Glasgow Math. J. **43** (2001) 199–208. © Glasgow Mathematical Journal Trust 2001. Printed in the United Kingdom

FIBREWISE HOMOLOGY

M. C. CRABB

Department of Mathematical Sciences, University of Aberdeen, Aberdeen AB24 3UE, UK
e-mail: m.crabb@maths.abdn.ac.uk

(Received 11 August, 1999)

Abstract. Methods from fibrewise homology theory are illustrated by computations of cohomology rings of certain mapping spaces arising in the geometry of loop groups, specifically the spaces of maps from S^1 to the classifying space $BSO(n)$ of $SO(n)$ and maps from S^2 to $BSU(n)$.

1991 *Mathematics Subject Classification*. 55R40, 55P35, 57T10.

1. Introduction. This note is concerned with the application of fibrewise *homology* theory to two specific computations: the calculation of (i) the mod 2 cohomology ring of the classifying space of the loop group $\mathcal{L}SO(n) = \mathrm{map}(S^1, SO(n))$, and (ii) the integral cohomology ring of the space $\mathrm{map}(S^2, BSU(n))$ of maps from S^2 to $BSU(n)$ (or "space" of principal $SU(n)$-bundles over the Riemann sphere, [**12**, p. 157]). The interest lies in the method, which interprets the complicated cohomology rings as duals of fibrewise homology groups admitting an easily described Hopf algebra structure; the results themselves are mostly known, at least in special cases.

The relationship between the two problems will be clarified if we identify the classifying space $B\mathcal{L}SO(n)$ of the loop group with $\mathcal{L}BSO(n) = \mathrm{map}(S^1, BSO(n))$; see [**1**]. (The space of maps is connected.) For any integer $m \geq 1$ and compact Lie group G, the space $\mathrm{map}(S^m, BG)$ fibres over BG by evaluation at the basepoint: $\mathrm{map}(S^m, BG) \to BG$, with fibre $\Omega^{m-1}G$. The following description of this fibration is well known; see, for example, [**6**].

LEMMA 1.1. *There is a fibre-homotopy equivalence*

$$\begin{array}{ccc} \mathrm{map}(S^m, BG) & \simeq & EG \times_G \Omega^{m-1}G \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

over BG, where the action of G on $\Omega^{m-1}G$ is by conjugation on G, and the righthand map $EG \times_G \Omega^{m-1}G \to BG$ is the projection to $EG/G = BG$.

One way of seeing this is to look at the homotopy functors classified by the two spaces. A pointed homotopy class $X \to \mathrm{map}(S^m, BG)$ classifies principal G-bundles over $X \times S^m$ together with a trivialization of the bundle over $* \times S^m$. (We denote any basepoint by $*$.) Such a bundle can be described, in the usual way, by its restriction to $X \times *$, which is a principal G-bundle $P \to X$ (with a trivialization over the basepoint of X), and a clutching map $S^{m-1} \to \mathrm{Aut}(P)$. It is precisely this data which is classified by $EG \times_G \Omega^{m-1}G$. □

In the two examples we are thus concerned with the cohomology of

$$\text{(i)}\quad \begin{array}{c} ESO(n)\times_{SO(n)} SO(n)\\ \downarrow\\ BSO(n)\end{array} \qquad \text{and} \qquad \text{(ii)}\quad \begin{array}{c} ESU(n)\times_{SU(n)} \Omega SU(n)\\ \downarrow\\ BSU(n)\end{array}.$$

More generally, let us fix a finite complex B and an n-dimensional vector bundle ξ over B, where either

(i) ξ is real and equipped with a Euclidean inner product, or

(ii) ξ is complex, with a Hermitian inner product.

In case (i), we write $O(\xi) \to B$ for the orthogonal bundle of ξ; its fibre at a point $b \in B$ is thus the group $O(\xi_b)$ of orthogonal automorphisms of the fibre ξ_b of the Euclidean bundle. The sub-bundle of special orthogonal groups is written as $SO(\xi) \to B$. In case (ii), we are interested in the bundles $U(\xi) \to B$ and $SU(\xi) \to B$ of unitary and special unitary groups. These are pointed fibre bundles (with base-point the identity in each fibre) and we can form the fibrewise loop spaces $\Omega_B U(\xi) \to B$ and $\Omega_B SU(\xi) \to B$, which are (locally trivial) bundles whose fibre at b is the loop space $\Omega U(\xi_b)$ or $\Omega SU(\xi_b)$, respectively.

