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Limited Sets and Bibasic Sequences

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Abstract. Bibasic sequences are used to study relative weak compactness and relative norm compactness of limited sets.

Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, and the closed linear span of a sequence (x_n) in X will be denoted by $[x_n]$. An operator $T\colon X \to Y$ will be a continuous and linear function. Let X^* denote the continuous linear dual of X, let $L(X,Y)$ denote the space of all continuous linear operators $T\colon X \to Y$, and let $K(X,Y)$ denote the compact linear maps. Let $L_{w^*}(X^*,Y)$ denote the $w^* - w$ continuous operators, and let $K_{w^*}(X^*,Y)$ denote the $w^* - w$ continuous compact operators.

A bounded subset A of a Banach space X is called a *limited subset* of X if for each w^*-null sequence (x_n^*) in X^*,

$$\lim_n \sup\{|x_n^*(x)| : x \in A\} = 0.$$

If A is a limited subset of X, then $T(A)$ is relatively compact for any operator $T\colon X \to c_0$. An operator $T\colon X \to Y$ is called *limited* if $T(B_X)$ is a limited subset of Y. We note that the operator T is limited if and only if T^* is w^*-norm sequentially continuous.

The subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence [1]. Every limited subset of X is weakly precompact [4].

It is known that ℓ_∞ contains limited sets that are not relatively weakly compact (see [21, Example 1.1.8]). In this paper we study limited sets that fail to be relatively norm or weakly compact.

A sequence (x_n, f_n^*) in $X \times X^*$ is called *bibasic* ([20, p. 85], [8]) if (x_n) is a basic sequence in X, (f_n^*) is a basic sequence in X^*, and $f_i^*(x_j) = \delta_{ij}$. If (x_n, f_n^*) is a bibasic sequence, $X_0 = [x_n]$, and $x_n^* = f_n^*|_{X_0}$ (*i.e.*, (x_n^*) is the sequence of coefficient functionals corresponding to the basic sequence (x_n)), then f_n^* is a continuous linear extension of x_n^* to all of X for each n. The bibasic sequence (x_n, f_n^*) is seminormalized if there are $p, q > 0$ so that $p \le \|x_n\| \le q$ for all n.

We denote the canonical unit vector basis of c_0 (resp. ℓ_1) by (e_n) (resp. (e_n^*)). If K is a subset of X, then $K - K$ is defined to be $\{x - y : x, y \in K\}$. For a subset A of X, let $\operatorname{co}(A)$ denote the convex hull of A. If U is a subspace of X, then

$$U^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in U\}.$$

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Suppose X has a basis (x_i) with the associated sequence of coordinate functionals (x_i^*). The basis (x_i) is *shrinking* if (x_i^*) is a basis for X^*. The unit vector bases of c_0 and ℓ_p, $1 < p < \infty$ are shrinking, and the unit vector basis of ℓ_1 is not shrinking.

A bounded subset A of X is called a V^**-subset* of X provided that

$$\lim_n \Big(\sup\{|x_n^*(x)| : x \in A\} \Big) = 0$$

for each weakly unconditionally converging series $\sum x_n^*$ in X^*.

A topological space S is called *dispersed* (or *scattered*) if every nonempty closed subset of S has an isolated point [19]. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [18]. The reader should consult [9] or [1] for undefined terminology and notation.

1 Limited Sets and Bibasic Sequences

In [8, Theorem 1] it is proved that every infinite dimensional Banach space contains a bounded bibasic sequence. We will show that special bibasic sequences exist in $X \times X^*$ whenever X contains a limited set that fails to be relatively weakly or norm compact. We begin with the following two lemmas.

Lemma 1.1 *If K is a limited subset of the Banach space X, (x_n, f_n^*) is a biorthogonal sequence in $K \times X^*$, and $\sup_n \|f_n^*\| < \infty$, then some subsequence of (f_n^*) is equivalent to the unit vector basis of ℓ_1.*

Proof Suppose that (f_n^*) has a w^*-Cauchy subsequence $(f_{n_i}^*)$. Let $y_i^* = f_{n_i}^* - f_{n_{i+1}}^*$ for each i. Then (y_i^*) is w^*-null and $y_i^*(x_{n_i}) \to 0$, since K is limited. However, $y_i^*(x_{n_i}) = 1$ for all i. Therefore, (f_n^*) has no w^*-Cauchy subsequence, and thus it has no weakly Cauchy subsequence. By Rosenthal's ℓ_1-theorem [9], some subsequence of (f_n^*) is equivalent to (e_n^*). ∎

