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CONNECTED ORDER 3 STANDARD REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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The concept of standard representations of simple Lie algebras was introduced by I. Z. Bouwer [1]. One of the difficulties was that of existence. The order zero standard representations are simply those having a dominant weight vector and these have been completely characterized, for example in [2]. The existence and complete characterization of all order 1 standard representations was carried out by Bouwer and a particular class of order 2 standard representations of the simple Lie algebra A_2 was constructed. In a later paper [3] the author established the existence of standard representations of arbitrary order for simple Lie algebras of sufficiently large rank. These representations were constructed by combining "disconnected" standard representations of orders 1 and 2. In this paper we establish the existence of a class of "connected" order 3 standard representations of simple Lie algebras—(the term "connected" in this context means that the set of simple roots, whose weight ladders in the lattice of weight functions of the representation are doubly infinite, form a connected subdiagram of the Lie algebra's Dynkin diagram).

Let $\Phi=\{\alpha_1, \alpha_2, \alpha_3\}$ denote a fundamental system of roots of the simple Lie algebra A_3 over the complex numbers and let $\Delta^+=\{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$ denote the set of all positive roots of A_3. Then one obtains a Cartan basis

$$B = \{H_\alpha, X_\beta, Y_\beta \mid \alpha \in \Phi; \beta \in \Delta^+\}$$

of A_3 with the usual multiplication constants—i.e. $[X_\alpha, Y_\alpha]=H_\alpha$, $[H_\alpha, X_\alpha]=2X_\alpha$, $[H_\alpha, Y_\alpha]=-2Y_\alpha$ for all $\alpha \in \Phi$, etc. Let $\Phi_1=\{\alpha_1, \alpha_2\}$, $\Phi_2=\{\alpha_2, \alpha_3\}$, $\Phi_3=\{\alpha_1, \alpha_2+\alpha_3\}$ and $\Phi_4=\{\alpha_1+\alpha_2, \alpha_3\}$. Then for $i=1,2,3,4$ $B_i=\{H_\alpha, X_\beta, Y_\beta \mid \alpha \in \Phi_i; \beta \in \Delta^+(\Phi_i)\}$ are Cartan bases for subalgebras $A_2(i)$ of A_3 each of which is isomorphic to the simple Lie algebra A_2. By inspection we obtain the 23 primitive cycles* of the cycle subalgebra* $C(A_3)$ namely H_{α_1}, H_{α_2}, H_{α_3}, $c_1=Y_{\alpha_1}X_{\alpha_1}$, $c_2=Y_{\alpha_2}X_{\alpha_2}$, $c_3=Y_{\alpha_3}X_{\alpha_3}$

$$c_4 = Y_{\alpha_1+\alpha_2}X_{\alpha_1+\alpha_2}, \quad c_5 = Y_{\alpha_2+\alpha_3}X_{\alpha_2+\alpha_3}, \quad c_6 = Y_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_1+\alpha_2+\alpha_3},$$
$$c_7 = Y_{\alpha_1+\alpha_2}X_{\alpha_1}X_{\alpha_2}, \quad c_8 = Y_{\alpha_2+\alpha_3}X_{\alpha_2}X_{\alpha_3}, \quad c_9 = Y_{\alpha_2}Y_{\alpha_1}X_{\alpha_1+\alpha_2},$$
$$c_{10} = Y_{\alpha_3}Y_{\alpha_2}X_{\alpha_2+\alpha_3}, \quad c_{11} = Y_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_1+\alpha_2}X_{\alpha_3}, \quad c_{12} = Y_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_1}X_{\alpha_2+\alpha_3},$$

* For definitions and basic properties of these concepts see Bouwer's paper [1].

$$c_{13} = Y_{\alpha_3}Y_{\alpha_1+\alpha_2}X_{\alpha_1+\alpha_2+\alpha_3},$$
$$c_{14} = Y_{\alpha_2+\alpha_3}Y_{\alpha_1}X_{\alpha_1+\alpha_2+\alpha_3},$$
$$c_{15} = Y_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_1}X_{\alpha_2}X_{\alpha_3},$$
$$c_{16} = Y_{\alpha_3}Y_{\alpha_2}Y_{\alpha_1}X_{\alpha_1+\alpha_2+\alpha_3},$$
$$c_{17} = Y_{\alpha_2+\alpha_3}Y_{\alpha_1}X_{\alpha_1+\alpha_2}X_{\alpha_3},$$
$$c_{18} = Y_{\alpha_3}Y_{\alpha_1+\alpha_2}X_{\alpha_1}X_{\alpha_2+\alpha_3},$$
$$c_{19} = Y_{\alpha_2}Y_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_1+\alpha_2}X_{\alpha_2+\alpha_3},$$
$$c_{20} = Y_{\alpha_2+\alpha_3}Y_{\alpha_1+\alpha_2}X_{\alpha_1+\alpha_2+\alpha_3}X_{\alpha_2}.$$

