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On the Diophantine Equation $n(n+d)\cdots(n+(k-1)d) = by^l$

Dedicated to Professor P. Ribenboim on the occasion of his 75th birthday

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Abstract. We show that the product of four or five consecutive positive terms in arithmetic progression can never be a perfect power whenever the initial term is coprime to the common difference of the arithmetic progression. This is a generalization of the results of Euler and Obláth for the case of squares, and an extension of a theorem of Győry on three terms in arithmetic progressions. Several other results concerning the integral solutions of the equation of the title are also obtained. We extend results of Sander on the rational solutions of the equation in n, y when $b = d = 1$. We show that there are only finitely many solutions in n, d, b, y when $k \geq 3$, $l \geq 2$ are fixed and $k + l > 6$.

1 Introduction

In this paper we consider the diophantine equation

$$\Pi = \Pi(n, d, k) = n(n+d)\cdots(n+(k-1)d) = by^l \tag{1.1}$$

in positive integers $n, d, y, b, l \geq 2, k \geq 2$ with $\gcd(n, d) = 1$, $P(b) \leq k$, where for any integer u with $|u| > 1$ we write $P(u)$ for the greatest prime factor of u and we put $P(\pm 1) = 1$. We also take b to be l-th power free.

First we take $l = 2$. In this case, if $k = 3$, $d = 1$, then (1.1) has no solution except when $n \in \{1, 2, 48\}$. We refer to [17] for the details and history. Fermat showed that there are no four squares in arithmetic progression. Euler proved the more general result that a product of four terms in an arithmetic progression can never be a perfect square. Obláth [11] extended this result to the case $k = 5$. Erdős [4] and Rigge [14], independently showed that a product of two or more consecutive integers is never a perfect square. Recently, Saradha and Shorey [19] proved that a product of four or more terms in an arithmetic progression can never be a perfect square provided that d is a power of a prime number.

Now we take $l \geq 3$. Erdős and Selfridge [5] proved the remarkable result that a product of two or more consecutive integers can never be a perfect power. When $d = 1$, it was proved by Saradha [16] for $k \geq 4$ and by Győry [8] for $k = 2, 3$

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that (1.1) has no solution with $P(y) > k$. Győry [9] showed that (1.1) for $k = 3$ does not hold whenever $P(b) \leq 2$. His proof depends on the works of Wiles [26], Darmon and Merel [3] and Ribet [13] on generalized Fermat equations. In this result by Győry $P(b) \leq 2$ cannot be replaced by $P(b) \leq 3$; for $(k, l) = (3, 3)$ equation (1.1) has infinitely many solutions with $P(b) = 3$, see Tijdeman [25]. Saradha and Shorey [18] showed that (1.1) with $k \geq 4$, $P(b) < k$ implies that d has a prime factor $\equiv 1$ mod l. Thus (1.1) with $k \geq 4$, $P(b) < k$ and $l \geq 3$ has no solution, if d has only the prime factors 2, 3 and 5.

In this paper we show that for $k = 4, 5$ and $b = 1$, (1.1) has no solution. In other words,

Theorem 1 *Equation* (1.1) *with* $k = 4, 5$ *and* $b = 1$ *does not hold.*

This gives an answer to a problem proposed by Guy; see **D17** in his book [6]. In fact, for $l \geq 3$ we show more.

Theorem 2

(i) *Let* $k = 4$. *Then* (1.1) *with* $l \geq 3$ *and* $P(b) \leq 2$ *implies that* l *has a prime factor* > 3 *and* $8 \parallel \Pi$.

(ii) *Let* $k = 5$. *Then* (1.1) *with* $l \geq 3$ *and* $P(b) \leq 2$ *implies that* l *has a prime factor* > 3. *Further, we have either* $8 \parallel \Pi$ *or* $16 \parallel \Pi$.

As will be seen below, for certain applications equation (1.1) is interesting also in the case when n and b are not necessarily positive integers, and $P(b) \leq 3$. We present some results (*cf.* Theorems 8–10) with these more general settings in Section 2. Theorems 1 and 2 will be simple consequences of our Theorems 8 and 9.

Now we consider the equation

$$x(x+1)\cdots(x+k-1) = \pm 2^{\alpha} z^{l} \tag{1.2}$$

in rational numbers x and $z \geq 0$, and integers $k \geq 2$, $l \geq 2$ and α with $-l < \alpha < l$. We may restrict ourselves to the case $0 \leq \alpha < l$ by replacing in (1.2) α, z by $l - \alpha, z/2$, respectively. If x and z are integers and $\alpha = 0$, then by the result of Erdős and Selfridge, we see that $x = -j$, $z = 0$ for $0 \leq j < k$ are the only solutions. These are also the solutions of (1.2) for each α; they will be called *trivial*. In what follows, we shall deal only with non-trivial solutions. Equation (1.2) was first considered by Sander [15], who studied it for $2 \leq k \leq 4$ and $\alpha = 0$. By putting $x = n/d$ and $z = y/y_1$ with $\gcd(n, d) = \gcd(y, y_1) = 1$, $d > 0$, $y \geq 0$ and $y_1 > 0$, we see that (1.2) reduces to

$$n(n+d)\cdots(n+(k-1)d) = \pm 2^{\beta} u^{l};\ v^{l} = 2^{\gamma} d^{k} \tag{1.3}$$

where $(u, v) = (y, y_1)$ and $\beta + \gamma = \alpha$ for some non-negative integers β and γ. Thus solving (1.2) for rational values x and $z \geq 0$ is equivalent to solving equation (1.1) with $P(b) \leq 2$ for integers n, $y \geq 0$ and $d > 0$ with the additional restriction that $2^{\gamma} d^{k}$ is an l-th power. With the help of our general Theorems 8 to 10 we shall prove:

Theorem 3 *Let* $2 \le k \le 18$ *and* $l \ge 3$ *with* $\gcd(l,k) = 1$. *Then* (1.2) *with* $z \ne 0$ *implies* $k = 2$ *and* $(x, z, \alpha) = (-1/2, 1/2, l-2), (-2, 1, 1), (1, 1, 1)$.