Lemma 1.2 ([17, Lemma 1(ii)]) *If (x_n) is an unconditional basis for X, (x_n^*) is the sequence of coefficient functionals corresponding to (x_n), and the subsequence $(x_{n_i}^*)$ of (x_n^*) is equivalent to (e_i^*), then $(x_{n_i}) \sim (e_i)$.*

It is known that ℓ_∞ contains limited sets that are not relatively weakly compact (see [21, Example 1.1.8]). Further, Haydon [15] has given an example of a $C(K)$ space that is a Grothendieck space and does not contain ℓ_∞. Such a space must contain limited sets that are not relatively weakly compact [21, pp. 27–28].

In [8, Proposition 1], Davis, Dean, and Lin showed that if X is an infinite dimensional space, then there is a bounded bibasic sequence (x_n, f_n^*) in $X \times X^*$ so that $(f_n^*) \not\sim (x_n^*)$. The following theorem, which extends [1, Theorem 4.5], shows that limited subsets that are not relatively weakly compact naturally generate such sequences. In the next theorem, $t\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function and $g_{t(n)}^* = x_{t(n)}^*|_{[x_{t(j)}: j \in \mathbb{N}]}$.

Theorem 1.3 *Suppose that K is a subset of the Banach space X. If K is a nonrelatively weakly compact limited set, then there is a bibasic sequence (x_n, f_n^*) in $K \times X^*$ and an*

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element $x^* \in X^*$ such that $(f_n^*) \sim (e_n^*)$ and $\lim x^*(x_n) > 0$. If (x_n) is unconditional, $t\colon \mathbb{N} \to \mathbb{N}$ and $g_{t(n)}^*$ are as above, then $(g_{t(n)}^*) \not\sim (f_{t(n)}^*)$.

Proof Suppose that K is a limited subset of X that is not relatively weakly compact. Let (x_n) be a sequence in K with no weakly convergent subsequence. By Pelczyinski's version of the Eberlein–Smulian theorem ([9, p. 41]), we may assume that (x_n) is basic and $\lim x^*(x_n) > 0$ for some $x^* \in X^*$. Note that (x_n) is a seminormalized sequence. Let (x_n^*) be the associated sequence of coefficient functionals, and for each n, let f_n^* be a Hahn–Banach extension of x_n^* to all of X. By Lemma 1.1, we may suppose without loss of generality that $(f_n^*) \sim (e_n^*)$.

Now suppose that (x_n) is a seminormalized unconditional basic sequence, (x_n) has no weakly convergent subsequence, and $(g_{t(n)}^*) \sim (f_{t(n)}^*)$. Consequently, we have an unconditional basic sequence (x_n) and a subsequence $(x_{t(n)}^*)$ of the coordinate functionals so that $(x_{t(n)}^*) \sim (e_n^*)$ in $[x_{t(n)}]^*$. Lemma 1.2 implies that $(x_{t(n)}) \sim (e_n)$. Therefore, $(x_{t(n)})$ is weakly null, a contradiction. ■

The following result, due to Schlumprecht [21], tells us when $\{e_n : n \in \mathbb{N}\}$ embeds as a limited subset of X.

Lemma 1.4 ([21, Theorem 1.3.2, p. 36]) *If (x_n) is a basic sequence in X that is equivalent to (e_n), then the following are equivalent:*

(a) *(x_n) is a limited sequence in X;*

(b) *if S is an infinite subset of $\mathbb{N}$, then $[x_n : n \in S]$ is not complemented in X.*

The following result studies the structure of limited sets that contain unconditional basic sequences (x_n) such that $(x_n^*) \sim (f_n^*)$.

