THE ASYMPTOTIC BEHAVIOUR OF THE LAURENT COEFFICIENTS

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Introduction. Let $G(z)$ be a function of a complex variable, regular in the annulus $0 \leqslant a \leqslant |z| < b \leqslant \infty$. We shall assume there exists a curve within the annulus for which

$$\lim_{|z| \to b} G(z) = \infty,$$

provided z is restricted to be a point of this curve. Under these restrictions $G(z)$ has a Laurent expansion of the form

$$G(z) = \sum_{-\infty}^{\infty} a_n z^n, \tag{1.1}$$

where the Laurent coefficients a_n have the integral representation

$$a_n = (2\pi i)^{-1} \int_C G(z)\, z^{-(n+1)}\, dz, \tag{1.2}$$

and C can be any contour, within the domain of regularity, that encloses $z = 0$. We shall also assume that the a_n are all real numbers. Using the usual complex conjugate notation, we can, therefore, write

$$\overline{G(z)} = G(\bar{z}). \tag{1.3}$$

The problem of determining the asymptotic behaviour of a_n as $n \to \infty$ is very old in mathematical literature and appears in many forms and disguises. It has been solved for specific classes of functions by many people using a multitude of methods. For certain lacunary type series a_n can have an almost chaotic behaviour for large n. It is, therefore, too much to expect that there exists a single method that will give the asymptotic behaviour of a_n for the class of all functions that possess Laurent expansions. We shall, therefore, in this paper, consider only a single method that yields the asymptotic behaviour of a_n for a particular class of generating functions $G(z)$ which will be called admissible. The major goal is to make the class of admissible functions as large as possible.

As far as we are aware the first attack on problems of this type, that could claim any degree of generality, is due to Darboux **(1)**. His class of admissible functions possessed Maclaurin expansions with a finite radius of convergence. On the circle of convergence $G(z)$ was allowed to have only a finite number of singularities of a particular type. The success of Darboux's method is almost

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entirely due to the fact that his assumptions allow us to deform the contour C beyond the circle of convergence and ultimately to prove that only small contours surrounding the singularities of $G(z)$ contribute to the asymptotic behaviour of a_n. This procedure is almost characteristic of all methods that depend for their success on the classification of the number and types of singularities the generating function $G(z)$ is allowed to have on the circle of convergence.

Such deformations are not available if the circle of convergence is a natural boundary of $G(z)$. In particular the partition function

$$G(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-1} = \sum_{n=0}^{\infty} p(n)\, z^n \tag{1.4}$$

falls into this classification. There exists, in the literature, several derivations of the behaviour of $p(n)$ as $n \to \infty$.

The success of known methods of studying the asymptotic behaviour of sequences, defined by integrals, lies almost entirely in the fact that the behaviour is determined by a knowledge of $G(z)$ in the neighbourhoods of certain points called critical points. Erdélyi **(2)** has pointed out that there does not exist any general theory of critical points and that only a few types of integrals have been studied. We shall provide such a theory for integrals of the type (1.2) and for which $G(z)$ belongs to our class of admissible functions.

Among recent papers dealing with problems of this type we note those of Szekeres **(8)**, Hayman **(4)**, and Moser and Wyman **(5; 6)**. Szekeres finds a complete asymptotic expansion for a specific example. Hayman introduced the idea of a class of admissible functions and found the first term of an asymptotic expansion for the Maclaurin coefficient of all members of his class of functions. Wyman and Moser find complete asymptotic expansions for the Maclaurin coefficients of a class of integral functions. We note these papers in particular because in each case there is a more or less common pattern of attack. In our present paper we shall attempt to generalize the general pattern of attack of the papers mentioned above.

2. Asymptotic expansions. In applying a method of critical points to integrals of the form (1.2) we place suitable restrictions on $G(z)$ that enable us to determine the asymptotic behaviour of a_n by considering only restricted portions of the contour C. It is thus clear that such a method implies a criterion by means of which we may recognize the portions of C that are to be retained and to recognize also the portions of C that may be discarded. This criterion is found in the choice of a suitable definition of an asymptotic expansion.

Since Poincaré's introduction of a definition of an asymptotic expansion, the concept has been generalized in many ways. Erdélyi **(2)** has given an elegant treatment of the concept which can be used to develop a general theory of asymptotic expansions. However, his treatment imposes certain restrictions that are neither necessary nor desirable for the type of problem we shall consider in this paper. In a private conversation Professor Erdélyi has given me a more general definition that completely fills our present needs. We shall, however, use only a very specialized version of Erdélyi's later definition.

Throughout the paper r shall denote a real variable lying in the fixed interval I given by $0 \leqslant a \leqslant r < b \leqslant \infty$. We shall be interested in the behaviour of certain real or complex valued functions $F(r)$ as $r \to b$. The end-point a plays no essential role in our discussion except in so far as all statements are required to have meaning only if r is restricted to be a value in I.

Definition 2.1. A real valued positive function $V(r)$ is called a *comparison function* if

$$\lim_{r \to b} V(r) = \infty.$$

Definition 2.2. Two real or complex valued functions $f(r)$ and $g(r)$ are said to be *asymptotically equal* at $r = b$, with respect to the comparison function $V(r)$, if

$$\lim_{r \to b} V^n (f(r) - g(r)) = 0 \tag{2.1}$$

for every non-negative integer n.

We shall write this relationship as

$$f(r) \approx g(r) \tag{2.2}$$

or in terms of the order relation o as

$$f(r) - g(r) = o(V^{-n}), \tag{2.3}$$

for every non-negative integer n.

Since we shall usually require b and $V(r)$ to remain fixed in our discussion we shall often delete the qualifying phrases in the definition of asymptotic equality.

If there exists a value r_0 in I and positive real numbers α, β such that

$$|f(r) - g(r)| = O(\exp(-\beta V^\alpha)) \tag{2.4}$$

for all $r \geqslant r_0$, then $f(r) \approx g(r)$.

Asymptotic equality is an equivalence relation and divides the class of complex functions defined on I into equivalence classes. In our development we shall usually replace a function $f(r) \approx 0$ by zero. If $b = \infty$, $V(r) = \log r$ then r, $(r^2 + 1)/r$ are examples of a pair of asymptotically equal functions.

Definition 2.3. Let $A_k(r)$, $k = 0, 1, 2, \ldots$ be an infinite sequence of complex valued functions. The formal series

$$\sum_{k=0}^{\infty} A_k V^{-k}$$

is called an *asymptotic expansion* of a function $F(r)$ if, for every non-negative integer m,

$$\lim_{r\to b} V^m\left(F - \sum_{k=0}^{m} A_k V^{-k}\right) = 0. \tag{2.5}$$

We write

$$F \sim \sum_{k=0}^{\infty} A_k V^{-k} \tag{2.6}$$

or

$$F - \sum_{k=0}^{m} A_k V^{-k} = o(V^{-m}). \tag{2.7}$$

The asymptotic equality of two functions is a special case of an asymptotic expansion with $A_k = 0$, $k \geqslant 1$. It is also easy to show that the definition of an asymptotic expansion implies $A_k = o(V)$, $k \geqslant 1$ and that the asymptotic expansion of a function is not unique. Any of the $A_k(r)$ may be replaced by an asymptotically equal function.

