A FAMILY OF BALANCED TERNARY DESIGNS

WITH BLOCK SIZE FOUR

DIANE DONOVAN

This paper shows the existence of an infinite family of cyclic balanced ternary designs where the block size is 4, the index 2 and each block contains precisely one repeated element.

A balanced ternary design (BTD) is a collection of $B$ multisets, called blocks, of size $K$, chosen from a set of $V$ elements where any element may occur 0, 1 or 2 times in any one block. Furthermore each of the $\binom{V}{2}$ pairs of distinct elements must occur a constant number of times, $\Lambda$, (the index). Balanced ternary designs first arose in a paper by Tocher in 1952 [6]. In addition to the above restrictions each element must occur a constant number of times throughout the design. It follows that

(1) $VR = BK$.

Let $\rho_l$ denote the number of blocks in which an element occurs $\ell$ times ($\ell=1$ or 2). Then

(2) $R = \rho_1 + 2\rho_2$,

and

(3) $\Lambda(V-1) = R(K-1) - 2\rho_2$.

Received 12 July 1985. I wish to thank my supervisor Dr. Elizabeth J. Billington for suggesting the problem and for her comments.
Moreover $p_1$ and $p_2$ are constant and the parameters of a BTD can be written $(V,B; p_1,p_2,R; K,\lambda)$.

Since 1952 a number of papers have been written on the existence of BTDs. In particular Saha and Dey [4], Saha [3], Billington [1] and Chandak [2] have produced papers on the existence of cyclic BTDs. Cyclic BTDs arise from a set of initial blocks which when developed modulo $V$ yield all blocks of the design. These initial blocks may be derived from a family of supplementary difference sets. (For a definition, see Wallis, Street and Wallis [7], page 280.)

In this paper we prove the existence of an infinite family of cyclic BTDs with parameters $K = 4$, $\lambda = 2$ and $B = Vp_2$. In general, when $K = 4$ and $\lambda = 2$ it is not possible to have blocks of the form \(xxyy\), thus $B \geq Vp_2$. By taking equation (1) and substituting $R = \frac{BK}{V}$ into equation (3), for $\lambda = 2$ and $K = 4$ we obtain

$$2V - 2 = 12 \frac{B}{V} - 2p_2,$$

but $\frac{B}{V} \geq p_2$ implies $V \geq 5p_2 + 1$. Therefore when considering the case $B = Vp_2$ and fixing $p_2$ we prove the existence of a BTD with $K = 4$, $\lambda = 2$ and minimal $V$, that is $V = 5p_2 + 1$.

The proof of this result uses a concept of pairing (see for example Stanton and Goulden [5].) We take $3p_2$ dots which represent the numbers $p_2 + 1$ to $V - (p_2 + 1)$. We draw an edge from the point, say $y$, to the point $y - x$ and write down the triple \(\{x,y-x,y\}\). From this we obtain the initial block \([0,0,x,y]\) which gives rise to the differences $\pm x$, $\pm y$ (each twice) and $\pm (y-x)$. The following example illustrates this method. Consider a BTD with parameters \((46,394; 18,9,34; 4,2)\). In Figure 1 the 27 dots represent the numbers 10 to 36:
From Figure 1 we can write down the nine triples and their corresponding initial blocks.

<table>
<thead>
<tr>
<th>TRIPLES</th>
<th>INITIAL BLOCKS</th>
<th>TRIPLES</th>
<th>INITIAL BLOCKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,19,20</td>
<td>[0,0,1,20]</td>
<td>9,27,36</td>
<td>[0,0,9,36]</td>
</tr>
<tr>
<td>2,30,32</td>
<td>[0,0,2,32]</td>
<td>3,30,33</td>
<td>[0,0,3,33]</td>
</tr>
<tr>
<td>4,17,21</td>
<td>[0,0,4,21]</td>
<td>5,17,22</td>
<td>[0,0,5,22]</td>
</tr>
<tr>
<td>6,28,34</td>
<td>[0,0,6,34]</td>
<td>7,28,35</td>
<td>[0,0,7,35]</td>
</tr>
<tr>
<td>8,23,31</td>
<td>[0,0,8,31]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is easily checked that each of the numbers 1 to 45 occurs twice as a difference arising from the nine initial blocks, and so these blocks when taken modulo 46 generate a BTD with the above parameters.

We now state our main result.

**THEOREM.** There exists an infinite family of cyclic BTDs with parameters \((5p_2+1,p_2(5p_2+1); 2p_2,p_2,4p_2; 4,2)\).