We shall see that (i) the mod 2 cohomology ring $H^*(O(\xi);\ \mathbb{F}_2)$ and (ii) the integral cohomology ring $H^*(\Omega_B U(\xi);\ \mathbb{Z})$ have the structure of Hopf algebras over $H^*(B;\ \mathbb{F}_2)$ or $H^*(B;\ \mathbb{Z})$. These Hopf algebras or, to be exact, their duals are described explicitly in Sections 3 and 4, respectively.

The cohomology of $\mathcal{L}BSO(n)$ and $\mathrm{map}(S^2, BSU(n))$ is obtained in Section 5 from these general results by taking B to be a finite skeleton of $BSO(n)$ or $BSU(n)$. The cohomology of $\mathcal{L}BSO(n)$ $(= B\mathcal{L}SO(n))$ is surely well known; for computations of $\mathcal{L}BG$ (or $B\mathcal{L}G$) for other connected compact Lie groups G see [**8**]. The computation for $\mathrm{map}(S^2,\ BSU(2))$ can be found in [**9**], [**10**].

2. Fibrewise homology and cohomology. The Pontrjagin ring structure of $H_*(O(n);\ \mathbb{F}_2)$ and $H_*(\Omega U(n);\ \mathbb{Z})$ is simpler to describe than the cohomology ring structure of $H^*(O(n);\ \mathbb{F}_2)$ and $H^*(\Omega U(n);\ \mathbb{Z})$. We shall approach the fibrewise computation in the same vein, and to do this we need to review some facts about fibrewise homology and cohomology. More details, including proofs, can be found in [**4**] (Part II, Section 15).

Fix a base space B, which will be a finite complex. We work with locally trivial fibrewise pointed spaces over B, which we shall call pointed fibre bundles, with fibre a finite pointed complex. These will be denoted generically by $X \to B$, $Y \to B$ and $Z \to B$. The fibre of $X \to B$ at $b \in B$ is written as X_b. Fixing coefficients, $\mathbb{F}_2$ in case (i) or $\mathbb{Z}$ in case (ii), we denote the Eilenberg-MacLane space $K(\mathbb{F}_2, n)$ or $K(\mathbb{Z}, n)$ by K_n.

We define the fibrewise cohomology groups for $i \in \mathbb{Z}$, following [**5**], as direct limits of sets of fibrewise pointed homotopy classes over B:

$$H^i_B\{X;\ Y\} := \varinjlim_n\ [\Sigma^n_B X;\ (B \times K_{n+i}) \wedge_B Y]_B,$$

where Σ_B is the fibrewise suspension. There are *composition and product* maps:

https://doi.org/10.1017/S0017089501020055 Published online by Cambridge University Press

$$H^i_B\{Y;\ Z\} \otimes H^j_B\{X;\ Y\} \to H^{i+j}_B\{X;\ Z\},$$
$$H^i_B\{X;\ Y\} \otimes H^j_B\{X';\ Y'\} \to H^{i+j}_B\{X \wedge_B X';\ Y \wedge_B Y'\},$$

with the usual properties. The *fibrewise homology category* over B has as morphisms $X \to Y$ the "homology maps" $H^0_B\{X;\ Y\}$ over B.

If B is a point, we drop the suffix and write simply $H^*\{X;\ Y\}$. In that case, $H^i\{X;\ S^0\}$ is the (reduced) cohomology $\tilde{H}^i(X)$ of the pointed space X, and $H^i\{S^0;\ Y\}$ is the homology $\tilde{H}_{-i}(Y)$ of Y.

In general, we refer to $H^*_B\{X;\ B \times S^0\}$ as the *fibrewise cohomology* of the fibrewise pointed space $X \to B$, and to $H^i_B\{B \times S^0;\ Y\}$ as the *fibrewise homology* of Y *over* B. The fibrewise cohomology group is easily seen to be just the (reduced) cohomology group of X modulo the subspace B included as the fibrewise basepoint:

$$H^i_B\{X;\ B \times S^0\} = \tilde{H}^i(X/B). \tag{2.1}$$

There is no similar classical interpretation of fibrewise homology groups.