Definition 2.4. If $F(r)$ and $H(r)$ are two complex valued functions such that

$$F/H \sim \sum_{k=0}^{\infty} A_k V^{-k} \tag{2.8}$$

then we write

$$F \sim H \sum_{k=0}^{\infty} A_k V^{-k}. \tag{2.9}$$

At first glance it might be thought that Definition (2.4) is redundant in the light of the fact that the $A_k(r)$ of Definition (2.3) are allowed to be functions of r. This, however, is not so because (2.9) implies only that $A_k = o(V)$ while

$$F \sim \sum_{k=0}^{\infty} (HA_k) V^{-k} \tag{2.10}$$

implies $HA_k = o(V)$. Hence (2.10) may or may not place a restriction on H that is not implied by (2.9).

The major distinction between our use at present of the meaning of an asymptotic expansion and Erdélyi's published version is the fact that we do not require $A_k V^{-k}$ to be an asymptotic sequence, even though V^{-k} is such a sequence. This situation could exist, for example, if some of the $A_k = 0$.

3. Relevant paths. Returning to (1.1), we may, without any loss of generality, assume that we are interested in the asymptotic behaviour of a_n for large positive n. The behaviour of a_n, for large negative n, could then be obtained from the generating function $G(1/z)$. If $G(z)$ has the form

$$G(z) = P_m(z) + Q(z^{-1}) \tag{3.1}$$

where $P_m(z)$ is a polynomial of degree m and Q is a regular function of its argument, then our problem is trivial. For such a function $a_n = 0, n \geqslant m + 1$. Hence we shall exclude all functions of the form (3.1) from the remainder of our discussion.

Using the polar form $z = r \exp(i\theta)$ we define $M(r, \theta)$ by

$$M(r, \theta) = |G(r \exp(i\,\theta))|. \tag{3.2}$$

Since we shall always operate within the domain of regularity of $G(z)$ for which (1.1) is true, r shall always lie in the interval I of the previous section. The function $M(r, \theta)$ is a periodic function of θ with a period of 2π and has continuous partial derivatives of all orders except possibly at the zeros of $G(z)$. The function $M(r)$ defined by

$$M(r) = \sup_{|z|=r} M(r, \theta) \tag{3.3}$$

is the so-called maximum modulus function and it is known that there always exists a point $z = z(r)$, on $|z| = r$, such that $|G(z(r))| = M(r)$. By varying r, $z = z(r)$ can be considered as the parametric representation of a curve in the complex z plane which is called a path of maximum modulus. The path of maximum modulus need not be unique. From the assumptions made, in the Introduction, on $G(z)$ it is clear that

$$\lim_{r \to b} M(r) = \infty$$

and hence $\log M(r)$ is a comparison function. The choice of $\log M(r)$ as a comparison function may, however, introduce a complexity that is neither necessary nor desirable. Hardy **(3)** has shown that $G(z) = \exp(\sin z)\exp(\exp z^2)$ has $M(r) = \exp|\sin r|\exp(\exp r^2)$. The corresponding path of maximum modulus is $\theta = 0, 2k\pi \leqslant r \leqslant (2k + 1)\pi$ and $\theta = \pi, (2k + 1)\pi \leqslant r \leqslant (2k + 2)\pi$. Hence the path of maximum modulus has an infinite number of discontinuities. For this reason, we shall see, the *a priori* choice of $\log M(r)$ as a comparison function often introduces a discontinuous picture of our problem that can easily be avoided. Previous authors have featured $\log M(r)$ in their discussions and have required $\theta = 0$ to be the continuous path of maximum modulus. Such a starting point greatly reduces, for no valid reason, the extent of our class of admissible generating functions $G(z)$.

Let us return to (3.2) and consider the stationary values of $M(r, \theta)$ which are given as solutions of the equation

$$\frac{\partial M}{\partial \theta} = 0. \tag{3.4}$$

There always exists at least one real solution of (3.4) $\theta = \theta(r)$. In fact our assumption that all a_n are real implies that $\theta = 0$ or $\theta = \pi$ are always solutions of (3.4) except possibly for the case when $M(r, 0) = 0$ or $M(r, \pi) = 0$. Implicit function theory tells us that if (r_0, θ_0) is a solution of (3.4), for which

$$\frac{\partial^2 M}{\partial \theta^2} \neq 0,$$

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then there always exist neighbourhoods $|r - r_0| < \delta$, $|\theta - \theta_0| < \delta$ for which the solution $\theta = \theta(r)$ is unique and has a continuous derivative. We may, therefore, consider such a solution $\theta = \theta(r)$ as the polar equation of a path, in the complex z plane, which lies within the domain of regularity of $G(z)$. If we traverse such a path, with increasing r, the path remains unique until we strike a point at which

$$\frac{\partial^2 M}{\partial \theta^2} = 0.$$

At such a point the path may cease to be unique and several branches may appear. However if r is sufficiently close to b, then (**9**, pp. 22-7), the number of such branch paths remains finite. Although $\theta(r)$ need not be differentiable at a branch point, the continuity of $\theta(r)$ is preserved. All paths determined as solutions of (3.4) shall be called the stationary paths of $G(z)$.

We interrupt our general development to consider the specific function $\exp(z/1 + z^2))$ which gives considerable insight into the general situation. The solutions of (3.4) are

(3.5) $\theta = 0,\ \theta = \pi,\ 0 \leqslant r \leqslant \sqrt{2} - 1$

(3.6) $\theta = 0,\ \theta = \pi,\ \theta = \pm \arccos((1 - r^2)/2r),\ \theta = \pi \pm \arccos((1 - r^2)/2r),\ \sqrt{2} - 1 \leqslant r \leqslant 1.$

The points, $(0, \sqrt{2} - 1)$, $(\pi, \sqrt{2} - 1)$, are branch points and there exist six stationary paths by means of which we may leave $z = 0$ and arrive at the boundary $|z| = 1$.

We note that the number of intersections of the stationary paths with the circle $|z| = r$ is a function of r and indicates the possibility that, for other generating functions $G(z)$, that the number of such intersections may tend to ∞ as $r \to b$. Let us now consider the stationary path $\theta = 0$, $0 \leqslant r \leqslant 1$. For a fixed value of r and variable θ, $M(r, 0)$ is a maximum of $M(r, \theta)$, for $0 \leqslant r < \sqrt{1} - 2$, and then $M(r, 0)$ changes its character to become a minimum of $M(r, \theta)$ for $\sqrt{2} - 1 < r < 1$. Again the example indicates the possibility that there may exist stationary paths $\theta = \theta(r)$ such that, along such a path, $M(r, \theta(r))$ may change its character from being a maximum of $M(r, \theta)$ to being a minimum of $M(r, \theta)$ an infinite number of times.

