G(2)$ is nilpotent of class precisely 2. It is a well-known result of Wright [1] that the nilpotency class of $G(n)$ $(n \geq 3)$ is at most $n + 1$. In this note we show that the nilpotency class of $G(n)$ $(n \geq 3)$ is at least $n + 1$. This settles a long standing conjecture of Wright [1].

Preliminaries

Let S denote the free associative ring freely generated by

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$x_1, x_2, \ldots$. Let $\rho = \big((1+x_1) \ldots (1+x_l)-1\big)^3$ $(l \geq 2)$ be a fixed element of S . A straight forward expansion shows that

(1) $$\rho = \sum_{(r,s,t)} \xi_r \xi_s \xi_t \quad (r, s, t \in \{1, \ldots, l\}) ,$$

where $\xi_m = \sum x_{k(1)} \ldots x_{k(m)}$, the sum being taken over all sequences $1 \leq k(1) < \ldots < k(m) \leq l$. For each sequence $1 \leq k(1) < \ldots < k(m) \leq l$, define

(2) $F_\rho\big(k(1), \ldots, k(m)\big)$ = sum of all monomials in (1) which have positive degree in each of $x_{k(1)}, \ldots, x_{k(m)}$ and degree zero in the other x's ;

(3) $F_\rho^{(d)}\big(k(1), \ldots, k(m)\big)$ = sum of all monomials in $F_\rho\big(k(1), \ldots, k(m)\big)$ which are of length precisely $m + d$ $(d \geq 0)$;

(4) $F^{(d)}_{\rho[k(i))}\big(k(1), \ldots, k(m)\big)$ = sum of all monomials in $F_\rho^{(d)}\big(k(1), \ldots, k(m)\big)$ whose first component is $x_{k(i)}$;

(5) $F^{(d)}_{\rho(k(i)]}\big(k(1), \ldots, k(m)\big)$ = sum of all monomials in $F_\rho^{(d)}\big(k(1), \ldots, k(m)\big)$ whose last component is $x_{k(i)}$;

(6) $F^{(d)}_{\rho[k(i),k(j)]}\big(k(i), \ldots, k(m)\big)$ = sum of all monomials in $F_\rho^{(d)}\big(k(1), \ldots, k(m)\big)$ whose first component is $x_{k(i)}$ and last component is $x_{k(j)}$; and

(7) $|F|$ = number of monomials in F .

LEMMA 1. *Let* $\rho = \big((1+x_1) \ldots (1+x_l)-1\big)^3$ $(l \geq 2)$ *be a fixed element of* S . *Then for each sequence* $1 \leq k(1) < \ldots < k(m) \leq l$,

(i) $\left|F^{(0)}_{\rho[k(i))}\big(k(1), \ldots, k(m)\big)\right|$ *is even for each* $i \in \{1, \ldots, m\}$;

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(ii) $\left|F^{(0)}_{\rho[k(i)]}\big(k(1), \ldots, k(m)\big)\right|$ *is even for each* $i \in \{1, \ldots, m\}$;

(iii) $\left|F^{(0)}_{\rho}\big(k(1), \ldots, k(m)\big)\right|$ *is even;*

(iv)

$$\left|F^{(0)}_{\rho[k(i),k(j)]}\big(k(1), \ldots, k(m)\big)\right| + \left|F^{(0)}_{\rho[k(j),k(i)]}\big(k(1), \ldots, k(m)\big)\right|$$

is even for all pairs $1 \le i < j \le m$; *and*

(v) $\left|F^{(1)}_{\rho[k(i),k(i)]}\big(k(1), \ldots, k(m)\big)\right|$ *is odd for each* $i \in \{1, \ldots, m\}$.

Proof of *(i)*. It is clearly enough to prove that $\left|F^{(0)}_{\rho[i]}(1, \ldots, m)\right|$ is even for each $i \in \{1, \ldots, m\}$. By definition every monomial in $F^{(0)}_{\rho[i]}(1, \ldots, m)$ is of the form $\left(x_iM_1\right)\left(M_2\right)\left(M_3\right)$, where x_iM_1, M_2, M_3 are terms from ξ_r, ξ_s, ξ_t respectively for some r, s, t . Thus we may pair $\left(x_iM_1\right)\left(M_2\right)\left(M_3\right)$ with $\left(x_iM_1\right)\left(M_3\right)\left(M_2\right)$ which by (1) is also present in $F^{(0)}_{\rho[i]}(1, \ldots, m)$.