Our homology computations will rest on the *Leray-Hirsch lemma.*

Lemma 2.2. *Suppose that there exist classes*

$$e_i \in H^{n_i}_B\{X;\ Y\} \quad (1 \le i \le m),$$

which restrict to a basis of the cohomology $H^*\{X_b;\ Y_b\}$ of the fibres for each $b \in B$. Then $\tilde{H}^*_B\{X;\ Y\}$ is free over the graded ring $R := H^*(B)$ on the basis $e_1, \ldots, e_m$.

This can be established in the same way as the classical Leray-Hirsch lemma in cohomology. For a discussion of this result and of the next lemma, see [**4**] (Part II, Lemma 15.14).

Lemma 2.3. *The following conditions on a pointed fibre bundle $X \to B$ are equivalent.*

(i) *The bundle is isomorphic in the fibrewise homology category to a trivial bundle $B \times F$ with fibre F a wedge of spheres.*

(ii) *There exist (homogeneous) classes e_i, $1 \le i \le m$, in the fibrewise cohomology group $H^*_B\{X;\ B \times S^0\} = \tilde{H}^*(X/B)$ that restrict to a basis of the cohomology $\tilde{H}^*(X_b)$ of each fibre, $b \in B$.*

(iii) *There exist (homogeneous) classes e'_i, $1 \le i \le m$, in the fibrewise homology group $H^*_B\{B \times S^0;\ X\}$ that restrict to a basis of the homology $\tilde{H}_*(X_b)$ of each fibre, $b \in B$.*

Definition 2.4. We say that a pointed fibre bundle (with fibre a finite pointed complex) $X \to B$ satisfying the equivalent conditions (2.3) is H-free over B.

The Künneth theorems for H-free pointed fibre bundles follow directly from Lemmas 2.2 and 2.3.

Lemma 2.5. *Let $X \to B$ and $Y \to B$ be H-free pointed fibre bundles (with fibres finite pointed complexes). Then*

https://doi.org/10.1017/S0017089501020055 Published online by Cambridge University Press

$$H_B^*\{X;\ Y\} = \mathrm{Hom}_R^*(H_B^*\{Y;\ B\times S^0\}, H_B^*\{X;\ B\times S^0\})$$
$$= \mathrm{Hom}_R^*(H_B^*\{B\times S^0;\ X\}, H_B^*\{B\times S^0;\ Y\}).$$

In particular, there is duality between homology and cohomology over B:

$$H_B^*\{B\times S^0;\ Y\} = \mathrm{Hom}_R^*(H_B^*\{Y;\ B\times S^0\}, R).$$

LEMMA 2.6. *Suppose that the four pointed fibre bundles (with fibres finite pointed complexes) X, X', Y and Y' over B are H-free. Then there is a Künneth isomorphism*

$$H_B^*\{X\wedge_B X';\ Y\wedge_B Y'\} = H_B^*\{X;\ Y\}\otimes_R H_B^*\{X';\ Y'\}.$$

3. The cohomology of $O(\xi)$. In this section we work in the framework (i) of Section 1; P will be used for the real projective space, and H will denote homology with $\mathbb{F}_2$-coefficients.

The reflection map $P(\mathbb{R}^n)\to O(n)$ includes the real projective space as the generating variety into the component of determinant -1. Writing x_i for the generator of $H_i(P(\mathbb{R}^n))$, $0\le i<n$, we have an inclusion

$$\begin{aligned} H_*(P(\mathbb{R}^n)) = \tilde{H}_*(P(\mathbb{R}^n)_+) = \bigoplus_{i=0}^{n-1}\mathbb{F}_2 x_i \hookrightarrow \\ \tilde{H}_*(O(n)_+) = \mathbb{F}_2[x_0,\dots,x_{n-1}]/(x_0^2=1,\ x_i^2=0 : 0<i<n). \end{aligned} \tag{3.1}$$

For the sake of the generalization, it is convenient to adjoin a basepoint to the two spaces, and this is denoted by a subscript "+". The class x_0 takes care of the two components.