Returning to the general case we will denote by L_k, $(k = 1, 2, \ldots, T)$, the stationary paths contained in the annulus $r_0 \leqslant |z| \leqslant r < b$. We note that the integer T may be a function of r. The corresponding polar equations are written as $\theta = \theta_k(r)$ and $M_k(r)$ is defined by

(3.7) $$M_k(r) = M(r, \theta_k(r)).$$

Assumption (1). For r sufficiently close to b there exists a continuous stationary path, with polar equation $\theta = \theta_1(r)$, by means of which we can reach the boundary $r = b$. Further $M(r, \theta)/M_1$ is bounded uniformly in θ.

Since the maximum modulus $M(r)$ is always attained for every value of r

at some value of θ assumption (1) implies that $M(r)/M_1$ is bounded as $r \to b$. However we have previously assumed that

$$\lim_{r \to b} M(r) = \infty.$$

This means that

$$\lim_{r \to b} M_1(r) = \infty.$$

Thus $\log M_1(r)$ is a comparison function by means of which we may give meaning to the concepts of asymptotic equality and asymptotic expansions. Throughout the remainder of the paper $\log M_1(r)$ will be our comparison function and $(\log M_1(r))^{-m}$ will be our asymptotic sequence.

Definition. A stationary path for which $M_k/M_1 \approx 0$ will be called an *irrelevant path.* All other stationary paths will be called *relevant paths.*

Assumption (2). We shall assume that the relevant paths can be identified in such a way that the following properties are true for r sufficiently close to b.

(a) The relevant paths are all continuous curves by means of which we may reach the boundary $r = b$.

(b) If $\theta = \theta_k(r)$ is a relevant path then a constant $K_k > 0$ and a non-negative integer m exists such that $M_k/M_1 \geqslant K_k (\log M_1)^{-m}$.

(c) The number N of relevant paths is fixed and independent of r.

(d) For every relevant path $\theta = \theta_k(r)$ the

$$\lim_{r \to b} \theta_k(r)$$

exists. Further if $\theta = \theta_k(r)$ and $\theta = \theta_s(r)$ are distinct relevant paths then

$$\lim_{r \to b} \theta_k(r) \neq \lim_{r \to b} \theta_s(r).$$

(e) Along every relevant path the function $M(r, \theta)$ has the property

$$\frac{\partial^2 M}{\partial \theta^2} < 0.$$

An effect of assumption (2) is that for r close to b the relevant paths can have no point of intersection. Such a point would be a branch point at which

$$\frac{\partial^2 M}{\partial \theta^2} = 0.$$

This would of course contradict the assumption. The assumption also guarantees that there can be no point of intersection of relevant paths even at $r = b$. We may also conclude that the number of intersections of the relevant paths with the circle $|z| = r < b$ is equal to the fixed number N. Hence without any loss of generality we may assume that the relevant paths are L_k, $(k = 1, 2, \ldots, N)$ and that they have been numbered in a counter-clockwise direction beginning

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with the featured path L_1. Along every relevant path the value $M_k(r)$ is, for fixed r and variable θ, a maximum value of $M(r, \theta)$ as $r \to b$. Finally, we also note that the functions $\theta_k(r)$ corresponding to relevant paths all have continuous first derivatives.

Since (any constant)$/M_1(r) \approx 0$ it is clear that $M_k(r)$, $(k = 1, 2, \ldots, N)$ must all be unbounded functions of r as $r \to b$. Since $M(r)/M_1(r)$ and $M_1(r)/M(r)$ are both bounded functions we must have

(3.8) $$\lim_{r \to b} \log M(r)/\log M_1(r) = 1.$$

This result can be extended to

(3.9) $$\lim_{r \to b} (\log M_k(r)/\log M_1(r)) = 1, \qquad k = 1, 2, \ldots, N.$$

It is not difficult to show that the function $\exp(\sin z)\exp(\exp z^2)$ obeys assumptions (1) and (2).

4. Functional behaviour along relevant paths. In the present section we shall investigate the functional behaviour of $\log G(z)$ and its derivatives along relevant paths. Since it turns out to be simpler to study the derivatives with respect to $\log z$ we introduce the operator H by

(4.1) $$H = z\frac{d}{dz}$$

and attach the usual meaning to the iterated operator H^m. Further, the symbol $H^m F(w)$ shall always mean $H^m F(z)]_{z=w}$. In this notation (3.4) becomes

(4.2) $$\text{Im } (H \log G(z)) = 0$$

and hence the quantity $H \log G(z)$ is always real along every relevant path L_k.

Let us denote the points of intersection of L_k with $|z| = r < b$ by

(4.3) $$z = z_k(r), \quad |z_k| = r, \quad k = 1, 2, \ldots, N.$$

For variable r, (4.3) provides a parametric representation of the relevant paths. Further $z_k(r)$ has a continuous derivative with respect to r as $r \to b$, since $\theta_k(r)$ has this property. Since $|G(z_k)| = M_k(r)$ and $M_k(r)$ is a maximum of $M(r, \theta)$ we must have $G(z_k) \neq 0$. Thus all of the functions $H^m \log G(z_k(r))$ have a continuous derivative with respect to r for r sufficiently close to b. The condition

$$\frac{\partial^2 M}{\partial \theta^2} < 0$$

can be translated to read

(4.4) $$Rl\ (H^2 \log G(z)) > 0,$$

along every relevant path.

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LEMMA 4.1. *The function $H \log G(z)$ is increasing in r and has no upper bound along any relevant path.*

Proof. Let z_0 be any fixed point along a relevant path and suppose z is a variable point along the same path. Along such a path $H \log G(z)$ has been shown to be real. If there exists a constant K such that $H \log G(z) < K$ then the identity

$$\log G(z) = \log G(z_9) + \int_{z_0}^{z} H \log G(z)\, d(\log z), \tag{4.5}$$

where the path of integration is the relevant path, implies

$$|G(z)| \leqslant |G(z_0)|\ (r/|z_0|)^K. \tag{4.6}$$

However, since our path of integration is a relevant path L_k we must have $|G(z)| = M_k(r)$. If b is finite, (4.6) implies that $M_k(r)$ is a bounded function of r as $r \to b$. This contradicts our assumption that L_k is a relevant path. If b is infinite the proof is somewhat more difficult but follows directly from known complex function theory as long as we remember that functions of the form (3.1) are excluded.

Along any relevant path we know that $H \log G(z)$ has a continuous first derivative with respect to r, if r is close enough to b. An easy computation gives

$$r \frac{d}{dr} H \log G(z) = Rl\ H^2 \log G(z) \left(1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right). \tag{4.7}$$

From (4.4) we have $d/dr\, H \log G(z) > 0$ along a relevant path and hence $H \log G(z)$ must be an increasing function of r along every such path. Since $H \log G(z)$ is an increasing continuous function of r, which does not have an upper bound, along every relevant path we must have

$$\lim_{r \to b} H \log G(z) = +\infty. \tag{4.8}$$

Thus far in our paper we have been discussing the functional behaviour of $G(z)$ and have not related this behaviour to the Laurent coefficients a_n. In order to make a start on obtaining this relationship we shall for the moment allow n to be a large positive real number. Ultimately n will assume its integer meaning in a_n. The results contained in Lemma 4.1 imply that, along any relevant path L_k, the equation

$$H \log G(z) = n \tag{4.9}$$

always has a unique solution $z = Z_k(n)$, for n sufficiently large. Considering n as a variable, the equations

$$z = Z_k(n) \tag{4.10}$$

provide a second parametric representation of L_k. Since it becomes convenient to use both of the parametric representations (4.3) and (4.10), simul-

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taneously in some discussions, we introduce the following convention to avoid confusion.