Proof of *(ii)*. As in *(i)* we may pair $\left(M_1\right)\left(M_2\right)\left(M_3x_i\right)$ and $\left(M_2\right)\left(M_1\right)\left(M_3x_i\right)$ together.

Proof of *(iii)*. Since

$$F^{(0)}_{\rho}\big(k(1), \ldots, k(m)\big) = \sum_{i=1}^{m} F^{(0)}_{\rho[k(i))}\big(k(1), \ldots, k(m)\big) ,$$

the proof follows by *(i)*.

Proof of *(iv)*. As in the proof of *(i)* it is enough to prove that $\left|F^{(0)}_{\rho[i,j]}(1, \ldots, m)\right| + \left|F^{(0)}_{\rho[j,i]}(1, \ldots, m)\right|$ is even for each pair $1 \le i < j \le m$. We note by (1) that every monomial in $F^{(0)}_{\rho[i,j]}(1, \ldots, m)$ is of the form

$$x_iM\left(B_1\right)M\left(C_1\right) \cdot M\left(A_1\right)M\left(B_2\right)M\left(C_2\right) \cdot M\left(A_2\right)M\left(B_3\right)x_j \;\left(\in \xi_r\cdot\xi_s\cdot\xi_t\right) ,$$

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and every monomial in $F^{(0)}_{\rho[j,i]}(1, \ldots, m)$ is of the form

$$x_j M(C_1) \cdot M(A_1)M(B)M(C_2) \cdot M(A_2)x_i \quad (\in \xi_r \cdot \xi_s \cdot \xi_t) ,$$

where

$$A_1 \cup A_2 = \{1, \ldots, i-1\} \quad (A_1 \cap A_2 = \emptyset) ,$$

$$B = B_1 \cup B_2 \cup B_3 = \{i+1, \ldots, j-1\} \quad (B_1 \cap B_2 = B_2 \cap B_3 = B_3 \cap B_1 = \emptyset) ,$$

$$C_1 \cup C_2 = \{j+1, \ldots, m\} \quad (C_1 \cap C_2 = \emptyset) ,$$

and

$$M(\alpha_1, \ldots, \alpha_q) = x_{\alpha_1} \ldots x_{\alpha_q} .$$

For each choice of B_1, B_2, B_3 $(B_2 \neq B)$, we may pair

$$\big(x_i M(B_1)M(C_1)\big) \quad \big(M(A_1)M(B_2)M(C_2)\big) \quad \big(M(A_2)M(B_3)x_j\big)$$

and

$$\big(x_i M(B_3)M(C_1)\big) \quad \big(M(A_1)M(B_2)M(C_2)\big) \quad \big(M(A_2)M(B_1)x_j\big) ,$$

so that $F^{(0)}_{\rho[i,j]}(1, \ldots, m) + F^{(0)}_{\rho[j,i]}(1, \ldots, m)$ equals the even number of terms plus the sum of paired terms of the form

$$\{\big(x_i M(C_1)\big)\big(M(A_1)M(B)M(C_2)\big)\big(M(A_2)x_j\big) + \big(x_j M(C_1)\big)\big(M(A_1)M(B)M(C_2)\big)\big(M(A_2)x_i\big)\}$$
$$= \text{sum of even number of terms} .$$

Proof of *(v)*. Once again it is enough to show that $\left|F^{(1)}_{\rho[i,i]}(1, \ldots, m)\right|$ is odd. By (1) every monomial in $F^{(1)}_{\rho[i,i]}(1, \ldots, m)$ is of the form

$$x_i M(B_1) \cdot M(A_1)M(B_2) \cdot M(A_2)x_i \quad (\in \xi_r \cdot \xi_s \cdot \xi_t) ,$$

where

$$A_1 \cup A_2 = \{1, \ldots, i-1\} \quad (A_1 \cap A_2 = \emptyset)$$

and

$$B_1 \cup B_2 = \{i+1, \ldots, m\} \quad (B_1 \cap B_2 = \emptyset) .$$

Since $s \geq 1$, there are $\sum_{k=1}^{m-1} \binom{m-1}{k}$ choices of $M(A_1)M(B_2)$ and for each choice of $M(A_1)M(B_2)$, $M(B_1)$ and $M(A_2)$ are uniquely determined. Thus $\left|F^{(1)}_{\rho[i,i]}(1, \ldots, m)\right| = \sum_{k=1}^{m-1} \binom{m-1}{k}$, which is an odd number (independent of the choice of $i \in \{1, \ldots, m\}$).