We can make exactly the same calculation over B. A subscript "$+B$" denotes adjunction of a basepoint in each fibre; that is, disjoint union with B. The projective bundle $P(\xi)$ is included as a sub-bundle in $O(\xi)$, and the pointed fibre bundle $P(\xi)_{+B}$ is included in $O(\xi)_{+B}$. Now

$$H_B^*\{P(\xi)_{+B};\ B\times S^0\} = H^*(P(\xi)) = R[[t]]/(t^n + w_1 t^{n-1} + \dots + w_n),$$

where the w_i are the Stiefel-Whitney classes of ξ, t is the Euler class of the Hopf line bundle, and R as in Section 2 is $H^*(B)$. From Lemma 2.5,

$$H_B^*\{B\times S^0;\ P(\xi)_{+B}\} = \mathrm{Hom}_R(H^*(P(\xi)), R) = \bigoplus_{i=0}^{n-1} R x_i,$$

where the basis (x_i) is dual to the basis (t^j) of the cohomology: $\langle x_i, t^j\rangle = \delta_{ij}$, $0\le i, j<n$. Note that the generator $x_i\in H_B^{-i}\{B\times S^0;\ P(\xi)_{+B}\}$ has *negative* degree $-i$. We use the inclusion $P(\xi)\hookrightarrow O(\xi)$ to map x_i to a class, which we denote by the same symbol, in $H_B^{-i}\{B\times S^0;\ O(\xi)_{+B}\}$.

The fibrewise multiplication

$$O(\xi)\times_B O(\xi)\to O(\xi) \qquad \text{or} \qquad O(\xi)_{+B}\wedge_B O(\xi)_{+B}\to O(\xi)_{+B}$$

determines a Pontrjagin multiplication on the fibrewise homology group

$$A := H^*_B\{B \times S^0;\ O(\xi)_{+B}\},$$

which thus becomes a graded R-algebra. Using the Leray-Hirsch lemma (2.2), we see that A is free as an R-module with basis $x_{i_1}x_{i_2}\dots x_{i_r}$, $0 \le i_1 < i_2 < \dots < i_r < n$.

By Lemma 2.6, $H^*_B\{B \times S^0;\ O(\xi)_{+B} \wedge_B O(\xi)_{+B}\}$ is equal to $A \otimes_R A$, and the diagonal determines a co-multiplication $\Delta : A \to A \otimes_R A$ that makes A a Hopf-algebra over R, because the group structure translates formally into the structure of a group object in the category of graded R-algebras. The diagonal can be computed on $P(\xi)$ and then extended to A as an R-algebra homomorphism. We have

$$\begin{aligned}\Delta x_i &= \sum_{0\le j,k<n} \langle \Delta x_i,\, t^j \otimes t^k\rangle x_j \otimes x_k \\ &= \sum_{0\le j,k<n} \langle x_i,\, t^{j+k}\rangle x_j \otimes x_k = \sum_{0\le j,k<n} w_{i,j+k}\, x_j \otimes x_k,\end{aligned} \tag{3.2}$$

where the coefficients $w_{i,j}$, $0 \le i < n$, $j \ge 0$, are defined by

$$t^j = \sum_{0\le i<n} w_{i,j} t^i \tag{3.3}$$

in $R[[t]]/(t^n + w_1 t^{n-1} + \dots + w_n)$. Thus, $w_{i,j} = \delta_{i,j}$ for $j < n$ and $w_{i,n} = w_{n-i}$.

To describe the ring structure of A we need to show that the multiplication is commutative and to compute the classes x_i^2. This can be done by including ξ as a summand of a trivial bundle: $\xi \oplus \xi^\perp = B \times \mathbb{R}^N$. Let us write the standard generators of $H_*(O(N))$ as X_i, $0 \le i < N$, and look at the maps in homology induced by the inclusion of $P(\xi)$ in $B \times P(\mathbb{R}^N)$ and $O(\xi)$ in $B \times O(N)$. Since $\langle x_i, t^j\rangle = w_{i,j}$, we have

$$x_i = \sum_{0\le j<N} w_{i,j} X_j = X_i + \sum_{n\le j<N} w_{i,j} X_j \qquad (0 \le i < n).$$

Hence $x_i x_j = x_j x_i$, $x_i^2 = 0$ for $i > 0$ and $x_0^2 = 1$, by (3.1).

The antipode involution $A \to A$, induced by the inverse $O(\xi) \to O(\xi)$, is the identity, because the inverse is the identity on the space of reflections $P(\xi)$.

It is also routine to calculate the action of the dual Steenrod squares: we have

$$\langle (\chi Sq)^k x_i,\, t^j\rangle = \langle x_i,\, Sq^k(t^j)\rangle = \langle x_i,\, \binom{j}{k} t^{j+k}\rangle = \binom{j}{k} w_{i,j+k}.$$

These results can be summarized as follows.

