A FAMILY OF BALANCED TERNARY DESIGNS

WITH BLOCK SIZE FOUR

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This paper shows the existence of an infinite family of cyclic balanced ternary designs where the block size is 4, the index 2 and each block contains precisely one repeated element.

A balanced ternary design (BTD) is a collection of $B$ multisets, called blocks, of size $K$, chosen from a set of $V$ elements where any element may occur 0, 1 or 2 times in any one block. Furthermore each of the $\binom{V}{2}$ pairs of distinct elements must occur a constant number of times, $\Lambda$, (the index). Balanced ternary designs first arose in a paper by Tocher in 1952 [6]. In addition to the above restrictions each element must occur a constant number of times throughout the design. It follows that

$$VR = BK.$$  

Let $\rho_l$ denote the number of blocks in which an element occurs $l$ times ($l=1$ or 2). Then

$$R = \rho_1 + 2\rho_2,$$

and

$$\Lambda(V-1) = R(K-1) - 2\rho_2.$$  

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Moreover \( \rho_1 \) and \( \rho_2 \) are constant and the parameters of a BTD can be written \((V, B; \rho_1, \rho_2, R; K, \Lambda)\).

Since 1952 a number of papers have been written on the existence of BTDs. In particular Saha and Dey [4], Saha [3], Billington [7] and Chandak [2] have produced papers on the existence of cyclic BTDs. Cyclic BTDs arise from a set of initial blocks which when developed modulo \( V \) yield all blocks of the design. These initial blocks may be derived from a family of supplementary difference sets. (For a definition, see Wallis, Street and Wallis [7], page 280.)

In this paper we prove the existence of an infinite family of cyclic BTDs with parameters \( K = 4, \Lambda = 2 \) and \( B = V\rho_2 \). In general, when \( K = 4 \) and \( \Lambda = 2 \) it is not possible to have blocks of the form \( xxyy \), thus \( B \geq V\rho_2 \). By taking equation (1) and substituting \( R = \frac{BK}{V} \) into equation (3), for \( \Lambda = 2 \) and \( K = 4 \) we obtain

\[ 2V - 2 = 12 \frac{B}{V} - 2\rho_2, \]

but \( \frac{B}{V} \geq \rho_2 \) implies \( V \geq 5\rho_2 + 1 \). Therefore when considering the case \( B = V\rho_2 \) and fixing \( \rho_2 \) we prove the existence of a BTD with \( K = 4, \Lambda = 2 \) and minimal \( V \), that is \( V = 5\rho_2 + 1 \).

The proof of this result uses a concept of pairing (see for example Stanton and Goulden [5].) We take \( 3\rho_2 \) dots which represent the numbers \( \rho_2 + 1 \) to \( V - (\rho_2 + 1) \). We draw an edge from the point, say \( y \), to the point \( y - x \) and write down the triple \( \{x, y - x, y\} \). From this we obtain the initial block \([0, 0, x, y]\) which gives rise to the differences \( \pm x, \pm y \) (each twice) and \( \pm (y - x) \). The following example illustrates this method.

Consider a BTD with parameters \((46, 394; 18, 9, 34; 4, 2)\). In Figure 1 the 27 dots represent the numbers 10 to 36.
From Figure 1 we can write down the nine triples and their corresponding initial blocks.

<table>
<thead>
<tr>
<th>TRIPLES</th>
<th>INITIAL BLOCKS</th>
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<tbody>
<tr>
<td>1,19,20</td>
<td>[0,0,1,20]</td>
<td>9,27,36</td>
<td>[0,0,9,36]</td>
</tr>
<tr>
<td>2,30,32</td>
<td>[0,0,2,32]</td>
<td>3,30,33</td>
<td>[0,0,3,33]</td>
</tr>
<tr>
<td>4,17,21</td>
<td>[0,0,4,21]</td>
<td>5,17,22</td>
<td>[0,0,5,22]</td>
</tr>
<tr>
<td>6,28,34</td>
<td>[0,0,6,34]</td>
<td>7,28,35</td>
<td>[0,0,7,35]</td>
</tr>
<tr>
<td>8,23,31</td>
<td>[0,0,8,31]</td>
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</table>

It is easily checked that each of the numbers 1 to 45 occurs twice as a difference arising from the nine initial blocks, and so these blocks when taken modulo 46 generate a BTD with the above parameters.

We now state our main result.

**THEOREM.** There exists an infinite family of cyclic BTDs with parameters \((5\rho_2+1,\rho_2(5\rho_2+1); 2\rho_2,\rho_2,4\rho_2; 4,2)\).