For small values of k we can remove the condition $\gcd(l,k) = 1$. In the cases $k = 2, 3, 4$ and $\alpha = 0$, Sander [15] proved that (1.2) has no solution. We find, however, in Theorem 4 below that for $k = l = 3$ there are two solutions which are missing from the corresponding Proposition 2 of [15]. Hence Conjecture 1 of Sander [15] stating that for $k \ge 3$, (1.2) with $\alpha = 0$ has only the trivial solutions, should be modified accordingly. We also completely solve (1.2) with $\alpha = 0$ for $k = 5$, a new result. Thus we have:

Theorem 4 *Let* $2 \le k \le 5$ *and* $l \ge 3$. *Then the only non-trivial solutions of* (1.2) *with* $\alpha = 0$ *are given by* $k = l = 3$ *and* $(x, z) = (-2/3, 2/3), (-4/3, 2/3)$.

For (1.2) when $\alpha \ne 0$ we show:

Theorem 5 *Let* k *and* l *be as in Theorem 4, with the assumption that* $l \ne 4$ *if* $k = 2$. *Let* $\alpha > 0$.

(i) *If* $k = 2$, *then equation* (1.2) *has the only non-trivial solutions*

$$(x, z, \alpha) = (-1/2, 1/2, l-2),\ (-2, 1, 1),\ (1, 1, 1).$$

(ii) *If* $k = 3, 4$ *then* (1.2) *has no non-trivial solution.*
(iii) *If* $k = 5$, *then* (1.2) *implies that* $l = 5$ *and* $\alpha \in \{3, 4\}$.

Remark The assumption that $l \ne 4$ if $k = 2$ is necessary. It is well-known that there are infinitely many triples (p, q, r) of positive integers with $\gcd(p, q, r) = 1$ satisfying $2p^4 - q^4 = r^2$ (see *e.g.*, [12, pp. 152–164]). By putting $x = q^4/r^2$, we see that (1.2) with $k = 2$, $\alpha = 1$ and $l = 4$ has infinitely many solutions in (x, z).

So far we have given complete solutions of (1.1) or (1.2) for small values of k. Now we present some finiteness results on (1.1). For a complete survey on such results we refer to [2, 9, 22, 23, 25]. By applying Faltings' theorem, Darmon and Granville [2] showed that (1.1) with $b = 1$, $k \ge 3$, $l \ge 4$ fixed has only finitely many solutions in n, d, y. We refine this result and extend it to the case $b > 1$.

Theorem 6 *For fixed* $k \ge 3$ *and* $l \ge 2$ *with* $k + l > 6$, *equation* (1.1) *has only finitely many solutions in* n, d, b, y.

Theorem 6 is best possible in the sense that for fixed $k \ge 3$, $l \ge 2$ with $k + l \le 6$, (1.1) has in each case infinitely many solutions; *cf.* Tijdeman [25]. From the proof of Theorem 6, we observe that the above result is valid for the solutions of (1.1) with $n < 0$ as well.

Shorey [22] proved that if $d > 1$ and $l \ge 4$ then the *abc*-conjecture implies that k is bounded by an absolute constant. We refine this result as

Theorem 7 *The abc-conjecture implies that* (1.1) *with* $d > 1$, $k \ge 3$ *and* $l \ge 4$ *has only finitely many solutions in* n, d, k, b, y, l.

We note that if we use an effective variant of the *abc*-conjecture, then the above theorem is also effective. The restriction $d > 1$ is obviously necessary; for $d = 1$ and $n = 1$, (1.1) is solvable for every $k \geq 2$.

2 A Generalization of Equation (1.1)

In this section we consider the following generalization of equation (1.1):

$$\Pi = \Pi(n, d, k) = n(n + d) \cdots (n + (k - 1)d) = by^l \tag{2.1}$$

in non-zero integers n, b and in $d > 0, y > 0, l \geq 2, k \geq 2$ with $\gcd(n, d) = 1$, $P(b) \leq k$. Further, to make the representation by^l unique we assume here and in Theorems 8–10 that y is not divisible by primes $\leq k$. Thus while considering (2.1), b is not taken as l-th power free. We note that if n, d, b, y is a solution of (2.1) then so is $-n - (k - 1)d, d, (-1)^k b, y$.

In what follows, $\nu_p(u)$ denotes the order of p in u for any prime p and non-zero integer u.

Theorem 8 *Suppose equation* (2.1) *holds.*

(i) *Let $k = 3$ and $l \geq 3$. Then either*

$$(n, d, b, y) \in \{(-4, 3, 8, 1), (-2, 3, -8, 1)\},$$

or $l \nmid \nu_3(b)$. Moreover, if $P(l) > 3$, then $\nu_2(b) \leq 5$.

(ii) *Let $k = 4$ and suppose that $P(l) > 3$. Then $\nu_3(b) > 0$, and either $\nu_2(b) = 0$, $l \nmid \nu_3(b)$ or $\nu_2(b) = 3$.*

(iii) *Let $k = 5$. Suppose that $P(l) > 3$, and that $l \mid \nu_5(b)$. Then $\nu_3(b) > 0$, and either $\nu_2(b) = 0$, $l \nmid \nu_3(b)$ or $\nu_2(b) = 3$ or 4.*

Remark For $k = 3$, $\nu_2(b) \leq 5$ is sharp as is shown by the example

$$2(2 + 7)(2 + 2 \cdot 7) = 3^2 \cdot 2^5.$$

Similarly, for $k = 4$, $\nu_2(b) = 3$ is sharp since $1 \cdot 2 \cdot 3 \cdot 4 = 3 \cdot 2^3$. For $(k, l) = (3, 3)$ and $\nu_3(b) = 1, 2$, it is known that (2.1) has infinitely many solutions which can be seen by taking $b = 3, 6, 36$; *cf.* Tijdeman [25].