Theorem 1.5 *If K is a limited subset of the Banach space X, then the following are equivalent:*

(i) *There is an isomorphic embedding $T\colon c_0 \to X$ so that $\{T(e_n)\} \subseteq K$.*

(ii) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional, $(f_n^*) \sim (x_n^*)$, and $[x_n]$ is not complemented in X.*

(iii) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional, $(f_n^*) \sim (x_n^*)$, $[f_n^*]$ is complemented in X^*, and $[x_n]$ is not complemented in X.*

(iv) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional, $(f_n^*) \sim (x_n^*)$, $\{y_k\}$ is not a limited subset of $[y_k]$ for each subsequence (y_k) of (x_n), and $[x_n]$ is not complemented in X.*

(v) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is shrinking and unconditional, $(f_n^*) \sim (x_n^*)$, $[f_n^*]$ is complemented in X^*, and $[x_n]$ is not complemented in X.*

(vi) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional and $\sum f^*(x_n) f_n^*$ converges for all $f^* \in X^*$, and $[x_n]$ is not complemented in X.*

Proof Suppose that (i) holds and let $x_n = T(e_n)$ for each n. Then (x_n) is an unconditional basic sequence and $(x_n) \sim (e_n)$. By Lemma 1.4, $[x_n]$ is not complemented in X. Let (x_n^*) be the associated sequence of functionals, and for each n, let f_n^* be a Hahn–Banach extension of x_n^* to all of X. Let $S\colon [e_n] \to [x_n]$ be an isomorphism from $[e_n]$ onto $[x_n]$. Then $S^*(x_n^*) = e_n^*$, S^* is an isomorphism, and $(x_n^*) \sim (e_n^*)$. Let $C > 0$ such that

$$C \sum |a_i| \leq \| \sum a_i x_i^* \|,$$

for each finite sequence $(a_1, \ldots, a_n)$ of real numbers. Then

$$C \sum |a_i| \leq \| \sum a_i f_i^* \mid_{[x_n]} \| \leq \| \sum a_i f_i^* \| \leq M \sum |a_i|,$$

where M is a bound for $(\|f_i^*\|)$. Therefore, $(f_n^*) \sim (e_n^*) \sim (x_n^*)$. Let (y_k) be a subsequence of (x_n) and let $X_0 = [y_n]$. Since X_0 is separable and limited subsets of separable spaces are relatively compact [4], (y_k) is not limited in X_0 (otherwise $\|y_n\| \to 0$, a contradiction). Thus (x_n, f_n^*) satisfies the conclusion of (ii) and (iv).

Suppose that (ii) holds and let (x_n, f_n^*) satisfy the conclusions of (ii). Use Lemma 1.1 and let $(f_{n_i}^*)$ be a subsequence of (f_n^*) so that $(f_{n_i}^*) \sim (e_i^*)$. Then $(x_{n_i}^*) \sim (e_i^*)$. Since (x_n) is unconditional, $(x_{n_i}) \sim (e_i)$, by Lemma 1.2. Then (x_{n_i}) is shrinking and $\sum x_{n_i}$ is weakly unconditionally converging. Therefore, $\sum f^*(x_{n_i}) f_{n_i}^*$ converges for all $f^* \in X^*$. Hence $[f_{n_i}^*]$ is complemented in X^*, by [20, Corollary 1.12 b, p. 93] (or by a result of [3], since $(f_{n_i}^*)$ is not a V^*-subset of X^*). Consequently, the bibasic sequence $(x_{n_i}, f_{n_i}^*)$ satisfies the conclusions of (i), (iii), (v), and (vi).

Note that (iii) implies (ii), (iv) implies (ii), and (v) implies (ii). Therefore (i), (ii), (iii), (iv), and (v) are equivalent.

Suppose (vi) holds and let (x_n, f_n^*) satisfy the conclusions of (vi). By [20, Proposition 1.14 b, p. 91], $[x_n]^\perp + [f_n^*] = X^*$. By [20, Theorem 1.10, p. 93] (or [20, Lemma 1.3, p. 91]), $(x_n^*) \sim (f_n^*)$, and (vi) implies (ii). ■

We note that if K is a limited subset of X, then the set $K - K$ is a limited subset of X.

Corollary 1.6 *Suppose that K is a non-relatively compact set limited subset of X. Then the statements* (i)–(vi), *replacing K with $K - K$ in Theorem 1.5, are equivalent.*

Proof Suppose (i) holds. Let $T\colon c_0 \to X$ be an isomorphic embedding so that $\{T(e_n)\} \subseteq K - K$ and let $x_n = T(e_n)$ for each n. Then (x_n) is an unconditional basic sequence, (x_n) is limited, and $(x_n) \sim (e_n)$. The proof that (i) implies (ii) in the previous theorem shows that $(x_n^*) \sim (e_n^*) \sim (f_n^*)$ and $[x_n]$ is not complemented in X.