Convention. If r is taken to be an independent variable then n shall mean the function $n = n(r)$ defined by

$$n(r) = H \log G(z_1(r)). \tag{4.11}$$

Thus $z_1(r) = Z_1(n)$ and we may consider both sets of points $z_k(r)$, $Z_k(n)$ as completely determined.

If n is taken to be an independent variable then r is taken to mean the function $r(n)$ defined by

$$r(n) = Z_1(n), \tag{4.12}$$

and again $z_1(r) = Z_1(n)$. As before $z_k(r)$, $Z_k(n)$ are completely determined by this convention. This notation implies

$$\lim_{r\to b} n(r) = \infty, \quad \lim_{n\to\infty} r(n) = b. \tag{4.13}$$

Assumption (3). We shall assume that, for r sufficiently close to b each of the following are true.

(a) There exist positive constants p, P_1, P_2, such that

$$P_1 (\log M_1) \leqslant n(r) \leqslant P_2 ((\log M_1)^{1+p}) \tag{4.14}$$

(b) $Z_k(n)/z_k(r) = 1 + 0(1/n)$.

(c) In the complex w plane there exists a fixed neighbourhood $|w| \leqslant h$ for which the functions $\log G(Z_k(n) \exp(\log M_1 w/n))$ are all regular functions of w. Further,

$$\lim_{r\to b} [\log G(Z_k(n) \exp(\log M_1(r)\, w/n)) - \log G(Z_k(n))]/\log M_1$$

exists, uniformly in w, for all w within and on the boundary of $|w| \leqslant h$. We shall denote this limit by $g_k(w)$ and shall assume $Rlg_k''(0) \neq 0$.

Definition 4.1. Any generating function having a Laurent expansion of the type (1.1) and for which assumptions (1), (2), and (3) are true will be called an admissible function.

The major result of this paper will be that the asymptotic behaviour of the Laurent coefficients of every admissible function can be determined. At this stage the conditions contained in assumption (3) must seem somewhat like pulling rabbits out of a hat. However these conditions do arise quite naturally in the method that we shall use to prove our central theorem.

Part (a) of assumption (3) makes all members of the original class of functions considered by Darboux inadmissible. As such our central theorem will be complimentary to the Darboux result. The second part of the assumption turns out to be sufficient to control the behaviour of the factor $z^{-(n+1)}$ in the integrand of (1.2). I suspect that this part of assumption (3) may be completely unnecessary and that every admissible function may have this property. The third part of the assumption involves functions that are reminiscent of the indicator functions of integral function theory. In some cases the indicator function can be derived from $g_k(w)$. Obviously the third part of this assumption places some restrictions on the locations of the zeros of $G(z)$.

To be a member of Hayman's class of admissible functions a generating function $G(z)$ has to have a unique path of maximum modulus that is $\theta = 0$. Certainly assumption (1) is considerably weaker than this condition. To this extent our class of admissible functions is considerably more extensive than is Hayman's. It is not, however, possible to say that our class of such functions completely contains Hayman's. Our assumptions have been designed to measure the contribution of any portion of the contour C, of (1.2), to any term in a complete asymptotic expansion. Hayman's assumptions were designed to measure the contributions of C to the first term of such an expansion. If we were to restrict ourselves to Hayman's objective we could, along the lines of the present paper, greatly extend Hayman's class of admissible functions.

When the outer boundary of convergence is finite our assumptions say, in essence, that $G(z)$ may have an infinite number of singularities on $r = b$. However, there can be only a finite number that dominate all the rest. The meaning of dominate is of course derived from our definition of asymptotic equality.

The two specific functions used for illustration in this paper are both admissible functions in our sense and neither is admissible in the Hayman sense. To illustrate another point we shall prove that

$$G(z) = [\exp(1/2(1 - z^2))]/(1 - z) \tag{4.15}$$

is an admissible function. For (4.15) it is not too difficult to prove that

$$z = r \quad \text{and} \quad z = -r \tag{4.16}$$

are the only two possible relevant paths. Since $z = r$ is the path of maximum modulus and since this path is continuous, for this example we choose

$$z_1(r) = r, \qquad z_2(r) = -r. \tag{4.17}$$

Hence

$$M_1(r) = \frac{\exp(\frac{1}{2}(1 - r^2))}{1 - r}, \; M_2(r) = \frac{\exp(\frac{1}{2}(1 - r^2))}{1 + r}. \tag{4.18}$$

Since

$$\lim_{r \to 1} \frac{(\log M_1(r))M_2(r)}{M_1(r)} = \frac{1}{8}, \tag{4.19}$$

we see that both paths are relevant. The fact that

$$\frac{\partial^2 M}{\partial \theta^2} < 0$$

on a relevant path is easily verified. This means assumption (1) is satisfied.

Turning to equation (4.9) the solutions are

$$Z_1(n) = 1 - \frac{1}{2(n)^{1/2}} - \frac{3}{8n} + \frac{1}{8(n)^{3/2}} + \ldots, \tag{4.20}$$

$$Z_2(n) = -1 + \frac{1}{2(n)^{1/2}} - \frac{1}{8n} - \frac{1}{8(n)^{3/2}} + \ldots. \tag{4.21}$$

Since $Z_1(n) = z_1(r) = r$ we find

$$z_2(r) = -1 + \frac{1}{2(n)^{1/2}} + \frac{3}{8n} - \frac{1}{8(n)^{3/2}} + \ldots \tag{4.22}$$

and

$$Z_2(n)/z_2(r) = 1 + 0(1/n). \tag{4.23}$$

Since $\log M_1(r) = \frac{1}{2}(n)^{\frac{1}{2}} + \ldots$, the first and second parts of assumption (3) are true. One can readily verify that $g_1(w) = g_2(w) = w/1 - w$ and that the remaining conditions of assumption (3) are true. Thus (4.15) gives a third example of an admissible function. It is interesting to note that (4.15) is not a member of Hayman's class of admissible functions. Actually (4.15) violates one of Hayman's assumptions that is not vital to the success of the method. This example provides an illustration of how Hayman's class of admissible functions could be extended along the lines of our present paper. The second relevant path contributes only to the second term of the asymptotic expansion.