For the cases $l = 3, 4$ we prove:

Theorem 9

(i) *Let $l = 3$. Then equation* (2.1) *with $k = 4$ has only the solutions*

$$(n, d, b, y) = (-6, 5, 216, 1), (-9, 5, 216, 1), (-3, 2, 9, 1), (1, 1, 24, 1), (-4, 1, 24, 1).$$

Further, (2.1) *has no solution with $k = 5$, $3 \mid \nu_5(b)$.*

(ii) *Let $l = 4$ and $4 \mid \nu_3(b)$, $4 \mid \nu_5(b)$. Then equation* (2.1) *does not hold with $k = 4, 5$.*

Theorem 10 *Let $d = 2^h d_1^l$ $(h \geq 0)$, $l \mid \nu_p(b)$ for each prime p with $3 \leq p \leq k$. Suppose that $2 \leq k \leq 18$ if $\nu_2(d) < 4$, and let $2 \leq k \leq 30$ otherwise,* i.e., *if $\nu_2(d) \geq 4$. Further, in the latter case we suppose that l has a prime factor > 3. Then the only solutions of* (2.1) *are as follows: $k = 2$ and $(n, d, b, y) = (-2, 1, 2, 1)$, $(1, 1, 2, 1)$, $(-1, 2, -1, 1)$.*

3 Notation and Lemmas

By equation (2.1), we observe that if a prime $p > k$ divides Π, then it divides only one term in Π and $\nu_p(\Pi) \equiv 0 \mod l$. Hence we deduce that

$$n + id = a_i x_i^l \tag{3.1}$$

with $P(a_i) \leq \max(P(b), k-1)$, $x_i > 0$, a_i l-th power free for $0 \leq i < k$. Also we have $\gcd(x_i, x_j) = 1$ for each $i \neq j$ if $2^l \geq k$. Further,

$$n + id = A_i X_i^l \tag{3.2}$$

with $P(A_i) \leq k$, $X_i > 0$, $\gcd\left(X_i, \prod_{p \leq k} p\right) = 1$ for $0 \leq i < k$. Note that $\gcd(X_i, X_j) = 1$ for each $i \neq j$. We need several lemmas for the proofs of our theorems. We begin with a result by Győry [9].

Lemma 1 *Equation* (1.1) *with $k = 3$, $l > 2$ and $P(b) \leq 2$ has no solution.*

Győry derives the above result as a consequence of the following statement (*cf.* [9, Theorem G]) on a generalized Fermat equation.

Lemma 2 *Let $l \geq 3$, $\alpha \geq 0$ be integers. Then the equation*

$$x^l + y^l = 2^\alpha z^l$$

in relatively prime integers $x, y, z \geq 1$ has no solution for $\alpha \neq 1$, and for $\alpha = 1$ the equation has only the trivial solution $x = y = z = 1$. Further, the equation

$$x^l - y^l = 2^\alpha z^l$$

has no solution in relatively prime integers $x, y, z \geq 1$.

The above result was established by Wiles [26] for $\alpha \equiv 0 \mod l$, by Darmon and Merel [3] for $\alpha \equiv 1 \mod l$, and by Ribet [13] for $\alpha \not\equiv 0, 1 \mod l$ and $l \geq 5$ prime. For the other cases, see Győry [9].

In [18], Saradha and Shorey gave the following result on a more general Fermat equation by using the contributions of Wiles [26], Ribet [13] and others. The first such results were due to Serre [10] and Kraus [21]; see also Sander [15, p. 432].

Lemma 3 *Let l be a positive integer having a prime factor > 3. Suppose that a, b, c are non-zero integers such that either $P(abc) \leq 3$ or a, b, c are composed of only 2 and 5. Then the equation*

$$ax^l + by^l = cz^l$$

in non-zero integers x, y, z with $\gcd(ax^l, by^l, cz^l) = 1$, $\nu_2(by^l) \geq 4$ has no solution.

Bennett and Skinner [1] proved the following:

Lemma 4 *The only solution to the equation*

$$x^l + y^l = 2z^2$$

in integers x, y, z, l with $\gcd(x, y, z) = 1$, $x > y$, $l \geq 4$ is $(x, y, z, l) = (3, -1, \pm 11, 5)$.

We shall also use the following consequence of Lemma 4.

Lemma 5 *The equation*

$$x^l - y^l = 2z^2$$

in integers x, y, z, l with $\gcd(x, y, z) = 1$, $x > y$ and $l \geq 4$ even with $l \neq 6$ has no solution.

Proof Suppose that the equation holds. We may assume that x, y, z are positive. Let $l = 2^k l_1$ with $k \geq 1$, l_1 odd. If $k \geq 2$, we arrive at a contradiction by (iii) of Lemma 7 below. Hence $k = 1$ and $l_1 \geq 5$ odd. We now deduce that either

$$x^{l_1} + y^{l_1} = 2z_1^2,\ x^{l_1} - y^{l_1} = z_2^2$$

or

$$x^{l_1} + y^{l_1} = z_1^2,\ x^{l_1} - y^{l_1} = 2z_2^2$$

with some positive integers z_1, z_2, which is impossible by Lemma 4. ∎

We also need results on several cubic and quartic equations. Cubic equations were extensively studied by Selmer [20] in a long paper. We present here results on these cubic equations which we come across in the proofs of our theorems. The study of quartic equations dates back to Euler. We refer to the book of Ribenboim [12, pp. 164–177], for the quartic equations we are interested in here.