Proposition 3.4. *As a Hopf algebra over $R = H^*(B)$, the fibrewise homology of $O(\xi)$ is*

$$H^*_B\{B \times S^0;\ O(\xi)_{+B}\} = R[x_0, \dots, x_{n-1}]/(x_0^2 = 1,\ x_i^2 = 0 : 0 < i < n),$$

with co-multiplication given by (3.2). The sub-algebra $R[x_1, \dots, x_{n-1}]$ is the fibrewise homology of $SO(\xi)$. The action of the dual Steenrod squares is determined by

https://doi.org/10.1017/S0017089501020055 Published online by Cambridge University Press

$$(\chi Sq)^k x_i = \sum_{k \le j < n} \binom{j}{k} w_{i,j+k}\, x_j. \qquad \square$$

In principle we have thus obtained, by duality, a description of the cohomology ring $H^*(O(\xi))$.

4. The cohomology of $\Omega_B U(\xi)$. We work now in the framework (ii) of Section 1. In this section H will denote homology with *integral* coefficients and P will be used for *complex* projective space.

The complex projective space $P(\mathbb{C}^n)$ is included as a generating variety in $\Omega U(n)$ and, if we write $\tilde{H}_{2i}(P(\mathbb{C}^n)_+) = \mathbb{Z}x_i$, then

$$\tilde{H}_*(\Omega U(n)_+) = \mathbb{Z}[x_0, x_0^{-1}][x_1, \ldots, x_{n-1}]$$

as a Pontrjagin ring. The components of $\Omega U(n)$ are indexed by the degree (in $\mathbb{Z}$) of the determinant: $\Omega U(n) \to \Omega S^1$. The classes x_i lie in the degree 1 component and multiplication by x_0 moves from one component to the next.

We can now argue almost exactly as in the previous section (ignoring, for the moment, the fact that $\Omega U(n)$ is not a finite complex). We have

$$H_B^*\{P(\xi)_{+B};\ B \times S^0\} = H^*(P(\xi)) = R[[t]]/(t^n + c_1 t^{n-1} + \ldots + c_n),$$

where the c_i are the Chern classes of ξ and t is the Euler class of the (dual) Hopf line bundle. The dual fibrewise homology is

$$H_B^*\{B \times S^0;\ P(\xi)_{+B}\} = \bigoplus_{i=0}^{n-1} Rx_i,$$

where $\langle x_i, t^j \rangle = \delta_{i,j}$, $0 \le i, j < n$, $x_i \in H^{-2i}$. Again, let us write

$$t^j = \sum_{0 \le i < n} c_{i,j} t^i \tag{4.1}$$

in $H^*(P(\xi))$. From the homotopy-commutativity of $\Omega_B U(\xi)$, we deduce that the Pontrjagin multiplication is commutative.

Proposition 4.2. *As a Pontrjagin algebra over R, the fibrewise homology of $\Omega_B U(\xi)$ is given by*

$$H_B^*\{B \times S^0;\ (\Omega_B U(\xi))_{+B}\} = R[x_0, x_0^{-1}][x_1, x_2, \ldots, x_{n-1}],$$

and admits a natural Hopf algebra structure with co-multiplication

$$\Delta x_i = \sum_{0 \le j, k < n} c_{i,j+k}\, x_j \otimes x_k$$

and antipode involution $x_i \mapsto \overline{x}_i$ determined by the n equations:

https://doi.org/10.1017/S0017089501020055 Published online by Cambridge University Press

$$\sum_{0\le j,\,k<n} c_{i,j+k}\, x_j\bar{x}_k = \begin{cases} 1 & \text{if } i=0,\\ 0 & \text{if } 0<i<n. \end{cases}$$ □

An explicit description of the dual $H^*(\Omega_B U(\xi))$ is not so clear, but formulae can be written down without too much trouble when $n = 2$. The space $\Omega_B U(\xi)$ is a disjoint union

$$\Omega_B U(\xi) = \coprod_{d\in\mathbb{Z}} \Omega_B^{(d)} U(\xi)$$

indexed by the degree d of the component of the fibre. Thus $\Omega_B^{(0)} U(\xi)$ is (homotopy-equivalent to) $\Omega_B SU(\xi)$. Let us specialize to the case $n = 2$ and write $y = x_0^{-1}x_1$, so that

$$H_B^*\{B \times S^0;\ (\Omega_B SU(\xi))_{+B}\} = R[y]\,.$$

Then

$$\Delta x_0 = x_0 \otimes x_0 - c_2 x_1 \otimes x_1, \qquad \Delta x_1 = x_0 \otimes x_1 - c_1 x_1 \otimes x_1 + x_1 \otimes x_0,$$

and so

$$\begin{aligned} \Delta y &= (\Delta x_0)^{-1}\Delta x_1 = (1 - c_2 y \otimes y)^{-1}(1 \otimes y - c_1 y \otimes y + y \otimes 1),\\ \Delta y^k &= (\Delta y)^k = (1 - c_2 y \otimes y)^{-k}(1 \otimes y - c_1 y \otimes y + y \otimes 1)^k. \end{aligned}$$