Suppose (ii) holds and let (x_n, f_n^*) in $(K - K) \times X^*$ satisfy the conclusions of (ii). Let (y_n) be a sequence in K and $\epsilon > 0$ such that $\|y_n - y_{n+1}\| > \epsilon$ for all n (since K is nonrelatively compact). Since limited sets are weakly precompact [4], we can suppose without loss of generality that (y_n) is weakly Cauchy. Let $x_n = y_n - y_{n+1}$, for each n. Since (x_n) is weakly null and not norm null, by Bessaga and Pelczyinski's selection principle ([2, 9]) we can suppose without loss of generality that (x_n) is a seminormalized basic sequence. Let (x_n^*) be the associated sequence of coefficient functionals, and for each n, let f_n^* be a Hahn–Banach extension of x_n^* to all of X^*. By

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Lemma 1.1 applied to the limited set $K - K$, (f_n^*) has a subsequence $(f_{n_i}^*)$ equivalent to (e_i^*). The proof of (ii) implies (i) in the previous theorem shows that $(x_{n_i}) \sim (e_i)$.

The proofs for the other implications are similar to the corresponding ones in the previous theorem. ■

Bourgain and Diestel proved that limited subsets of Banach spaces not containing ℓ_1 are relatively weakly compact ([4, Proposition 7]). The following lemmas will be used to give an alternative proof of this result.

Lemma 1.7 ([22, Lemma 1]) *Let A be a bounded subset of a Banach space Y. Then A is relatively weakly compact if and only if, given any sequence (x_n) in A, there exists a sequence (y_n) with $y_n \in co\{x_i : i \geq n\}$ that converges weakly.*

The following result is from [13]. We include a proof for the convenience of the reader.

Lemma 1.8 ([13, Theorem 3.12]) *The following are equivalent:*

(i) *If $T : Y \to X$ is an operator and $T^* : X^* \to Y^*$ is w^*-norm sequentially continuous, then T is weakly compact (resp. compact).*

(ii) *Same as* (i) *with $Y = \ell_1$.*

(iii) *Every limited subset of X is relatively weakly compact (resp. relatively compact).*

Proof (*weakly compact*) Certainly (i) implies (ii). Now suppose that (ii) holds, A is a limited subset of X, and let (x_n) be a sequence in A. Define $T : \ell_1 \to X$ by $T(b) = \sum b_i x_i$. Since the closed absolutely convex hull of (x_i) is a limited subset of X, T is limited and T^* is w^*-norm sequentially continuous. Thus T is weakly compact, and $(T(e_n^*)) = (x_n)$ has a weakly convergent subsequence.

Suppose that (iii) holds and let $T : Y \to X$ be an operator so that T^* is w^*-norm sequentially continuous. Then $T(B_Y)$ is a limited subset of X, and thus relatively weakly compact. ■

Theorem 1.9 ([4]) *If $\ell_1 \not\hookrightarrow X$, then every limited subset of X is relatively weakly compact.*

Proof Let $T : Y \to X$ such that $T^* : X^* \to Y^*$ is w^*-norm sequentially continuous. Let (x_n^*) be a sequence in B_{X^*}. Since $\ell_1 \not\hookrightarrow X$, every bounded sequence in X^* has a w^*-convergent convex block ([16, Lemma 3A, p. 4], [21, Lemma 2.2.1, p. 47]). Let (y_n^*) be a w^*-convergent convex block of (x_n^*). Let (k_n) be a strictly increasing sequence of natural numbers and (a_n) a sequence of positive real numbers with $\sum_{i=k_n}^{k_{n+1}-1} a_i = 1$, so that

$$y_n^* = \sum_{i=k_n}^{k_{n+1}-1} a_i x_i^*.$$

Then $(T^*(y_n^*))$ is norm convergent. Note that $y_n^* \in co\{x_i^* : i \geq n\}$ and $T^*(y_n^*) \in co\{T^*(x_i^*) : i \geq n\}$ for each n. Then $(T^*(x_n^*))$ is relatively weakly compact, by Lemma 1.7. Then T^*, and thus T, is weakly compact. By Lemma 1.8 , every limited subset of X is relatively weakly compact. ■

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The proof of the previous theorem shows that if each sequence in B_{X^*} has a w^*-convex block sequence, then every limited subset of X is relatively weakly compact.