Lemma 6 *The equations*

$$x^3 + 2y^3 = 3z^3; \quad x^3 + 4y^3 = 3z^3$$

have no solution in non-zero integers x, y, z with $\gcd(x, y, z) = 1$ and $|xyz| > 1$, and the equations

$$x^3 + y^3 = 3z^3; \quad x^3 + y^3 = 4z^3; \quad x^3 + 4y^3 = 9z^3$$

have no solution in non-zero integers x, y, z with $\gcd(x, y, z) = 1$.

Lemma 7 *Let* x, y, z *be positive integers with* $\gcd(x, y, z) = 1$ *and* $\alpha \geq 0$ *an integer.*

(i) *If* $x^4 - 2^\alpha y^4 = z^2$, *then* $\alpha \equiv 1 \mod 4$.
(ii) *If* $2^\alpha x^4 - y^4 = z^2$, *then* $\alpha \equiv 1 \mod 4$.
(iii) $x^4 - y^4 = 2^\alpha z^2$ *is impossible.*
(iv) *If* $x^4 + y^4 = 2^\alpha z^2$, *then* $x = y = z = 1$.

Lemma 8 *Let* (2.1) *be valid, and suppose that* $l \mid \nu_p(b)$ *for each prime* p *with* $3 \leq p \leq k$. *There exist indices* i, j *with* $0 \leq i < j < k$ *such that in* (3.1)

$$a_i = \tau_i 2^{\alpha_i},\ a_j = \tau_j 2^{\alpha_j},\ j - i = 2^\delta \text{ for } 2 \leq k \leq 18 \tag{3.3}$$

and

$$a_i = \tau_i,\ a_j = \tau_j,\ j - i = 2^\varrho 3^\varepsilon \text{ or } 2^\varrho 5^\varepsilon \text{ for } 2 \leq k \leq 30 \text{ if } d \text{ is even.} \tag{3.4}$$

Here $\alpha_i, \alpha_j, \delta, \varrho, \varepsilon$ *denote some non-negative integers, and* τ_i, τ_j *may assume* ± 1.

Proof of Lemma 8 The assertion can be easily checked for $k \leq 6$. We explain the case $k = 7$ and d odd. We observe that in this case $7 \nmid a_i$ for $0 \leq i < k$. Further we have either 5 not dividing any a_i or 5 dividing a_0, a_5 or 5 dividing a_1, a_6. Suppose $5 \nmid a_i$ for $0 \leq i < k$. Then the statement follows with $i = 0$, $j = 1$ if $3 \nmid a_0 a_1$; $i = 1$, $j = 2$ if $3 \mid a_0$; $i = 0$, $j = 2$ if $3 \mid a_1$. Hence we may assume that 5 divides either a_0, a_5 or a_1, a_6. The assertion follows with $i = 2$, $j = 3$ if $3 \nmid a_2 a_3$, with $i = 3$, $j = 4$ if $3 \mid a_2$, and with $i = 4$, $j = 5$ if $3 \mid a_3$. This procedure has been programmed and (3.3) is checked. When d is even we observe that all the a_is are odd and we check that (3.4) is valid for $k \leq 30$. The largest cases $k = 29, 30$ took 8 hours of computation. ■

4 Proofs of Theorems 8–10

We will use the notation introduced in the previous section without any further mention.

Proof of Theorem 8 Suppose equation (2.1) holds. For $k = 3, 4$ and for $k = 5$ with $l \mid \nu_5(b)$, (3.2) can be modified such that $n + id = A_i X_i^l$ with

$$A_i = \tau_i 2^{\alpha_i} 3^{\beta_i},\ \tau_i = \pm 1,\ X_i > 0,\ \gcd(X_i, 6) = 1 \text{ for } 0 \leq i < k. \tag{4.1}$$

Let $\alpha = \max(\alpha_0, \ldots, \alpha_{k-1})$.

(i): Let $k = 3$. First we show that in this case $l \nmid \nu_3(b)$. We assume that

$$(n, d, b, y) \notin \{(-4, 3, 8, 1), (-2, 3, -8, 1)\}. \tag{4.2}$$

Suppose to the contrary that $l \mid \nu_3(b)$. Then $\beta_i = lt_i$ with non-negative integers t_i among which at least two are zero. By Lemma 1, we may suppose that $n < 0$ and

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$n + 2d > 0$. In view of $\gcd(n, d) = 1$ we infer that $(n, n + d) = (n + 2d, n + d) = 1$ and $(n, n + 2d) = 1$ or 2.

If $n + d$ is even, then $n, n + 2d$ are odd and we deduce from (4.1) that

$$n + d = \tau_1 2^{\alpha_1} (3^{t_1} X_1)^l, \; n = -(3^{t_0} X_0)^l, \; n + 2d = (3^{t_2} X_2)^l$$

with $\tau_1 = \pm 1$ and $\alpha_1 \geq 1$. Then

$$-(3^{t_0} X_0)^l + (3^{t_2} X_2)^l = \tau_1 2^{\alpha_1 + 1} (3^{t_1} X_1)^l. \tag{4.3}$$

By Lemma 2 we obtain that there is no solution in this case.

If $n + d$ is odd then there are two subcases to be distinguished. If n and $n + 2d$ are also odd then we arrive at equation (4.3) with $\alpha_1 = 0$, which leads to a contradiction.

Assume now that n and $n + 2d$ are even. Then we get from (4.1) that

$$n + d = \tau_1 (3^{t_1} X_1)^l, \quad n = -2^{\alpha_0} (3^{t_0} X_0)^l, \quad n + 2d = 2^{\alpha_2} (3^{t_2} X_2)^l \tag{4.4}$$

where $\tau_1 = \pm 1$, $\alpha_0, \alpha_2 \geq 1$ such that one of α_0, α_2 equals 1, and $3^{t_0} X_0, 3^{t_1} X_1, 3^{t_2} X_2$ are relatively prime odd positive integers. If $\alpha_2 = 1$, then we obtain that

$$-2^{\alpha_0 - 1} (3^{t_0} X_0)^l + (3^{t_2} X_2)^l = \tau_1 (3^{t_1} X_1)^l,$$

which by Lemma 2 gives $\alpha_0 = 2$, $\tau_1 = -1$, $t_i = 0$ for $i = 0, 1, 2$ and $X_0 = X_1 = X_2 = 1$. We infer from (4.4) that $(n, d, b, y) = (-4, 3, 8, 1)$ which is excluded.