5. Further preliminary results. From assumption 3(c) it is easily seen that $\log G(Z_k \exp u)$ is a regular function of the complex variable u providing $|u| \leqslant h\ (\log M_1)/n$. In this neighbourhood we may expand $\log G$ $(Z_k \exp u)$ into a Maclaurin expansion and obtain

$$\log G(Z_k \exp u) = \log G(Z_k) + H \log G(Z_k)u + H^2 \log G(Z_k)\frac{u^2}{2!} + \sum_{m=3}^{\infty} H^m \log G(Z_k)\frac{u^m}{m!}. \tag{5.1}$$

Since $H \log G(Z_k) = n$ we may, by choosing $u = \log(z/Z_k)$, say

$$\log G(z) = \log G(Z_k) + n \log(z/Z_k) + H^2 \log G(Z_k)\frac{(\log z/Z_k)^2}{2!} + \sum_{m=3}^{\infty} H^m \log G(Z_k)(\log(z/Z_k))^m/m!, \tag{5.2}$$

providing $|\log(z/Z_k)| \leqslant h\ (\log M_1)/n$. If we denote the set of points satisfying $|\log(z/Z_k)| \leqslant h \log(M_1)/n$ by D_k, then the point $z = z_k(r)$ lies in D_k.

Assumption (3) tells us that

$$\lim_{r\to b} (\log G(Z_k \exp(\log M_1(r)\, w/n)) - \log G(Z_k))/\log M_1(r) = g_k(w), \tag{5.3}$$

and that the limit exists uniformly in w, for $|w| \leqslant h$. We may, therefore,

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differentiate (5.3) m times with respect to w and then place $w = 0$. This gives

$$\lim_{r\to b} ((\log M_1(r))^{m-1} H^m \log G(Z_k)/n^m) = g_k^{(m)}(0). \tag{5.4}$$

From (5.4) the following results can easily be obtained.

$$g_k(0) = 0, \qquad g_k'(0) = 1. \tag{5.5}$$

$$Rl\ g_k''(0) > 0. \tag{5.6}$$

There exist constants P_3 and P_4 such that

$$P_3 (\log M_1(r)) \leqslant |H^2 \log G(Z_k)| \leqslant P_4 ((\log M_1(r))^{1+2p}). \tag{5.7}$$

$$\lim_{r\to b} |H^2 \log G(Z_k)| = \infty. \tag{5.8}$$

$$\lim_{r\to b} \arg(H^2 \log G(Z_k)) = \arg g_k''(0). \tag{5.9}$$

$$H^2 \log G(Z_k) = 0(H^2 \log G(Z_1)). \tag{5.10}$$

$$|H^m \log G(Z_k)/m!| = 0(n^m/h^m (\log M_1)^{m-1}). \tag{5.11}$$

If we define ψ_k by

$$\psi_k = \lim_{r\to b} \arg(H^2 \log G(Z_k)) \tag{5.12}$$

then (5.6) and (5.9) tell us that

$$|\psi_k| < \tfrac{1}{2}\pi. \tag{5.13}$$

We shall define an infinitesimal ϵ by

$$\epsilon = |H^2 \log G(Z_1)|^{-\alpha} \tag{5.14}$$

where, for reasons that shall appear later, we restrict α to satisfy

$$(6p + 2)/6(2p + 1) < \alpha < \tfrac{1}{2}. \tag{5.15}$$

The constant p is taken from assumption (3). Since the $Rl\ (H^2 \log G(Z_k)) > 0$ we may introduce $(H^2 \log G(Z_k))^{\frac{1}{2}}$ in an unambiguous manner by taking the branch with a positive real part. The quantity $\phi_{0,k}(r)$ defined by

$$\phi_{0,k}(r) = |H^2 \log G(Z_1)|^{-\alpha} (H^2 \log G(Z_k))^{\frac{1}{2}} \tag{5.16}$$

has the properties that

$$|\phi_{0,k}(r)| \sim \frac{|g_k''(0)|^{\frac{1}{2}}}{|g_1''(0)|^{\alpha}} (n^2/\log M_1)^{\frac{1}{2}-\alpha} = 0[(\log M_1)^{(1+2p)(\frac{1}{2}-\alpha)}] \tag{5.17}$$

and

$$\lim_{r\to b} |\phi_{0,k}(r)| = \infty, \tag{5.18}$$

and

$$\lim_{r\to b} |\arg \phi_{0,k}(r)| = \tfrac{1}{2}|\psi_k| < \tfrac{1}{4}\pi. \tag{5.19}$$

Since

$$\lim_{r\to b} n\epsilon = \infty,$$

we see that $1/n$ is a higher order infinitesimal than ϵ. Further, the fact that

$$\lim_{r\to b} n\epsilon/\log M_1(r) = 0$$

implies that any point z lying on the line joining $z_k \exp i\epsilon$ to $Z_k \exp i\epsilon$ is a member of the domain D_k. For such points z we may prove

(5.20) $$\log (z/Z_k) = i\epsilon + 0(1/n).$$

Further, since $1/n$ is a higher order infinitesimal than ϵ we may, to this order of approximation, write

(5.21) $$\log (z/Z_k) = i\epsilon.$$

This fact plus (5.2) enables us to prove that

(5.22) $$\log(G(z)/z^n) = \log(G(Z_k)/Z_k^n) - \frac{\epsilon^2}{2} H^2 \log G(Z_k) + 0(n^3\epsilon^3/(\log M_1)^2).$$

From (5.4)

(5.23) $$\lim_{r\to b} [(\log M_1)\, H^2 \log G(Z_k)/n^2] = g_k''(0)$$

and from (5.6) $g_k''(0) \neq 0$. Hence

(5.24) $$H^2 \log G(Z_k) = n^2(g_k''(0) + o(1))/\log M_1.$$

Thus (5.22) may be written

(5.25) $$\log(G(z)\, Z_k^n/z^n\, G(Z_k)) = \frac{\epsilon^2}{2} H^2 \log G(Z_k)(1 + O(n\epsilon/\log M_1)).$$

We have already seen that

$$\lim_{r\to b} (n\epsilon/\log M_1) = 0.$$

Hence (5.25) can be shown to imply that a constant $P_5 > 0$ exists such that

(5.26) $$|G(z)\, Z_k^n/G(Z_k)\, z^n| = O(\exp(-P_5(\log M_1)^{1-2\alpha})).$$

Hence $|G(z)\, Z_k^n/G(Z_k) z^n| \approx 0$ with respect to $\log M_1(r)$. Since $G(Z_k)/M_1(r)$ and $(r/Z_k)^n$ are bounded functions of r this means

(5.27) $|G(z)\, r^n/M_1(r) z^n| \approx 0$, with respect to $\log M_1(r)$.

In particular (5.27) is true for the points $z_k \exp (i\epsilon)$ and $Z_k \exp (i\epsilon)$. Obviously the same result may be obtained by replacing ϵ by $-\epsilon$. We use (5.27) to prove the following lemma.

LEMMA 5.1. *The quantity $I(r)$ defined by*

(5.28) $$I(r) = (r^n/M_1(r)) \int_C G(z)\, z^{-(n+1)}\, dz$$

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is ≈ 0, *with respect to log* $M_1(r)$, *providing* L *is taken to be one of the following paths of integration.*

(a) The arc of the circle $|z| = r$ joining the points $z_k \exp(i\epsilon)$ to $z_{k+1} \exp(-i\epsilon)$
(b) The line joining $z_k \exp(i\epsilon)$ to $Z_k \exp(i\epsilon)$.
(c) The line joining $z_k \exp(-i\epsilon)$ to $Z_k \exp(-i\epsilon)$.