**Proof.** We consider four separate cases.

Case 1: Let \(\rho_2 \equiv 0 \pmod{4}\), so say \(\rho_2 = 4m\) and \(V = 20m + 1\) for any positive integer \(m\). Consider the pairing diagram in Figure 2; from it we can write down \(4m\) sets of triples and from these \(4m\) initial blocks.
<table>
<thead>
<tr>
<th>TRIPLES</th>
<th>INITIAL BLOCKS</th>
</tr>
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<tbody>
<tr>
<td>$1,10m+1,10m+2$</td>
<td>$[0,0,1,10m+2]$</td>
</tr>
<tr>
<td>$4m,10m+1,14m+1$</td>
<td>$[0,0,4m,14m+1]$</td>
</tr>
<tr>
<td>$4\ell,14m-2\ell+1,14m+2\ell+1$</td>
<td>$[0,0,4\ell,14m+2\ell+1]$ for $1\leq s \leq m-1$</td>
</tr>
<tr>
<td>$4\ell+1,14m-2\ell+1,14m+2\ell+2$</td>
<td>$[0,0,4\ell+1,14m+2\ell+2]$ for $1\leq s \leq m-1$</td>
</tr>
<tr>
<td>$4\ell+2,8m-2\ell-3,8m+2\ell-1$</td>
<td>$[0,0,4\ell+2,8m+2\ell-1]$ for $0\leq s \leq m-1$</td>
</tr>
<tr>
<td>$4\ell+3,8m-2\ell-3,8m-2\ell$</td>
<td>$[0,0,4\ell+3,8m+2\ell]$ for $0\leq s \leq m-1$</td>
</tr>
</tbody>
</table>

It can easily be checked that the numbers $1$ to $20m$ occur twice as differences arising from the above initial blocks and so these blocks generate a cyclic BTD with parameters $(20m+1,80m^2+4m; 8m,4m,16m; 4,2)$.

Similar pairing diagrams, which have been omitted for brevity, yield the initial blocks stated in cases 2, 3 and 4.

Case 2: Let $p_2 \equiv 1 \pmod{4}$ so that $p_2 = 4m + 1$ and $V = 20m + 6$ for any non-negative integer $m$. The initial blocks

$[0,0,4m+1,16m+4]$ ; $[0,0,1,8m+4]$ ; $[0,0,4\ell,8m+2\ell+3]$, for $1\leq s \leq m-1$; $[0,0,4\ell+1,8m+2\ell+4]$, for $1\leq s \leq m-1$; $[0,0,4\ell+2,14m+2\ell+4]$, for $0\leq s \leq m-1$; $[0,0,4\ell+3,14m+2\ell+5]$, for $0\leq s \leq m-1$; $[0,0,4m,14m+3]$;

generate a cyclic BTD with parameters

$(20m+6,80m^2+44m+6; 8m+2,4m+1,16m+4; 4,2)$ for all non-negative integers $m$.

Case 3: Let $p_2 \equiv 2 \pmod{4}$ so that $p_2 = 4m + 2$ and $V = 20m + 11$ for any non-negative integer $m$. If $m = 0$ the initial blocks $[0,0,1,7]$ and $[0,0,2,8]$ generate a cyclic BTD with parameters $(11,22; 4,2,8; 4,2)$. If $m = 1$ the initial blocks $[0,0,1,17]$, $[0,0,2,20]$, $[0,0,3,21]$, $[0,0,4,23]$, $[0,0,5,24]$ and $[0,0,6,22]$ generate a cyclic BTD with parameters $(31,186; 12,6,24; 4,2)$. If $m \geq 2$ the initial blocks
Ternary designs

Figure 2.

Diagram showing ternary designs with numbers from 4m+1 to 16m.
generate a cyclic BTD with parameters

\[(20m+11, 80m^2+84m+22; 8m+4, 4m+2, 16m+8; 4, 2)\]

for \(m \geq 2\).

**Case 4:** Let \(p_2 \equiv 3 \pmod{4}\) and so \(p_2 = 4m + 3\) and \(V = 20m + 16\) for any non-negative integer \(m\). The initial blocks

\[
[0, 0, 4m+2, 14m+12]; [0, 0, 1, 10m+7];
\]
\[
[0, 0, 4m, 14m+2l+11], \text{ for } 1 \leq l \leq m;
\]
\[
[0, 0, 4l+1, 14m+2l+12], \text{ for } 1 \leq l \leq m;
\]
\[
[0, 0, 4l+2, 8m+2l+6], \text{ for } 0 \leq l \leq m-1;
\]
\[
[0, 0, 4l+3, 8m+2l+7], \text{ for } 0 \leq l \leq m-1;
\]
\[
[0, 0, 4m+3, 14m+11];
\]

generate a cyclic BTD with parameters

\[(20m+16, 80m^2+124m+48; 8m+6, 4m+3, 16m+12; 4, 2)\].

This completes the proof. \(\square\)

**References**


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