Finally, if $\alpha_0 = 1$ then

$$-(3^{t_0} X_0)^l + 2^{\alpha_2 - 1} (3^{t_2} X_2)^l = \tau_1 (3^{t_1} X_1)^l,$$

and Lemma 2 implies that $\alpha_2 = 2$, $\tau_1 = 1$, $t_i = 0$ for $i = 0, 1, 2$, $X_0 = X_1 = X_2 = 1$. Then, by (4.4) we get $(n, d, b, y) = (-2, 3, -8, 1)$ which is excluded. So if (4.2) holds, then $l \nmid \nu_3(b)$.

Assume now that l has a prime factor > 3. We may suppose that d is odd since otherwise $\nu_2(b) = 0$. Further, we have

$$A_0 X_0^l + A_2 X_2^l = 2 A_1 X_1^l. \tag{4.5}$$

If $\alpha = 1$, then clearly $\nu_2(b) \leq 2$. Assume that $\alpha > 1$. We observe that

$$(\alpha_0, \alpha_1, \alpha_2) \in \{(\alpha, 0, 1), (1, 0, \alpha), (0, \alpha, 0)\}.$$

Now we apply Lemma 3 to (4.5) to get $\alpha \leq 4$. Hence $\nu_2(b) \leq 5$.

(ii): Let $k = 4$. In case of $\nu_3(b) = 0$, *i.e.* if

$$(\beta_0, \beta_1, \beta_2, \beta_3) = (0, 0, 0, 0)$$

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we can use the results just proved for $k = 3$ to show that there is no solution. Hence $\nu_3(b) > 0$. Further, we have

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \{(0,0,0,0), (\alpha,0,1,0), (1,0,\alpha,0), (0,\alpha,0,1), (0,1,0,\alpha)\}.$$

If d is even, then $\nu_2(b) = 0$. Suppose $l \mid \nu_3(b)$. Then we have

$$(\beta_0, \beta_1, \beta_2, \beta_3) \in \{(tl-1,0,0,1), (1,0,0,tl-1)\}$$

for some integer $t > 0$. In both cases we see that

$$\tau_1\tau_2(X_1X_2)^l - \tau_0\tau_3(3^tX_0X_3)^l = (n+d)(n+2d) - n(n+3d) = 2d^2.$$

As $X_i > 0$ ($i = 0,1,2,3$), this equation has no non-trivial solution by Lemmas 4 and 5. Thus $l \nmid \nu_3(b)$.

Let d be odd. Then $\alpha > 1$, whence $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \neq (0,0,0,0)$. We take $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha,0,1,0)$, in the other cases the proof is similar. We apply Lemma 3 to the equation

$$2A_0X_0^l + A_3X_3^l = 2n + (n+3d) = 3(n+d) = 3A_1X_1^l$$

to get $\alpha \le 2$. As clearly $\alpha \ge 2$, thus $\nu_2(b) = 3$ and part (ii) is proved.

(iii): Let $k = 5$ and $P(b) < 5$. Then we have

$$\begin{aligned}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \{&(0,0,0,0,0), (\alpha,0,1,0,2), (2,0,1,0,\alpha),\\ &(1,0,\alpha,0,1), (0,\alpha,0,1,0), (0,1,0,\alpha,0)\}.\end{aligned}$$

Suppose $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0,0,0,0,0)$. Then we argue as in the case $k = 4$ to see that $l \nmid \nu_3(b)$. In the next two possibilities, we observe that $\alpha > 2$. On the other hand, applying Lemma 3 to the equations obtained from the equalities

$$2n + (n+3d) = 3(n+d) \text{ and } (n+d) + 2(n+4d) = 3(n+3d)$$

we get $\alpha \le 2$, a contradiction. For the last three quintuples we apply Lemma 3 to the equations obtained from

$$(n+id) + (n+(i+2)d) = 2(n+(i+1)d),\ i = 1,0,2,$$

respectively, to find $\alpha \le 2$. As $\alpha \ge 2$, we get $\nu_2(b) = 3$ or 4.

Further, if $\beta_i = 0$ for $i = 0,\dots,4$, then by part (ii) of the theorem, (2.1) has no solution with $k = 5$. Thus $\nu_3(b) > 0$, and (iii) is also proved. ■

Proof of Theorem 9 Suppose that (2.1) holds. Since $k = 4$ or 5, we can modify (3.2) so that $n + id = A_i X_i^l$ and

$$A_i = \tau_i 2^{\alpha_i} 3^{\beta_i} \text{ with } \tau_i = \pm 1,\ X_i > 0,\ \gcd(X_i, 6) = 1 \text{ for } i = 0, 1, \dots, k-1.$$

Further, we have $\gcd(X_i, X_j) = 1$ whenever $i \neq j$.

Assume first that $k = 4$. Let $\alpha = \max(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and observe that $\alpha = 0$ or $\alpha \geq 2$. Moreover, we have

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \{(0,0,0,0), (\alpha,0,1,0), (1,0,\alpha,0), (0,\alpha,0,1), (0,1,0,\alpha)\}.$$

Let $l = 3$. Suppose first that $(\beta_0, \beta_1, \beta_2, \beta_3) = (0,0,0,0)$. Then we may apply part (i) of Theorem 8 to the first three factors of the left hand side of (2.1) to prove that there is no solution in this case. Thus we may assume that

$$(\beta_0, \beta_1, \beta_2, \beta_3) \in \{(1,0,0,\beta), (\beta,0,0,1), (0,\beta,0,0), (0,0,\beta,0)\}$$

with $\beta \geq 1$.