Proof. Our notation implies that $z_k(r)$, $z_{k+1}(r)$ are two points of intersection of two consecutive relevant paths with $z = r$. By assumption (2)

$$\lim_{r\to b} \arg z_k(r)$$

exist for $k = 1, 2, \ldots, N$ and, further, the values of these limits are distinct. Hence our numbering system can be assumed, without any loss of generality, to imply that

$$\lim_{r\to b} \arg z_k(r) < \lim_{r\to b} \arg z_{k+1}.$$

Further, since ϵ was defined by (5.14) we have

$$\lim_{r\to b} \arg(z_k(r) \exp(i\epsilon)) = \lim_{r\to b} \arg z_k(r).$$

This of course means that for r close to b

$$\arg z_k(r) < \arg (z_k(r) \exp(i\epsilon)) < \arg (z_{k+1}(r) \exp(-i\epsilon)) < \arg z_{k+1}(r).$$

This means that there can be no points of intersection of a relevant path and $|z| = r$ contained in $\arg (z_k(r) \exp(i\epsilon) \leqslant \theta \leqslant \arg (z_{k+1}(r) \exp(-i\epsilon)$. Thus on the path of integration given by part (a) we must have $|G(z)|$ less than or equal to the value of $|G(z)|$ at a maximum which is on an irrelevant path, or $|G(z)|$ is less than or equal to the value of $|G(z)|$ at one of the end-points. In either case $I(r) \approx 0$. To make our proof consistent even if the points z_N and z_1 are involved we adopt the convention that $|z_{N+1}| = |z_1|$ but $\arg z_{N+1} = \arg z_1 + 2\pi$.

For the straight line paths of integration (5.27) is sufficient to establish the result. As one would suspect, Lemma 5.1 enables us to recognize the portions of the contour C of (1.2) that can be discarded.

To complete our preliminary results we now proceed to study the behaviour of certain functions which enter into the derivation of the central theorem of the next section of our paper. Let ϕ be a complex variable and suppose we restrict ϕ to be on the line, in the complex ϕ plane, joining the points $\phi_{0,k}(r)$ to $-\phi_{0,k}(r)$ where $\phi_{0,k}$ is given by (5.16). We define $a_{m,k}$, ($m = 3, 4, \ldots$, $k = 1, 2, \ldots, N$), by

$$(5.29) \qquad a_{m,k} = (\log M_1(r))^{\frac{1}{2}(m-2)} \mathrm{H}^m \log G(Z_k)\ (i\,\phi)^m / (m!(\tfrac{1}{2}\mathrm{H}^2 \log G(Z_k))^{m/2})$$

then (5.11) and (5.24) combined with (5.29) give

$$(5.30) \qquad |a_{m,k}| = 0((Q|\phi|)^m),$$

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where Q is a positive constant. For this reason the functions $f_k(r, \phi, u)$, of the complex variable u, defined by

(5.31) $$f_k(r, \phi, u) = \sum_{m=1}^{\infty} a_{m+2,k}u^m$$

are regular functions of u as long as $Q|\phi|\,|u| < 1$. From the fact that $|\phi| \leqslant |\phi_{0,k}|$, (5.15) implies that if $|u| \leqslant 2\,(\log M_1(r))^{-\frac{1}{2}}$ then

$$\lim_{r\to b} |\phi u| = 0.$$

Thus the point $u = (\log M_1(r))^{-\frac{1}{2}}$ is certainly within the domain of regularity of $f_k(r, \phi, u)$. Let us now go on to consider the functions $F_k(r, \phi, u)$ defined by

(5.32) $$F_k\,(r, \phi, u) = \exp f_k\,(r, \phi, u).$$

All of the functions $F_k(r, \phi, u)$ have Maclaurin expansions about $u = 0$ and the radii of convergence of these expansions are certainly all $\geqslant 2\,(\log M_1(r))^{-\frac{1}{2}}$ We write

(5.33) $$F_k(r, \phi, u) = 1 + \sum_{m=1}^{\infty} b_{m,k}(r, \phi)\,u^m.$$

We can easily establish that

(5.34) $b_{2m+1,k}$ is a polynomial in ϕ and is an odd function of ϕ.

(5.35) $b_{2m,k}$ is a polynomial in ϕ and is an even function of ϕ.

(5.36) $|b_{m,k}| = 0((P_6|\phi|)^{3m})$, for large $|\phi|$, and P_6 and the order relation involve constants that are independent of r.

The result (5.36) follows from a lemma of Moser and Wyman **(7)**. Further, (5.15) implies that

$$\lim_{r\to b} |\phi_0|^3\,(\log M_1(r))^{-\frac{1}{2}} = 0.$$

Hence as long as $|u| \leqslant 2\,(\log M_1(r)^{-\frac{1}{2}}$ we must have

(5.37) $$\left|\sum_{s+1}^{\infty} b_{m,k}(r, \phi)\,u^m\right| = O((p_6|\phi|)^{3s+3}\,(\log M_1(r))^{-\frac{1}{2}(s+1)}).$$

From the fact that

$$\int_{-\infty\exp\,i\beta}^{\infty\exp\,i\beta} (\exp(-\phi^2))\,\phi^m\,d\phi$$

exists as long as $|\beta| < \pi/4$ it is easily shown that

(5.38) $$\left|\int_{-\phi_{0,k}}^{\phi_{0,k}} \left(\sum_{s+1}^{\infty} b_{m,k}(r, \phi)\,u^m\right) \exp(-\phi^2)\,d\phi\right| = O((\log M_1(r))^{-\frac{1}{2}(s+1)})$$

where in (5.38) the constants entering into the order relation may depend on s but not on r. In particular if $R_{s,k}$ is given by

(5.39) $$R_{s,k} = \int_{-\phi_{0,k}}^{\phi_{0,k}} \sum_{s+1}^{\infty} (b_{m,k}(r, \phi) (\log M_1(r))^{-\frac{1}{2}m}) \exp(-\phi^2)\, d\phi$$

then

(5.40) $$|R_{s,k}| = 0((\log M_1(r))^{-(s+1)}).$$

This completes the preliminary results necessary for the proof of our central theorem.

6. The asymptotic formula. From (1.2) the Laurent coefficients a_n are given by

(6.1) $$a_n = (2\pi i)^{-1} \int_C G(z)\, z^{-(n+1)}\, dz,$$

where C can be any contour within the domain of regularity of $G(z)$ and enclosing $z = 0$. We shall assume n is a large positive integer. Hence the convention of § 4 that a choice of n determines the value $r = r(n) = |Z_1(n)|$, where $Z_1(n)$ is the featured solution of

(6.2) $$H \log G(z) = n.$$

We may also consider the points $z_k(r)$, $Z_k(n)$, $(k = 1, 2, \ldots, N)$ as determined. The contour C is now chosen as follows.