As a basis for $H^*(\Omega_B^{(d)} U(\xi))$ we take the dual (over R) $b_j^{(d)}$, $j \ge 0$, to the homology basis $x_0^d y^i$, $i \ge 0$. Now $\langle x_0^d y^k, b_i^{(d)} b_j^{(d)}\rangle = \langle \Delta(x_0^d y^k), b_i^{(d)} \otimes b_j^{(d)}\rangle$. This yields the following description of the cup product, which was first obtained (for the case $d = 0$, $c_1 = 0$) by Masbaum, [**9**], [**10**].

Proposition 4.3. *For $n = 2$, the ring $H^*(\Omega_B^{(d)} U(\xi))$ $(= S$ say$)$ is free as an R-module, with basis $b_i^{(d)}$, $i \ge 0$, and multiplication described in terms of the formal power series $\beta^{(d)}(X) = \sum b_i^{(d)} X^i \in S[[X]]$ by the identity*

$$\beta^{(d)}(X) \cdot \beta^{(d)}(Y) = (1 - c_2 XY)^d \beta^{(d)}(F(X, Y)) \in S[[X, Y]],$$

where $F(X, Y)$ is the formal group law $(X + Y - c_1 XY)/(1 - c_2 XY)$. □

Although the discussion in Section 2 was restricted to bundles with fibre a finite complex, there is no real problem in dealing with the cohomology of bundles such as $\Omega_B U(\xi)$ with fibre of finite type. But in fact there is a nice, $U(n)$-equivariant, geometric filtration (due to Mitchell, [**11**]) of $\Omega U(n)$ by compact sub-ENRs, as discussed, for example, in [**3**], [**13**]. The filtration originates from the generating variety $P(\xi) \subseteq \Omega_B U(\xi)$. The image of the k-fold product $P(\xi) \times_B \cdots \times_B P(\xi) \to \Omega_B U(\xi)$ is a sub-bundle, which we denote by $S_k(\xi) \subseteq \Omega_B U(\xi)$, with fibre a compact algebraic variety, and its fibrewise homology can be computed by another application of the Leray-Hirsch lemma as the module of homogeneous polynomials of degree k:

$$H_B^*\{B \times S^0;\ S_k(\xi)_{+B}\} = (R[x_0, \ldots, x_{n-1}])_k. \tag{4.4}$$

To construct a filtration we use the section ζ of $\Omega_B^{(n)} U(\xi)$ that restricts to the inclusion $S^1 \to U(\xi_b)$ of the centre in each fibre, $b \in B$. Multiplication by ζ gives a bundle isomorphism between $\Omega_B^{(d)} U(\xi)$ and $\Omega_B^{(d+n)} U(\xi)$, and we can filter $\Omega_B^{(d)} U(\xi)$ by the finite-dimensional bundles $\zeta^{-r} S_{d+rn}(\xi)$ with $d + rn \geq 0$. In particular, the component of degree 0 is filtered as

$$B = S_0(\xi) \subseteq \zeta^{-1} S_n(\xi) \subseteq \cdots \subseteq \zeta^{-r} S_{rn}(\xi) \subseteq \cdots \subseteq \Omega_B^{(0)} U(\xi).$$

Writing z for the class in $H_B^0\{B \times S^0;\ (\Omega_B U(\xi))_{+B}\}$ determined by the section ζ, we have, by the Leray-Hirsch lemma again,

$$H_B^*\{B \times S^0;\ (\zeta^{-r} S_{d+rn}(\xi))_{+B}\} = z^{-r}(R[x_0, \ldots, x_{n-1}])_{d+rn}. \tag{4.5}$$

To compute this class z we can use a splitting principle argument.