Using one of the equalities

$$A_0 X_0^3 + A_2 X_2^3 = 2A_1 X_1^3, \quad A_1 X_1^3 + A_3 X_3^3 = 2A_2 X_2^3,$$

or

$$2A_0 X_0^3 + A_3 X_3^3 = 3A_1 X_1^3, \quad A_0 X_0^3 + 2A_3 X_3^3 = 3A_2 X_2^3,$$

we can reduce each of the 20 cases arising to one of the cubic equations in Lemma 6. Hence by this lemma, we get all the solutions listed in the theorem in this case.

Let $l = 4$. If $4 \mid \beta_i$ for some i and $\beta_j = 0$ for each $j \neq i$, then we may use again part (i) of Theorem 8 to conclude that there is no solution. There remains the case $(\beta_0, \beta_1, \beta_2, \beta_3) = (\beta_0, 0, 0, \beta_3)$ with positive β_0, β_3 such that $4 \mid (\beta_0 + \beta_3)$. Further, we have $3 \nmid d$. In what follows, we assume that $\beta_0 = 4t - 1$ ($t \in \mathbb{N}$), $\beta_3 = 1$. In the opposite case $\beta_0 = 1$, $\beta_3 = 4t - 1$ ($t \in \mathbb{N}$), we can argue in a similar way.

When

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \{(0,0,0,0), (1,0,\alpha,0), (\alpha,0,1,0)\},$$

by using the relation $2A_0 X_0^4 + A_3 X_3^4 = 3A_1 X_1^4$, we get the equations

$$2(3^{2t-1} X_0^2)^2 = \tau_0 \tau_1 X_1^4 - \tau_0 \tau_3 X_3^4,$$

$$4(3^{2t-1} X_0^2)^2 = \tau_0 \tau_1 X_1^4 - \tau_0 \tau_3 X_3^4,$$

$$2^{\alpha+1}(3^{2t-1} X_0^2)^2 = \tau_0 \tau_1 X_1^4 - \tau_0 \tau_3 X_3^4,$$

respectively, which are all impossible by Lemma 7.

In case of $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, \alpha, 0, 1)$ we apply $A_0 X_0^4 + 2A_3 X_3^4 = 3A_2 X_2^4$ to obtain

$$\tau_0 3^{4t-2} X_0^4 + \tau_3 4 X_3^4 = \tau_2 X_2^4,$$

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which by $2 \nmid X_i$ $(i = 0, 1, 2, 3)$ leads to a contradiction mod 16.

Finally, suppose that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 1, 0, \alpha)$. We have $(A_0, A_1, A_2, A_3) = (\tau_0 3^{4t-1}, \tau_1 2, \tau_2, \tau_3 2^{\alpha} 3)$. By $2A_0X_0^4 + A_3X_3^4 = 3A_1X_1^4$, we get

$$\tau_0 3^{4t-2} X_0^4 + \tau_3 2^{\alpha-1} X_3^4 = \tau_1 X_1^4.$$

As $2 \nmid X_i$, the equation mod 16 gives $\alpha = 4$. However, the relation $A_0X_0^4 + 2A_3X_3^4 = 3A_2X_2^4$ yields

$$\tau_0 3^{4t-2} X_0^4 + \tau_3 32 X_3^4 = \tau_2 X_2^4,$$

which is impossible mod 16.

Now we take $k = 5$. If $l = 3$ then we may apply the statement proved above for $k = 4$ to the first four factors of the left hand side of (2.1) and the assertion follows. When $l = 4$, the problem can be reduced to the case $k = 4$ by considering the first or last four factors of the left hand side of (2.1), according as $3 \mid n$ or $3 \nmid n$. Then the statement immediately follows from the result just proved above for $k = 4$. ■

Proof of Theorem 10 Suppose first that $\nu_2(d) < 4$ and $2 \le k \le 18$. Write $d_1 = 2^{h_1} d_2$ with d_2 odd. Hence $d = 2^{h+lh_1} d_2^l$; put $h_0 = h + lh_1$. By Lemma 8, there exist i, j with $0 \le i < j < k$ such that (3.1) and (3.3) hold. Further, if p^{β} divides x_i and x_j for some prime p and integer $\beta > 0$, then $p^{\beta l} \mid (j-i)d$ implying $p^{\beta l} \mid (j-i)$. Hence $p^{\beta l} \le 17$ giving $p = 2, \beta = 1, l = 3$; $p = 2, \beta = 1, l = 4$. Thus we have

$$\tau_j 2^{\alpha_j} x_j^l - \tau_i 2^{\alpha_i} x_i^l = 2^{\delta} d = 2^{\delta + h_0} d_2^l.$$

Now we write $2^{\alpha_i} x_i^l = 2^{t_i} z_i^l$ and $2^{\alpha_j} x_j^l = 2^{t_j} z_j^l$ with z_i and z_j odd. Then by the preceding observation we see that $\gcd(z_i, z_j) = 1$. Also

$$2^{\delta + h_0} d_2^l = \tau_j 2^{t_j} z_j^l - \tau_i 2^{t_i} z_i^l.$$

Since d_2, z_i, z_j are all odd, it follows that exactly two among $\delta + h_0, t_i, t_j$ are equal, and the third is greater than the others. Suppose $t_i = t_j$. Then

$$2^{\delta + h_0 - t_i} d_2^l = \tau_j z_j^l - \tau_i z_i^l.$$

Hence by Lemma 2 we have $d_2 = z_j = z_i = 1$, giving $d = 2^{h_0}$, $\delta + h_0 - t_i = 1$, $n + id = \tau_i 2^{\delta + h_0 - 1}$, $n + jd = \tau_j 2^{\delta + h_0 - 1}$.