In order to construct an order 3 standard representation of A_3 it suffices to construct an algebra homomorphism $\gamma: C(A_3)\to C$ which satisfies condition 4.6 and condition (iv) of Definitions 3.1 in Bouwer's paper. Clearly, if such an algebra homomorphism exists, when it is restricted to each subalgebra $C(A_2(i))$ it will again be an algebra homomorphism satisfying the restricted conditions. Thus $\gamma\downarrow C(A_2(i))$ must coincide with the solutions obtained by Bouwer in §6 of [1]. Writing out these solutions with parameters z_1, z_2, z_3 and z_4 and equating "overlapping" values we obtain 6 quadratic equations relating these four parameters. For example we have $c_1 \in C(A_2(1)) \cap C(A_2(3))$ and hence

$$\gamma(c_1) = (z_1-1)(z_1+\gamma(H_{\alpha_1})) = (z_3-1)(z_3+\gamma(H_{\alpha_1})).$$

Solving we obtain $z_1=z_3$ or $z_1=1-\gamma(H_{\alpha_1})-z_3$. Continuing in this fashion we obtain eight possible solutions relating the four parameters, however, only one of these, namely $z=z_1=z_3=z_2+\gamma(H_{\alpha_2})=z_4+\gamma(H_{\alpha_2})$ does not impose any restrictions on the possible values of the $\gamma(H_{\alpha_i})$'s. Using this particular solution we obtain consistent values for $\gamma(c_i)$ for $i=1, 2, \ldots, 14$. Making use of the fact that γ is to be an algebra homomorphism on $C(A_3)$ and using the "c-basis" of $C(A_3)$ introduced by Bouwer [1; p. 358] we obtain necessary values of $\gamma(c_i)$ for $i=15, 16, \ldots, 20$. For example, in terms of the "c-basis" of $C(A_3)$. $c_{14}c_7=c_{20}c_1-c_8c_1+c_5c_1-c_5c_4+c_{14}c_4+c_7c_5$. Thus $\gamma(c_{14}c_7)=\gamma(c_{20}c_1-c_8c_1+c_5c_1-c_5c_4+c_{14}c_4+c_7c_5)$ or $\gamma(c_{20})\gamma(c_1)=\gamma(c_{14})\gamma(c_7)+\gamma(c_8)\gamma(c_1)-\gamma(c_5)\gamma(c_1)+\gamma(c_5)\gamma(c_4)-\gamma(c_{14})\gamma(c_4)-\gamma(c_7)\gamma(c_5)$. $\gamma(c_{20})(z-1)(z+\gamma(H_{\alpha_1}))=(z-1)z(z+\gamma(H_{\alpha_1}))^2(z-1-\gamma(H_{\alpha_2}))(z-1-\gamma(H_{\alpha_2})-\gamma(H_{\alpha_3}))$. By condition 6.4 mentioned above $(z-1)(z+\gamma(H_{\alpha_1}))\neq 0$. Therefore, $\gamma(c_{20})=z(z+\gamma(H_{\alpha_1}))(z-1-\gamma(H_{\alpha_2}))(z-1-\gamma(H_{\alpha_2})-\gamma(H_{\alpha_3}))$.

In this manner we arrive at the following tentative solution:

$$\gamma(H_{\alpha i}) = \lambda_i \qquad i = 1, 2, 3 \qquad\qquad \lambda_i \text{ arbitrary scalar.}$$
$$\gamma(c_1) = (z-1)(z+\lambda_1)$$
$$\gamma(c_2) = z(z-1-\lambda_2)$$
$$\gamma(c_3) = (z-\lambda_2)(z-1-\lambda_2-\lambda_3)$$

$$\gamma(c_4) = (z+\lambda_1)(z-1-\lambda_2)$$
$$\gamma(c_5) = z(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_6) = (z+\lambda_1)(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_7) = \gamma(c_9) = z(z+\lambda_1)(z-1-\lambda_2)$$
$$\gamma(c_8) = \gamma(c_{10}) = z(z-\lambda_2)(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_{11}) = \gamma(c_{13}) = (z+\lambda_1)(z-\lambda_2)(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_{12}) = \gamma(c_{14}) = z(z+\lambda_1)(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_{15}) = \gamma(c_{16}) = \gamma(c_{17}) = \gamma(c_{18}) = z(z+\lambda_1)(z-\lambda_2)(z-1-\lambda_2-\lambda_3)$$
$$\gamma(c_{19}) = \gamma(c_{20}) = z(z+\lambda_1)(z-1-\lambda_2)(z-1-\lambda_2-\lambda_3)$$

It suffices then to verify that for this set of values condition (c) of Theorem 4.4 [1] is satisfied. This was verified by direct computation. (It should be noted that the number of these computations although still large was considerably reduced by the use of some symmetrics in the proposed solution).

Using the results from [3] one can then obtain connected order 3 standard representations for any simple Lie algebra whose Dynkin diagram contains as a subdiagram the Dynkin diagram of A_3.

The above construction theoretically can be used to yield connected order n standard representations of A_n, however, the computations involved in verifying condition c) of Theorem 4.4 become prohibitive. Unfortunately we have been unable to replace condition (c) by a more tractable condition.

Bibliography

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