Lemma 4.6. *The cohomology class $z \in H_B^0\{B \times S^0;\ (\Omega_B U(\xi))_{+B}\}$ is given by*

$$z = \prod_{j=1}^{n} \Big(\sum_{0 \leq i < n} (-l_j)^i x_i \Big),$$

where c_i is the ith elementary symmetric function in $l_1, \ldots, l_n$. In particular, for $n = 2$ we have $z = x_0^2 - c_1 x_0 x_1 + c_2 x_1^2$.

For suppose that ξ is a direct sum of complex line bundles: $\lambda_1 \oplus \ldots \oplus \lambda_n$. Let $\iota_j : P(\lambda_j) \to P(\xi)$ be the inclusion, and let $x_0(\lambda_j)$ denote the canonical fibrewise homology generator in $H_B^0\{B \times S^0;\ P(\lambda_j)_{+B}\}$. Then z is the product $\iota_1(x_0(\lambda_1)) \cdots \iota_n(x_0(\lambda_n))$. In cohomology ι_j maps $t^i \in H^*(P(\xi))$ to $t^i = (-l_j)^i \in H^*(P(\lambda_j)) = R[[t]]/(t + l_j)$, where $l_j = c_1(\lambda_j)$. Dually in homology we have $\iota_j(x_0(\lambda_j)) = \sum_{0 \leq i < n} (-l_j)^i x_i$. This establishes the result. □

Remarks 4.7. (i) If λ is a complex line bundle over B, then we can identify $U(\lambda \otimes \xi)$ with $U(\xi)$ by taking the tensor product with the identity on a fibre of λ. In the case $n = 2$ the corresponding generators are related by

$$x_0(\lambda \otimes \xi) = x_0(\xi) + c_1(\lambda).x_1(\xi),\ x_1(\lambda \otimes \xi) = x_1(\xi).$$

(ii) The connective complex K-theory (and, indeed, the MU-theory) of $\Omega_B U(\xi)$ can be calculated by the same method, with little more than changes in notation. For Grothendieck's description of the cohomology of the projective bundle $P(\xi)$ as a free module over the cohomology of the base generalizes to any complex-oriented cohomology theory. See **[4]** (Part II, Section 15) for a brief introduction to fibrewise K-theory.

5. Classifying spaces. We shall deal only with $O(n)$ and $\Omega U(n)$; the two cases of (i) $SO(n)$ and (ii) $\Omega SU(n)$ discussed in Section 1 are slightly easier. As in Lemma 1.1, a group G is understood to act on $\Omega^{m-1} G$ by conjugation on G. The cohomology of $EO(n) \times_{O(n)} O(n)$ is simply the inverse limit of the groups $H^*(O(\xi))$ where ξ is the

https://doi.org/10.1017/S0017089501020055 Published online by Cambridge University Press

restriction of the universal bundle to a finite skeleton B of the classifying space $BO(n)$; for the inverse limit in a given dimension stabilizes. We can think of $H^*(EO(n) \times_{O(n)} O(n))$ as the $O(n)$-equivariant *Borel cohomology* $H^*_{O(n)}(O(n))$ of the group $O(n)$. (See [**7**] for an account of Borel homology and cohomology and [**4**], for example, for a discussion of the relation between fibrewise and equivariant homology.) The *Borel homology* group ${}_{O(n)}H^*\{S^0;\ O(n)_+\}$ is, similarly, the inverse limit of the fibrewise homology groups $H^*_B\{B \times S^0;\ O(\xi)_{+B}\}$, because the limit again stabilizes in each dimension.

PROPOSITION 5.1. *The $O(n)$-equivariant Borel homology of $O(n)$ is*

$$ {}_{O(n)}H^*\{S^0;\ O(n)_+\} = \mathbb{F}_2[[w_1, \dots, w_n]][x_0, x_1, \dots, x_{n-1}]/(x_0^2 = 1,\ x_i^2 = 0\ :\ 0 < i < n) $$

as a Hopf algebra, over the graded ring $H^*_{O(n)}(*) = \mathbb{F}_2[[w_1, \dots, w_n]]$, with co-multiplication given by (3.2). □