Suppose d is even. Then $n + id = \tau_i$, $n + jd = \tau_j$ and $\delta + h_0 = 1$. Hence $h_0 = 1$ giving $d = 2$, $n + 2i = -1$, $n + 2j = 1$. It is easy to check that the only solution to (2.1) is given by $k = 2$ and $(n, d, b, y) = (-1, 2, -1, 1)$. Suppose d is odd. Then $h_0 = 0$ giving $d = 1$ and $\delta - t_i = 1$, $n + i = \tau_i 2^{\delta - 1}$, $n + j = \tau_j 2^{\delta - 1}$. Using $\delta \le 4$, we get for $k = 2$ the solutions as $(n, d, b, y) = (-2, 1, 2, 1)$ and $(1, 1, 2, 1)$, and for $k \ge 3$ we check that there exists a prime $p > 2$ with $p \parallel \Pi(n, 1, k)$. Hence equation (2.1) does not hold if $k \ge 3$. The argument for the cases $\delta + h_0 = t_i$ or $\delta + h_0 = t_j$ is similar.

Now let $\nu_2(d) \ge 4$ and $2 \le k \le 30$, and suppose that l is divisible by a prime > 3. Then by (3.4) we have

$$\tau_j x_j^l - \tau_i x_i^l = 2^{\varrho} 3^{\varepsilon} d \text{ or } 2^{\varrho} 5^{\varepsilon} d.$$

We apply Lemma 3 to see that (2.1) has no solution in this case. ■

5 Proofs of Theorems 1–5

Proof of Theorem 1 Let $k = 4, 5$. When $l = 2$, the assertion is the result of Euler for $k = 4$ and of Obláth for $k = 5$. We observe from (1.1) that whenever $l \geq 3$, we may assume l to be a prime. The assertion for any prime $l \geq 5$ follows from Theorem 8 and for $l = 3$ from Theorem 9. Hence the theorem follows for any $l \geq 2$. ■

Proof of Theorem 2 (i) Let $k = 4$. Suppose (1.1) holds with $l \geq 3$ and $P(b) \leq 2$. From part (ii) of Theorem 8, we find that $8 \parallel \Pi$ whenever l has a prime factor > 3. For $l = 3, 4$ we apply Theorem 9 to see that (1.1) cannot hold with $P(b) \leq 2$. Hence the statement follows.

(ii) Let $k = 5$. We apply part (iii) of Theorem 8 for $l \geq 5$ and Theorem 9 for $l = 3, 4$ to obtain the assertion. ■

Proof of Theorem 3 Suppose $2 \leq k \leq 18$, $l \geq 3$ with $\gcd(l, k) = 1$ and equation (1.2) holds. Then (1.3) is valid. Further, from the second equality of (1.3) it follows that $d = 2^h d_1^l$ $(h \geq 0)$ since $\gcd(l, k) = 1$. By Theorem 10 we get that (1.2) has only the solutions

$$(x, z, \alpha) = (-1/2, 1/2, l-2),\ (-2, 1, 1),\ (1, 1, 1),$$

and the theorem is proved. ■

Proof of Theorem 4 Assume (1.2) with $\alpha = 0$. Hence (1.3) is valid with $\beta = \gamma = 0$. By Theorem 3, we need to consider only the cases $k = l = 3, 4, 5$ and $k = 2, l = 4$. For $k = l = 3$, using part (i) of Theorem 8 we get $(n, d) \in \{(-4, 3), (-2, 3)\}$, which gives $x = -4/3, -2/3$. By part (ii) of Theorem 9 and part (iii) of Theorem 8 we can exclude the possibilities $k = l = 4$ and $k = l = 5$, respectively. Finally, let $k = 2, l = 4$. Then by (1.3) and (3.1) we get

$$a_1 x_1^4 - a_0 x_0^4 = v^2,$$

with $a_0 a_1 = \pm 1$. However, by using Lemma 7 one can easily see that there is no solution in this case. ■

Proof of Theorem 5 Assume (1.2) with $\alpha > 0$. Then (1.3) is valid. By Theorem 3 we get all solutions for $k = 2$, $l \geq 3$ prime, and we need to consider only $k = l = 3, 4, 5$ and $k = 2, l = 8$. Using part (i) of Theorem 8 we get that there is no solution with $k = l = 3$. For $k = l = 4$ and $k = l = 5$ by part (ii) of Theorem 9 and part (iii) of Theorem 8, respectively, we get that the former case is excluded while in the latter case $\alpha = 3, 4$.

Finally, suppose that $k = 2$ and $l = 8$. Now (1.3) yields that $n = \pm 2^{\beta_0} x_0^4$ and $n + d = \pm 2^{\beta_1} x_1^4$ with $(\beta_0, \beta_1) = (\beta, 0)$ or $(0, \beta)$, $x_0, x_1 > 0$ and $\gcd(x_0, x_1) = 1$. Moreover, γ is even, and $v^4 = 2^{\gamma/2} d$. Thus we obtain the equation

$$\pm 2^{\beta_0} x_0^4 + 2^{-\gamma/2} v^4 = \pm 2^{\beta_1} x_1^4.$$

By Lemma 2 we get $x_0 = x_1 = v = 1$, whence $d = 1$ and $n = 1$ or $n = -2$. Thus we obtain the solutions $(x, z, \alpha) = (1, 1, 1), (-2, 1, 1)$ which were already found, and the theorem follows. ■

6 Proofs of Theorems 6 and 7

Proof of Theorem 6 Since k and l are fixed and the a_i in (3.1) are l-th power free with $P(a_i) \le k$, the coefficients a_i may assume only finitely many values. Fix a_i for $i = 0, \ldots, k-1$.