Corollary 1.10

(i) *Suppose K is a dispersed compact Hausdorff space and $\ell_1 \not\hookrightarrow X$. Then every limited subset of $C(K, X)$ is relatively weakly compact.*

(ii) *Suppose $\ell_1 \not\hookrightarrow X$, $\ell_1 \not\hookrightarrow Y$, and $L(X, Y^*) = K(X, Y^*)$. Then every limited subset of $X \otimes_\pi Y$ is relatively weakly compact.*

(iii) *Suppose $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon–Nykodim Property. Then every limited subset of $K_{w^*}(X^*, Y)$ is relatively weakly compact.*

(iv) *Suppose $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon–Nykodim Property. Then every limited subset of $X \otimes_\epsilon Y$ is relatively weakly compact.*

Proof (i) Suppose $\ell_1 \not\hookrightarrow C(K)$ and $\ell_1 \not\hookrightarrow X$. Then $\ell_1 \not\hookrightarrow C(K, X)$ [6]. Apply Theorem 1.9.

(ii) Suppose $\ell_1 \not\hookrightarrow X$, $\ell_1 \not\hookrightarrow Y$, and $L(X, Y^*) = K(X, Y^*)$. By [12, Theorem 3], $\ell_1 \not\hookrightarrow X \otimes_\pi Y$. Apply Theorem 1.9.

(iii) If $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon–Nykodim Property, then $\ell_1 \not\hookrightarrow K_{w^*}(X^*, Y)$ ([7, Theorem 1.7]). Apply Theorem 1.9.

(iv) If $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon–Nykodim Property, then $\ell_1 \not\hookrightarrow X \otimes_\epsilon Y$ ([7, Theorem 1.14]). Apply Theorem 1.9. ∎

A bounded subset A of X^* is called an L-subset of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A; *i.e.,*

$$\lim_n \sup\{|x^*(x_n)| : x^* \in A\} = 0.$$

Lemma 1.11 ([14, Lemmas 4.1 and 4.2]) *Suppose A is a bounded subset of X^*. Then A is an L-subset of X^* if and only if $T(A)$ is relatively compact for every $T \in L_{w^*}(X^*, c_0)$.*

Theorem 1.12 *If A is a bounded set that is not an L-subset of X^*, then there exists a sequence (x_n^*) in A and a weakly null basic sequence (x_n) in X so that $x_m^*(x_n) = \delta_{nm}$.*

Proof Suppose A is a bounded set that is not an L-subset of X^*. Then there exists an operator $T \in L_{w^*}(X^*, c_0)$ such that $T(A)$ is not relatively compact, by Lemma 1.11. Since limited subsets of separable spaces are relatively compact [4], $T(A)$ is not a limited subset of c_0. By [21, Lemma 1.3.1], there is a w^*-null sequence (y_n^*) in ℓ_1 and a sequence (x_n^*) in A such that $\langle y_n^*, T(x_m^*)\rangle = \delta_{nm}$. Let $x_n = T^*(y_n^*)$. Then (x_n) is weakly null in X and $x_m^*(x_n) = \delta_{nm}$. Since (x_n) is weakly null and not norm null, by Bessaga and Pelczyinski's selection principle ([2, 9]), we can assume without loss of generality that (x_n) is basic. ∎

Proposition 1.13 *Suppose A is a subset of X^* such that for every $\epsilon > 0$ there exists an L-subset A_ϵ of X^* with $A \subseteq A_\epsilon + \epsilon B_{X^*}$. Then A is an L-set.*

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Proof Suppose A satisfies the hypothesis. Let $\epsilon > 0$ and A_ϵ as in the hypothesis. Suppose $T \in L_{w^*}(X^*, c_0)$, $\|T\| \leq 1$. Then

$$T(A) \subseteq T(A_\epsilon) + \epsilon B_{c_0},$$

and $T(A_\epsilon)$ is relatively compact, by Lemma 1.11. Then $T(A)$ is relatively compact ([9, p. 5]), and thus A is an L-set. ■

The following results are related to the Dunford–Pettis property. We recall that a Banach space X has the Dunford–Pettis property (DPP) if every weakly compact operator T with domain X is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and (x_n^*) in X^* [10]. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. If X is a Grothendieck space with the DPP, then a bounded subset of X is weakly precompact if and only if it is limited [21].

The following theorem was proved in [13].