The Borel cohomology $H^*_{U(n)}(\Omega U(n)) = H^*(EU(n) \times_{U(n)} \Omega U(n))$ is again the inverse limit of groups $H^*(\Omega_B U(\xi))$. More care is needed in the discussion of the Borel homology of the infinite-dimensional space $\Omega U(n)$. The homology group of a CW-complex of finite type is the direct limit of the homology groups of its finite skeleta. We can define the equivariant homology of the dth component $\Omega^{(d)}U(n)$ to be the direct limit of the equivariant homology groups of the compact $U(n)$-ENRs $\zeta^{-r}S_{d+rn}(\mathbb{C}^n)$, for $d + rn \geq 0$, in the filtration described in Section 4. (Here ζ is simply the inclusion of the centre $S^1 \to U(n)$ and $S_k(\mathbb{C}^n)$ is the image in $\Omega^{(k)}U(n)$ of the k-fold product of the generating variety $P(\mathbb{C}^n)$.) From (4.5) we find that

$$ {}_{U(n)}H^*\{S^0;\ (\zeta^{-r}S_{d+rn}(\mathbb{C}^n))_+\} = z^{-r}\mathbb{Z}[[c_1, \dots, c_n]][x_0, x_1, \dots, x_{n-1}]_{d+rn}. $$

Forming the direct limit, and assembling the components, we obtain the equivariant homology of $\Omega U(n)$; it is non-zero in every even dimension, positive or negative.

PROPOSITION 5.2. *The $U(n)$-equivariant Borel homology of $U(n)$ is*

$$ {}_{U(n)}H^*\{S^0;\ (\Omega U(n))_+\} = \mathbb{Z}[[c_1, \dots, c_n]][x_0, x_1, \dots, x_{n-1}][z^{-1}] $$

as a Hopf algebra over the graded ring $H^*_{U(n)}(*) = \mathbb{Z}[[c_1, \dots, c_n]]$, with co-multiplication as in Proposition 4.2. The generator x_i has dimension $-2i$ and the class z, in dimension 0*, is defined in Lemma 4.6.* □

The Borel cohomology rings of $O(n)$ and $\Omega U(n)$ may be deduced, in principle, by taking duals.

ACKNOWLEDGEMENTS. It is a pleasure to thank A. Kono and W. A. Sutherland for stimulating conversations and seminars during their visits to Aberdeen. I am indebted to A. L. Cook for numerous discussions on fibrewise Hopf spaces, [**2**], in which he pointed out, in particular, the importance of the Hopf algebra structure in the fibrewise theory. My thanks are due also to the referee, who requested a more detailed discussion of the filtration of $\Omega U(n)$ and thus corrected an error in the original formulation of Proposition 5.2.

REFERENCES

1. M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London* **308A** (1982), 523–615.

2. A. L. Cook and M. C. Crabb, Fibrewise Hopf structures on sphere-bundles, *J. London Math. Soc.* **48** (1993), 365–384.

3. M. C. Crabb, On the stable splitting of $U(n)$ and $\Omega U(n)$, in *Algebraic Topology, Barcelona 1986* (ed. J. Aguadé, R. Kane), Lecture Notes in Math., **1298** (1987), 35–53.

4. M. C. Crabb and I. M. James, *Fibrewise Homotopy Theory* (Springer-Verlag, 1998).

5. M. C. Crabb and W. A. Sutherland, The space of sections of a sphere-bundle, I, *Proc. Edinburgh Math. Soc. (2)* **29** (1986), 383–403.

6. M. C. Crabb and W. A. Sutherland, Counting homotopy types of gauge groups, *Proc. London Math. Soc.* **81** (2000), 747–768.

7. J. P. C. Greenlees, Generalized Eilenberg-Moore spectral sequences for elementary abelian groups and tori, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 77–89

8. A. Kono and K. Kozima, The adjoint action of a Lie group on the space of loops, *J. Math. Soc. Japan* **45** (1993), 495–510.

9. G. Masbaum, Sur l'algèbre de cohomologie entière du classifiant du groupe de jauge, *C. R. Acad. Sci. Paris Série I* **307** (1988), 339–342.

10. G. Masbaum, On the cohomology of the classifying space of the gauge group over some 4-complexes, *Bull. Soc. Math. France* **119** (1991), 1–31.

11. S. A. Mitchell, A filtration of the loops on $SU(n)$ by Schubert varieties, *Math. Z.* **193** (1986), 347–362.

12. A. Pressley and G. Segal, *Loop Groups* (Oxford University Press, Oxford, 1986).

13. G. Segal, Loop groups and harmonic maps, in *Advances in Homotopy Theory* (ed. S. M. Salamon, B. Steer, W. A. Sutherland), London Math. Soc. Lecture Note Series **139** (1989), 153–164.