We take j consecutive terms from the product $\Pi(n, d, k)$ in (1.1), say $n + id, n + (i+1)d, \ldots, n + (i + j - 1)d$ with $i = 0$, $j = k$ if $k = 3$ or 4, and $i \ge 0$, $j = 5$ if $k \ge 5$. It follows from (3.1) and

$$(n + id) + (n + (i+2)d) = 2(n + (i+1)d)$$

that

$$(2a_{i+1}x_{i+1})^l = (2a_{i+1})^{l-1}(a_i x_i^l + a_{i+2}x_{i+2}^l). \tag{6.1}$$

Further, if $k \ge 4$, then we get similarly

$$(2a_{i+3}x_{i+3})^l = (2a_{i+3})^{l-1}(-a_i x_i^l + 3a_{i+2}x_{i+2}^l), \tag{6.2}$$

and if $k \ge 5$, then

$$(a_{i+4}x_{i+4})^l = (a_{i+4})^{l-1}(-a_i x_i^l + 2a_{i+2}x_{i+2}^l). \tag{6.3}$$

Denote by $F_1(x_i, x_{i+2})$, $F_2(x_i, x_{i+2})$, $F_3(x_i, x_{i+2})$ the right-hand side of (6.1), (6.2) and (6.3), respectively.

By assumption, $\gcd(n, d) = 1$. Hence it is easy to see that $\gcd(n+id, n+(i+2)d) \mid 2$, which implies that $\gcd(x_i, x_{i+2}) = 1$.

First consider the case when $k \ge 5$. Then, by assumption $k + l > 6$, hence we get $l \ge 2$. Multiplying the equations (6.1) to (6.3) and putting

$$F(x_i, x_{i+2}) = \prod_{t=1}^{3} F_t(x_i, x_{i+2}),$$

we arrive at the equation

$$F(x_i, x_{i+2}) = z^l \tag{6.4}$$

with $z = 4a_{i+1}a_{i+3}a_{i+4}x_{i+1}x_{i+3}x_{i+4}$. Here F is a homogeneous polynomial in x_i, x_{i+2} with integral coefficients and with $3l \ge 6$ pairwise linearly independent linear factors over $\bar{\mathbb{Q}}$. Hence by [2, Theorem 1] we see that x_i, x_{i+2}, z, and hence also x_{i+1}, x_{i+3}, x_{i+4} may assume only finitely many integral values. Since this is true for any five consecutive terms in the product $\Pi(n, d, k)$, we see that all x_i with $0 \leq i < k$ assume only finitely many values. Thus n, d are bounded, and so b, y are also bounded.

Next assume that $k = 4$. Then, by assumption, $l \geq 3$. In this case (6.1) and (6.2) imply (6.4) with the choice $i = 0$,

$$F(x_0, x_2) = \prod_{t=1}^{2} F_t(x_0, x_2) \text{ and } z = 4a_1a_3x_1x_3.$$

Then Theorem 1 of [2] applies again to (6.4) and proves our theorem.

Finally, if $k = 3$ and $l \geq 4$, then we can take in (6.4) $F(x_0, x_2) = F_1(x_0, x_2)$ and $z = 2a_1x_1$ and the assertion follows in the same way as before. ■

Proof of Theorem 7 We denote by $c_1, \ldots, c_8$ explicitly computable absolute constants. We assume (1.1) with $d > 1$, $k \geq 3$ and $l \geq 4$. We take $(n, d, k) \neq (2, 7, 3)$. Then by a theorem of Shorey and Tijdeman [24], $P(\Pi) > k$, whence $P(y) > k$. By a result of Shorey [22], the *abc*-conjecture implies that $k \leq c_1$. We fix k with $3 \leq k \leq c_1$. For $0 \leq i < j < k-1$, we have

$$(j-i)(n+(k-1)d) + (k-1-j)(n+id) = (k-1-i)(n+jd).$$

It is easy to see that the greatest common divisor of these three terms is at most k^2. Now we use (3.1) in the above equality and divide by the greatest common divisor to get

$$e_{k-1}x_{k-1}^l + e_i x_i^l = e_j x_j^l, \tag{6.5}$$

where e_{k-1}, e_i, e_j are coprime positive integers composed only of primes not exceeding k. Since $P(y) > k$, at least one of the numbers $x_0, \ldots, x_{k-1}$, say x_i, has a prime factor greater than k. Put $X = \max(x_{k-1}, x_i, x_j)$. We now apply the *abc*-conjecture to (6.5) with $\varepsilon = 1/4$ to get

$$X^l \leq c_2 \Big(\prod_{p \leq k} p \Big)^{5/4} \Big(\prod_{p \mid x_{k-1}x_i x_j} p \Big)^{5/4} \leq c_3 X^{3 \cdot 5/4}.$$

Thus

$$X^{l-3.75} \leq c_3.$$

As $X > 1$ and $l \geq 4$, we obtain $l \leq c_4$ whence $X^l \leq c_5$. This means that in (6.5) x_{k-1}^l, x_i^l and x_j^l can assume only finitely many values. We fix such possible values of x_{k-1}^l, x_i^l and x_j^l. Then (6.5) becomes an S-unit equation for the set of primes $S = \{p \mid p \leq k\}$, which equation has only finitely many solutions in e_{k-1}, e_i, e_j, moreover $\max(e_{k-1}, e_i, e_j) \leq c_6$ (*cf.* [7]). Consequently,

$$n + (k-1)d = a_{k-1}x_{k-1}^l \leq k^2 e_{k-1} x_{k-1}^l \leq c_7.$$

Thus n, d, b, y are all bounded by c_8. ■

Remark In the above proof we used an effective version of the *abc*-conjecture, when c_2 is explicitly computable. For $l \geq 7$, we could also use the weak *abc*-conjecture with $\varepsilon = 1$ and c_2=1.

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