This completes the proof of Lemma 1.

The ring $R(n)$ $(n \geq 2)$

Let $n \geq 2$ be a fixed positive integer and let $R(n)$ denote the ring (with 1) of characteristic 2 generated by $y_1, \ldots, y_n$ and satisfying *only* the following three conditions and their consequences:

(I) $y_{i(1)} \cdots y_{i(n+2)} = 0$;

(II) $y_{i(1)} \cdots y_{i(l)} = 0$ if $l \leq n$ and $i(j) = i(k)$ for some $j \neq k$;

(III) $y_i y_1 \cdots \hat{y}_i \cdots y_n y_i = y_i y_{1\sigma} \cdots \hat{y}_{i\sigma} \cdots y_{n\sigma} y_i$ for all $i = 1, \ldots, n$ and all permutations σ of $\{1, \ldots, n\}$ such that $i\sigma = i$.

Let J denote the ideal of $R(n)$ generated by all elements of the form

$$\rho(l) = \left(\left(1+y_{k(1)}\right) \cdots \left(1+y_{k(l)}\right)-1\right)^3 \quad (l \geq 2) .$$

For each $l = 2, 3, \ldots$, there are only finitely many, say $r(l)$, such elements. Thus we may write

$$J = \text{ideal}_{R(n)}\{\rho(l, q(l)), q(l) \in \{1, \ldots, r(l)\}, l = 2, 3, \ldots\} .$$

We refer the reader to (2) where in the expansion of

$$\rho = \left(\left(1+x_1\right) \cdots \left(1+x_l\right)-1\right)^3 \quad (l \geq 2) ,$$

the terms $F_\rho(k(1), \ldots, k(m))$ have been defined for each sequence $1 \leq k(1) < \ldots < k(m) \leq l$. Correspondingly, we may write

$$\rho(l,\ q(l)) = \left((1+y_{j(1)}) \cdots (1+y_{j(l)})-1\right)^3$$
$$= \sum F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) ,$$

where the summation is taken over all sequences $1 \le k(1) < \ldots < k(m) \le l$.

We clearly have

$$J \subseteq \text{ideal}_{R(n)}\{F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right)\} = K .$$

We note that

if $m \le n-1$, then by (II),

$$F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) = F^{(0)}_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right)$$

with $|\{j(k(1)),\ \ldots,\ j(k(m))\}| = m$;

if $m \ge n+2$, then by (I),

$$F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) = 0 ;$$

if $m = n + 1$, then by (I),

$$F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) = \begin{cases} 0 \text{ , or} \\ F^{(0)}_{\rho(l,q(l))[j(k(i)),j(k(p))]}\left(j(k(1)),\right. \\ \left.\ldots,\ j(k(m))\right) \ \left(\text{with } j(k(i)) = j(k(p))\right. \\ \qquad\qquad\qquad \left.\text{for some } i \ne p\right) \end{cases}$$
$$= 0 \text{ by (III) and the pairing used to prove Lemma 1 } (iv);$$

and if $m = n$, then

$$F_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) = \begin{cases} 0 \text{ , or} \\ F^{(0)}_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) \\ + F^{(1)}_{\rho(l,q(l))}\left(j(k(1)),\ \ldots,\ j(k(m))\right) \\ \left(\text{with } |\{j(k(1)),\ \ldots,\ j(k(m))\}| = m\right) . \end{cases}$$