Contour C. The points in the z plane $Z_k \exp(-i\epsilon)$, $Z_k \exp(i\epsilon)$ are joined by arcs of the circles $|z| = |Z_k|$. The points $Z_k \exp(-i\epsilon)$ and $Z_k \exp(i\epsilon)$ are joined, respectively, to the points $z_k \exp(-i\epsilon)$ and $z_k \exp(i\epsilon)$ by straight lines. Finally $z_k \exp(i\epsilon)$ is joined to $z_{k+1} \exp(-i\epsilon)$ by the arc of the circle $|z| = |z_k| = r$. The convention adopted in a previous section concerning z_{N+1} assures that C becomes a closed contour. Further, the previous work on relevant paths also ensures that the contour C does not traverse any of its points more than once.

From Lemma 5.1 the only portions of C that can contribute to the asymptotic expansion of $r^n a_n/M_1(r)$ are those portions that intersect a relevant path. For this reason we may write

(6.3) $$r^n a_n/M_1(r) \sim \sum_{k=1}^{N} I_k$$

where

(6.4) $$I_k = (2\pi i M_1(r))^{-1} r^n \int_{C_k} G(z)\, z^{-(n+1)}\, dz$$

and C_k is the arc of the circle $|z| = |Z_k|$ joining $Z_k \exp(-i\epsilon)$ to $Z_k \exp(i\epsilon)$. For this reason we study

(6.5) $$K_k = (2\pi i)^{-1} \int_{C_k} G(z)\, z^{-(n+1)}\, dz.$$

The substitution $z = Z_k \exp i\theta$ makes (6.5) become

(6.6) $$K_k = (2\pi)^{-1} Z_k^{-n} \int_{-\epsilon}^{\epsilon} G(Z_k \exp i\theta) \exp(-in\theta)\, d\theta.$$

Expanding $\log G(Z_k \exp i\theta)$ in a Maclaurin expansion about $\theta = 0$ we have

(6.7) $$\log G(Z_k \exp i\theta) - in\theta = \log G(Z_k) - \tfrac{1}{2} H^2 \log G(Z_k)\, \theta^2 + \sum_{m=3}^{\infty} H^m \log G(Z_k)(i\theta)^m/m!$$

Hence the substitution

(6.8) $$\phi = (\tfrac{1}{2}H^2 \log G(Z_k))^{\frac{1}{2}}\theta$$

into (6.6) yields

(6.9) $$2\pi K_k = G(Z_k) Z_k^{-n} (\tfrac{1}{2} H^2 \log G(Z_k))^{-\frac{1}{2}} \int_{-\phi_{0,k}}^{\phi_{0,k}} F_k(r, \phi, (\log M_1(r))^{-\frac{1}{2}}) \exp(-\phi^2)\, d\phi$$

where $\phi_{0,k}$ is given by (5.16) and $F_k(r, \phi, u)$ by (5.32). From (5.33) and (5.39) we may write

(6.10) $$\int_{-\phi_{0,k}}^{\phi_{0,}} F_k(r, \phi, (\log M_1(r))^{-\frac{1}{2}}) \exp(-\phi^2)\, d\phi$$
$$= \int_{-\phi_{0,k}}^{\phi_{0,k}} \exp(-\phi^2)(1 + \sum_{m=1}^{2s} b_{m,k}(r, \phi)(\log M_1(r))^{-m/2})\, d\phi + R_{2s,k}$$
$$= \int_{-\phi_{0,k}}^{\phi_{0,k}} \exp(-\phi^2)(1 + \sum_{m=1}^{s} b_{2m,k}(\log M_1(r))^{-m})\, d\phi + O((\log M_1(r))^{-s-\frac{1}{2}}$$

because of (5.34) and (5.40). Further, since $b_{2m,k}(r, \phi)$ is a polynomial in ϕ we include only terms which are exponentially small by replacing $\phi_{0,k}$ by $\infty \exp(i\psi_k)$, and without any loss in generality we may consider ϕ to be now a real variable and take limits of integration from $-\infty$ to ∞. If we let $A_{m,k}(r)$ be given by

(6.11) $$A_{m,k}(r) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\phi^2)\, b_{2m,k}(r, \phi)\, d\phi$$

then we have proven

(6.12) $$\int_{-\phi_{0,k}}^{\phi_{0,k}} F_k(r, \phi, (\log M_1(r))^{-\frac{1}{2}}) \exp(-\phi^2)\, d\phi \sim \sqrt{\pi}\left(1 + \sum_{m=1}^{\infty} A_{m,k}(\log M_1(r))^{-m}\right)$$

and (6.9) gives

(6.13) $$K_k \sim G(Z_k) Z_k^{-n} (2\pi H^2 \log G(Z_k))^{-\frac{1}{2}} \left(1 + \sum_{m=1}^{\infty} A_{m,k}(\log M_1(r))^{-m}\right).$$

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This in turn gives the complete asymptotic expansion for I_k in (6.4) and a_n from (6.3). Although by assumption $G(Z_k)/M_1(r)$ is not asymptotically equal to zero, it is possible that

$$G(Z_k)/M_1(r) = O\left(\frac{1}{(\log M_1(r))^s}\right),$$

and in such a case I_1 and I_k would be terms of different orders. For this reason it is better to leave (6.13) as our major formula and in each specific case write out (6.3) after the different orders of the K_k have been established. We shall illustrate the procedure in the next section. Thus we have established the following central theorem of our paper.

CENTRAL THEOREM. *A complete asymptotic expansion of the Laurent coefficients of any generating function $G(z)$ that is admissible can be obtained by the formulae established in this section.*

In fact it is possible to show that the first two terms of (6.13) are given by

$$(6.14)\quad K_k \sim \frac{G(Z_k)}{Z_k^n(2\pi H^2 \log G(Z_k))^{\frac{1}{2}}} \left[1 + \frac{3H^2 \log G(Z_k)\, H^4 \log G(Z_k) - 5(H^3 \log G(Z_k))^2}{24\,(H^2 \log G(Z_k))^3}\right].$$

In using these formulae we must examine the solutions of

$$(6.15)\qquad H \log G(z) = n$$

and from these we must select the featured path $z = Z_1(n)$ and establish all of the relevant paths $z = Z_k(n)$, $(k = 1, 2, \ldots, N)$. The variable r is determined by

$$(6.16)\qquad r = |Z_1(n)|$$

and all quantities involved in the asymptotic expansion may then be computed.