Theorem 1.14 ([13, Theorem 3.4]) *If $T\colon Y \to X$ is an operator and $LT\colon Y \to c_0$ is compact for all weakly compact operators $L\colon X \to c_0$, then T is weakly precompact.*

Theorem 1.14 implies that every operator $T\colon Y \to X$ with completely continuous adjoint is weakly precompact. Indeed, suppose that $T^*\colon X^* \to Y^*$ is completely continuous. If $L\colon X \to c_0$ is weakly compact, then T^*L^*, and thus LT, is compact. Hence T is weakly precompact by Theorem 1.14.

The following result gives a characterization of Banach spaces with the DPP.

Theorem 1.15 *Let X be a Banach space. The following are equivalent:*

(i) *X has the DPP.*
(ii) *If $T\colon Y \to X$ is an operator, then T is weakly precompact if and only if $T^*\colon X^* \to Y^*$ is completely continuous, for all Banach spaces Y.*
(iii) *Same as* (ii) *with $Y = \ell_1$.*

Proof (i) ⇒ (ii) Suppose X has the DPP. If $T\colon Y \to X$ is a weakly precompact operator, then $T^*\colon X^* \to Y^*$ is completely continuous, by [10, Theorem 1]. The converse follows from Theorem 1.14.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i) Suppose (iii) holds. Let (x_n) be weakly Cauchy in X and (x_n^*) be weakly null in X^*. Define $T\colon \ell_1 \to X$ by $T(b) = \sum b_n x_n$, $b = (b_n) \in \ell_1$. The operator T maps the unit ball of ℓ_1 into the absolutely closed convex hull of $\{x_n : n \in \mathbb{N}\}$, a weakly precompact set ([21, p. 27]). Hence T is weakly precompact. Note that $T^*\colon X^* \to \ell_\infty$, $T^*(x^*) = (x^*(x_i))$, $x^* \in X^*$. Since T^* is completely continuous,

$$|x_n^*(x_n)| \leq \|T^*(x_n^*)\| = \sup_i |x_n^*(x_i)| \to 0,$$

and thus X has the DPP. ■

The following theorem gives a characterization of dual Banach spaces with the DPP and improves [5, Proposition 10].

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Theorem 1.16 *Let X be a Banach space. The following are equivalent:*

(i) *X^* has the DPP.*
(ii) *For all Banach spaces Y, if $S\colon Y \to X^*$ is an operator, then S is weakly precompact if and only if $S^*\colon X^{**} \to Y^*$ is completely continuous.*
(iii) *Same as* (ii) *with $Y = \ell_1$.*
(iv) *For all Banach spaces Y, if $T\colon X \to Y$ is an operator, then $T^*\colon Y^* \to X^*$ is weakly precompact if and only if $T^{**}\colon X^{**} \to Y^{**}$ is completely continuous.*
(v) *Same as* (iv) *with $Y = c_0$.*
(vi) ([5]) *For all Banach spaces Y, if $T\colon X \to Y$ is a weakly compact operator, then $T^{**}\colon X^{**} \to Y$ is completely continuous.*
(vii) *Same as* (vi) *with $Y = c_0$.*

Proof (i), (ii), and (iii) are equivalent by Theorem 1.15.

(ii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (vii) are clear.

(iv) $\Rightarrow$ (vi) is clear. We note that if T is weakly compact, then $T^{**}(X^{**}) \subseteq Y$.

(vi) $\Rightarrow$ (vii) is clear.

(vii) $\Rightarrow$ (i) Suppose (x_n^*) is weakly null in X^* and (x_n^{**}) is weakly null in X^{**}. Define $T\colon X \to c_0$ by $T(x) = (x_n^*(x))$. Then $T^*\colon \ell_1 \to X^*$, $T^*(e_n^*) = x_n^*$, and $T^*(b) = \sum b_i x_i^*$, for $b = (b_i) \in \ell_1$. Further, T^* maps the unit ball of ℓ_1 into the absolutely closed convex hull of $\{x_i^* : n \in \mathbb{N}\}$, a relatively weakly compact set ([11, p. 51]). Then T^*, and thus T, is weakly compact. If $x^{**} \in X^{**}$, then $T^{**}(x^{**}) = (x^{**}(x_i^*))$. Since T^{**} is completely continuous,

$$|x_n^{**}(x_n^*)| \le \|T^{**}(x_n^{**})\| = \sup_i |x_n^{**}(x_i^*)| \to 0,$$

and thus X^* has the DPP. ■

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