Thus

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$$J \subseteq K = \text{ideal}_{R(n)}\Big\{F^{(0)}_{\rho(l,q(l))}\big(j(k(1)), \ldots, j(k(m))\big) \text{ with } \big|\{j(k(1)), \ldots, j(k(m))\}\big| = m \le n \text{ and } F^{(1)}_{\rho(l,q(l))}\big(j(k(1)), \ldots, j(k(n))\big) \text{ with } \{j(k(1)), \ldots, j(k(n))\} = \{1, \ldots, n\}\Big\} .$$

In particular if $\sum_{i=1}^{n} \varepsilon_i\left(y_iy_1 \ldots \hat{y}_i \ldots y_ny_i\right) \in K$, $\varepsilon_i \in \{0, 1\}$, then by Lemma 1 *(i)*, *(ii)*, *(iii)* and (III) it follows that

$$\sum_{i=1}^{n} \varepsilon_i\left(y_iy_1 \ldots \hat{y}_i \ldots y_ny_i\right) \in \text{ideal}_{R(n)}\Big\{F^{(1)}_{\rho(l,q(l))}\big(j(k(1)), \ldots, j(k(n))\big), \text{ with } \{j(k(1)), \ldots, j(k(n))\} = \{1, \ldots, n\}\Big\}$$

$$= \text{ideal}_{R(n)}\left\{\sum_{i=1}^{n} y_iy_1 \ldots \hat{y}_i \ldots y_ny_i\right\} \text{ by Lemma 1 } (v) ;$$

hence $\varepsilon_i = 0$ for all i or $\varepsilon_i = 1$ for all i . We summarize these observations in the following:

LEMMA 2. *If* $\sum_{i=1}^{n} \varepsilon_i\left(y_iy_1 \ldots \hat{y}_i \ldots y_ny_i\right)$, $\varepsilon_i \in \{0, 1\}$, *lies in the ideal* J , *then either* $\varepsilon_1 = \ldots = \varepsilon_n = 0$ *or* $\varepsilon_1 = \ldots = \varepsilon_n = 1$. *In particular* $y_1y_2 \ldots y_ny_1 \notin J$.

The main result

Let $M(n+1)$ denote the multiplicative group of 2×2 matrices over $R(n)/J$ generated by $X, Y_1, \ldots, Y_n$ $(n \ge 2)$, where

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad Y_i = \begin{pmatrix} 1+y_i & 0 \\ y_i & 1 \end{pmatrix} ,$$

$i = 1, \ldots, n$.

THEOREM. *$M(n+1)$ is a homomorphic image of $G(n+1)$ and the nilpotency class of $M(n+1)$ is at least $n + 2$.*

Proof. It is easily verified that X and Y_i's are of order 2 .

An arbitrary element of $M(n+1)$ is of the form $W = \begin{pmatrix} \tau & 0 \\ \mu & 1 \end{pmatrix}$, where τ is a unit. Since $(1+\tau)^3 \in J$ it follows that $1 + \tau + \tau^2 + \tau^3 \in J$ and $\tau^4 = 1$. Thus

$$W^4 = \begin{pmatrix} \tau^4 & 0 \\ \mu\left(1+\tau+\tau^2+\tau^3\right) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which implies that $M(n+1)$ is a homomorphic image of $G(n+1)$. Next we note that

$$\left[\begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}, \begin{pmatrix} 1+y_i & 0 \\ y_i & 1 \end{pmatrix}\right] = \begin{pmatrix} 1+y_i & 0 \\ \rho\left(1+y_i\right)+y_i & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ \rho^* & 1 \end{pmatrix}$$

where

$$\rho^* = \left(\rho\left(1+y_i\right)+y_i\right)\left(1+y_i\right) + \rho\left(1+y_i\right) + y_i = \left(\rho\left(1+y_i\right)+y_i\right)y_i = \rho\left(1+y_i\right)y_i = \rho y_i .$$

Thus by obvious induction

$$\left[X, Y_1, Y_2, \ldots, Y_n, Y_1\right] = \begin{pmatrix} 1 & 0 \\ y_1y_2\cdots y_ny_1 & 1 \end{pmatrix}$$

which is not the identity matrix by Lemma 2. Thus $M(n+1)$ is nilpotent of class at least $n + 2$ as was to be shown.

Reference

[1] C.R.B. Wright, "On groups of exponent four with generators of order two", *Pacific J. Math.* 10 (1960), 1097-1105.

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