**Proof.** We consider four separate cases.

Case 1: Let \(p_2 \equiv 0 \pmod{4}\), so say \(p_2 = 4m\) and \(V = 20m + 1\) for any positive integer \(m\). Consider the pairing diagram in Figure 2; from it we can write down \(4m\) sets of triples and from these \(4m\) initial blocks.
324  Diane Donovan

<table>
<thead>
<tr>
<th>TRIPLES</th>
<th>INITIAL BLOCKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,10m+1,10m+2</td>
<td>[0,0,1,10m+2]</td>
</tr>
<tr>
<td>4m,10m+1,14m+1</td>
<td>[0,0,4m,14m+1]</td>
</tr>
<tr>
<td>4l,14m-2l+1,14m+2l+1</td>
<td>[0,0,4l,14m+2l+1] for 1≤l≤m-1</td>
</tr>
<tr>
<td>4l+1,14m-2l+1,14m+2l+2</td>
<td>[0,0,4l+1,14m+2l+2] for 1≤l≤m-1</td>
</tr>
<tr>
<td>4l+2,8m-2l-3,8m+2l-1</td>
<td>[0,0,4l+2,8m+2l-1] for 0≤l≤m-1</td>
</tr>
<tr>
<td>4l+3,8m-2l-3,8m-2l</td>
<td>[0,0,4l+3,8m+2l] for 0≤l≤m-1</td>
</tr>
</tbody>
</table>

It can easily be checked that the numbers 1 to 20m occur twice as differences arising from the above initial blocks and so these blocks generate a cyclic BTD with parameters \((20m+1,80m^2+4m; 8m,4m,16m; 4,2)\).

Similar pairing diagrams, which have been omitted for brevity, yield the initial blocks stated in cases 2, 3 and 4.

Case 2: Let \(\rho_2 \equiv 1 \pmod{4}\) so that \(\rho_2 = 4m + 1\) and \(V = 20m + 6\) for any non-negative integer \(m\). The initial blocks

\[
[0,0,4m+1,16m+4] ; [0,0,1,8m+4] ; \\
[0,0,4l,8m+2l+3] , for 1≤l≤m-1; \\
[0,0,4l+1,8m+2l+4] , for 1≤l≤m-1; \\
[0,0,4l+2,14m+2l+4] , for 0≤l≤m-1; \\
[0,0,4l+3,14m+2l+5] , for 0≤l≤m-1; \\
[0,0,4m,14m+3] ;
\]

generate a cyclic BTD with parameters

\((20m+6,80m^2+44m+6; 8m+2,4m+1,16m+4; 4,2)\) for all non-negative integers \(m\).

Case 3: Let \(\rho_2 \equiv 2 \pmod{4}\) so that \(\rho_2 = 4m + 2\) and \(V = 20m + 11\) for any non-negative integer \(m\). If \(m = 0\) the initial blocks \([0,0,1,7]\) and \([0,0,2,8]\) generate a cyclic BTD with parameters \((11,22; 4,2,8; 4,2)\).

If \(m = 1\) the initial blocks \([0,0,1,17]\, [0,0,2,20]\, [0,0,3,21]\, [0,0,4,23]\, [0,0,5,24]\) and \([0,0,6,22]\) generate a cyclic BTD with parameters \((31,186; 12,6,24; 4,2)\). If \(m \geq 2\) the initial blocks
[0,0,4m+2,14m+8] ; [0,0,1,10m+7];
[0,0,4l,14m+2l+7], for 1 ≤ l ≤ m;
[0,0,4l+1,14m+2l+8], for 1 ≤ l ≤ m;
[0,0,4l+2,8m+2l+5], for 0 ≤ l ≤ m-2;
[0,0,4l+3,8m+2l+6], for 0 ≤ l ≤ m-2;
[0,0,4m-2,14m+6]; [0,0,4m-1,14m+7];

generate a cyclic BTD with parameters

\((20m+11,80m^2+84m+22; 8m+4,4m+2,16m+8; 4,2)\) for \( m \geq 2 \).

Case 4: Let \( p_2 \equiv 3 \pmod{4} \) and so \( p_2 = 4m + 3 \) and \( V = 20m + 16 \)
for any non-negative integer \( m \). The initial blocks

[0,0,4m+2,14m+12]; [0,0,1,10m+7];
[0,0,4l,14m+2l+11], for 1 ≤ l ≤ m;
[0,0,4l+1,14m+2l+12], for 1 ≤ l ≤ m;
[0,0,4l+2,8m+2l+6], for 0 ≤ l ≤ m-1;
[0,0,4l+3,8m+2l+7], for 0 ≤ l ≤ m-1;
[0,0,4m+3,14m+11];

generate a cyclic BTD with parameters

\((20m+16,80m^2+124m+48; 8m+6,4m+3,16m+12; 4,2)\).

This completes the proof.

References

[1] Elizabeth J. Billington, "Balanced n-ary designs: A combinatorial
survey and some new results" Ars Combinatoria 17A (1984), 37-72.

Ciencia Indica VII(m) No 3 (1981), 205-208.


triple systems and cyclic triple systems", Aequationes Math. 22


Department of Mathematics
University of Queensland
St. Lucia, Q. 4067,
Australia.