7. Specific examples. Since we have already shown that $[\exp(\frac{1}{2}(1 - z^2))]/(1 - z)$ is an admissible function we shall use this generating function as our first illustration. From (4.20) and (4.21) we have

$$(7.1)\qquad r = Z_1(n) = 1 - \frac{1}{2n^{\frac{1}{2}}} - \frac{3}{8n} + \frac{1}{8n^{3/2}} + \ldots$$

$$(7.2)\qquad Z_2(n) = -1 + \frac{1}{2n^{\frac{1}{2}}} - \frac{1}{8n} - \frac{1}{8n^{3/2}} + \ldots$$

Thus

$$(7.3)\qquad M_1(r) = 2n^{\frac{1}{2}}\, e^{(2n^{\frac{1}{2}}-1)/4}\left(1 - \frac{5}{16n^{\frac{1}{2}}} + \ldots\right) = G(Z_1).$$

$$G(Z_2) = \tfrac{1}{2}e^{(2n^{\frac{1}{2}}+1)/4}\left(1 + \frac{7}{16n^{\frac{1}{2}}} + \dots\right). \tag{7.4}$$

$$H^2\log G(Z_1) = 4n^{3/2}\left(1 - \frac{2}{n^{\frac{1}{2}}}\right). \tag{7.5}$$

$$H^2\log G(Z_2) = 4n^{3/2}\,(1 + \dots). \tag{7.6}$$

$$H^3\log G(Z_1) = 24n^2\,(1 + \dots). \tag{7.7}$$

$$H^4\log G(Z_1) = 192\,\mathrm{n}^{5/2}\,(1 + \dots). \tag{7.8}$$

$$K_1 \sim \frac{e^{n^{\frac{1}{2}}+1/4}}{(2\pi)^{\frac{1}{2}}n^{1/4}}\left(1 + \frac{5}{12n^{\frac{1}{2}}} + \dots\right). \tag{7.9}$$

$$K_2 \sim \frac{(-1)^n\, e^{n^{\frac{1}{2}}+0.25}}{4(2\pi)^{\frac{1}{2}}n^{3/4}}\,(1 + \dots). \tag{7.10}$$

Thus we have an illustration of a case where K_2 is a different order term than K_1 and K_2 affects only the second term of the asymptotic expansion. From (7.9) and (7.10) we find

$$a_n \sim \frac{e^{n^{\frac{1}{2}}+1/4}}{(2\pi)^{\frac{1}{2}}n^{1/4}}\left(1 + \left(\frac{5}{12} + \frac{(-1)^n}{4}\right)\frac{1}{n^{\frac{1}{2}}} + \dots\right). \tag{7.11}$$

We have so far in the paper only used admissible functions which have Maclaurin expansions. One of the simplest examples of an admissible function that has no Maclaurin expansion about $z = 0$ and does have a Laurent expansion is the generating function $\exp\left(\frac{1}{2}x(z - z^{-1})\right)$ for the Bessel Functions $J_n(x)$. We shall assume x is fixed, real, and positive. To use our method on this particular generating function is a little like shooting sparrows with cannons. It is of course well known that the series definition of $J_n(x)$ will act as an asymptotic expansion under these conditions. However it does act to illustrate the procedure for examples of similar type but more complicated in nature. For this generating function there is only one relevant path $\theta = 0$.

$$r = |Z_1(n)| = \frac{n + (n^2 - x^2)^{\frac{1}{2}}}{x}. \tag{7.12}$$

Hence

$$M_1(r) = G(Z_1) = \exp\left[(n^2 - x^2)^{\frac{1}{2}}\right]. \tag{7.13}$$

$$H^2\log G(Z_1) = (n^2 - x^2)^{\frac{1}{2}}. \tag{7.14}$$

$$H^3\log G(Z_1) = n. \tag{7.15}$$

$$H^4\log G(Z_1) = (n^2 - x^2)^{\frac{1}{2}}. \tag{7.16}$$

$$J_n(x) \sim \frac{x^n \exp[(n^2 - x^2)^{\frac{1}{2}}]}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}(n^2 - x^2)^{1/4}[n + (n^2 - x^2)^{\frac{1}{2}}]^n}\left(1 - \frac{2n^2 + 3x^2}{24(n^2 - x^2)^{3/2}} + \dots\right). \tag{7.17}$$

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As a final example we use the generating function $\exp(z/(1+z^2))$ to illustrate the method when the relevant paths are not straight lines. For this generating function there are two relevant paths. These are

$$Z_1(n) = i + \frac{1-i}{2n^{\frac{1}{2}}} - \frac{1}{4n} + \ldots. \tag{7.18}$$

$$Z_2(n) = -i + \frac{1+i}{2n^{\frac{1}{2}}} - \frac{1}{4n} + \ldots = \bar{Z}_1(n). \tag{7.19}$$

Since $z_1(r) = Z_1(n)$, $z_2(r) = Z_2(n)$ the fact that $\exp(z/(1+z^2))$ is admissible can easily be verified. Hence

$$G(Z_1) = [\exp(\tfrac{1}{2}(n^{\frac{1}{2}}(1+i)))]\,(1+\ldots). \tag{7.20}$$

$$G(Z_2) = [\exp(\tfrac{1}{2}(n^{\frac{1}{2}}(1-i)))]\,(1+\ldots). \tag{7.21}$$

$$Z_1^{-n} = [\exp(\tfrac{1}{2}(n^{\frac{1}{2}}(1+i) - i\,n\pi))]\,(1+\ldots). \tag{7.22}$$

$$Z_2^{-n} = [\exp(\tfrac{1}{2}(n^{\frac{1}{2}}(1-i) + in\pi))]\,(1+\ldots). \tag{7.23}$$

$$H^2 \log G(Z_1) = 2(1-i)n^{3/2}\,(1+\ldots). \tag{7.24}$$

$$H^2 \log G(Z_2) = (2(1+i)n^{3/2}\,(1+\ldots)\,. \tag{7.25}$$

and

$$a_n \sim \frac{2e^{n^{\frac{1}{2}}} \cos[n^{\frac{1}{2}} - \frac{1}{8}(4n-1)\pi]}{(2\pi)^{\frac{1}{2}}(2n)^{3/4}}. \tag{7.26}$$

8. Conclusion. The major result of this paper has been to give a set of conditions on a complex function $G(z)$ by means of which we can recognize whether or not $G(z)$ belongs to our class of admissible functions. If $G(z)$ does belong to such a class then our central theorem tells us that the complete asymptotic behaviour of the Laurent coefficients can be determined. It would be desirable to have results that tell us that certain large classes of functions are admissible. For example, it is possible to show that all functions of the form $P_m(z)\exp(S_t(z) + Q(1/z))$ are admissible providing $P_m(z)$, $S_k(z)$ are polynomials and $Q(1/z)$ is a regular function of its argument. From our experience we would say that there exists a very extensive class of integral functions all of which are admissible. Problems of this type would be worth investigation so that in specific examples one could tell, almost at a glance, whether or not the generating function is admissible. In such cases one can almost use our central theorem to write down the asymptotic behaviour of the Laurent coefficients.

In our terminology Hayman's class of admissible functions required that $G(z)$ possess a unique relevant path $\theta = 0$ that contributes to the first term of the asymptotic expansion. For this reason he was able to prove results stating that the product of admissible functions is admissible. In our case the two functions involved might have quite different relvant paths and such a theorem is not possible. For example, the functions exp $(z/(1-z^2))$ and $(\exp(z/(z^2-1)))/(1-z)$ are both admissible but their product $1/(1-z)$ is not admissible.

When one considers the class of functions for which the method of Darboux applies and the class of functions for which the method outlined in this paper applies, we now have a very extensive class of functions for which we can consider the problem of determining the behaviour of a_n, as $n \to \infty$, as a solved problem.

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