SOME INTEGRAL FORMULAS FOR HYPER-SURFACES IN EUCLIDEAN SPACES

BANG-YEN CHEN

1. Introduction

Let $M$ be an oriented hypersurface differentiably immersed in a Euclidean space of $n + 1 \geq 3$ dimensions. The $r$-th mean curvature $K_r$ of $M$ at the point $P$ of $M$ is defined by the following equation:

$$ \det(\delta_{ij} + a_{ij}) = \sum_{r=0}^{n} \binom{n}{r} K_r t^r $$

where $\delta_{ij}$ denotes the Kronecker delta, $\binom{n}{r} = n! / r! (n - r)!$, and $a_{ij}$ are the coefficients of the second fundamental form. Throughout this paper all Latin indices take the values $1, \cdots, n$, Greek indices the values $1, \cdots, n + 1$, and we shall also follow the convention that repeated indices imply summation unless otherwise stated. Let $p$ denote the oriented distance from a fixed point $0$ in $E^{n+1}$ to the tangent hyperplane of $M$ at the point $P$, and $dV$ denote the area element of $M$. Let $e_1, \cdots, e_n$ be an ordered orthonormal frame in the tangent space of the hypersurface $M$ at the point $P$, and denote by $x_i$ the scalar product of $e_i$ and the position vector $X$ of the point $P$ with respect to the fixed point $0$ in $E^{n+1}$. The main purpose of this paper is to establish the following theorems:

**Theorem 1.** Let $M$ be an oriented hypersurface with regular smooth boundary differentiably immersed in a Euclidean space $E^{n+1}$. Then we have

$$ \int_M p^{n-1} X \cdot \nabla K_r dV + n \int_M p^{n-1}(K_r - K_t K_r p) dV + (m - 1) \int_M p^{n-2} K_r x_i x_i a_{ij} dV $$

$$ = \int_{\partial M} p^{n-1} K_r X \cdot \star dX, \quad r = 0, 1, \cdots, n - 1, $$

where $m$ is any real number, $\nabla K_r$ is the gradient of $K_r$, $\partial M$ is the boundary of $M$ and $\star$ denotes the star operator.

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Formula (2) was obtained by Amur [1] for \( m = 1 \) in an alternating form.

**Theorem 2.** Under the same assumption of Theorem 1, we have

\[
\sum_{i=0}^{r} (-1)^i \binom{n}{r-i} p^{m-i} K_{i-1} x_i a_i n_0 \left( \prod_{k=1}^{i} a_{h_k} \cdot 1_k \right) e_n \, dV
\]

\[
= (n-r) \binom{n}{r} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} p^{m} K_{i} e_i dV - \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} \sum_{\beta} \partial^{m} K_{i}^{*} U_{\beta},
\]

where \( e \) denotes the unit outer normal vector. In particular, we have

\[
m \int_{M} p^{m-1} X_i K_n dV = n \int_{M} p^{m} e K_n dV - (1/n!) \int_{M} p^{m} \sigma_{n-1},
\]

and

\[
m \int_{M} p^{m-1} a_{i} x_i e_j dV = n \int_{M} p^{m} K_i e_i dV + \int_{M} p^{m} e dX.
\]

Formula (4) was obtained by Flanders [4] for \( m = 1 \) and \( M \) is closed.

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**2. Preliminaries**

In a Euclidean space \( E^{n+1} \) of \( n+1 \geq 3 \) dimensions, let us consider a fixed right-handed rectangular frame \( X, e_1, \ldots, e_{n+1} \), where \( X \) is a point in \( E^{n+1} \), and \( e_1, \ldots, e_{n+1} \) is an ordered set of mutually orthogonal unit vectors such that its determinant is

\[
[e_1, \ldots, e_{n+1}] = 1,
\]

so that \( e_a \cdot e_b = \delta_{ab} \). Let \( F \) denote the bundle of all such frames. We also use \( X \) to denote the position vector of the point \( P \) with respect to a fixed point 0 in \( E^{n+1} \). Then we have

\[
dX = \theta_a e_a, \quad de_a = \theta_{ab} e_b
\]

where \( d \) denotes the exterior differentiation, and \( \theta_a \), \( \theta_{ab} \) are Pfaffian forms. Since \( d^2 X = d( dX ) = d( de_a ) = 0 \), exterior differentiation of equations of (7) find that

\[
d\theta_a = \theta_b \wedge \theta_{ab}, \quad d\theta_{ab} = \theta_{a\beta} \wedge \theta_{ab}, \quad \theta_{ab} + \theta_{ba} = 0,
\]

where \( \wedge \) denotes the exterior product.
Let $M$ be a hypersurface twice differentiably immersed in $E^{n+1}$. Consider the set $B$ consisting of frames $X, e_1, \cdots, e_n, e$ in $E^{n+1}$ satisfying the conditions $X \in M$ and $e_1, \cdots, e_n$ are vectors tangent to $M$ at $X$. Then we have a canonical mapping, said $\lambda$, from $B$ into $F$. Let $\lambda^*$ denote the dual mapping of $\lambda$. By setting
\[
\omega_a = \lambda^* \theta_a, \quad \omega_{a\beta} = \lambda^* \theta_{a\beta}.
\]
from (8) we have
\[
d\omega_a = \omega_{\beta} \wedge \omega_{\beta a}, \quad d\omega_{a\beta} = \omega_{a\gamma} \wedge \omega_{\gamma\beta}, \quad \omega_{a\beta} + \omega_{\beta a} = 0.
\]
From the definition of $B$, it follows that $\omega_{n+1} = 0$ and $\omega_1, \cdots, \omega_n$ are linear independent. Thus the first equation of (10) gives
\[
\omega_i \wedge \omega_{i,n+1} = 0.
\]
From which we can write
\[
\omega_{n+1,i} = a_i \omega_i, \quad a_{ij} = a_{ij}.
\]
Throughout a point in the space $E^{n+1}$, let $V_1, \cdots, V_n, J$ be $n+1$ vectors in the space $E^{n+1}$, and $V_1 \times \cdots \times V_n$ denote the vector product of the $n$ vectors $V_1, \cdots, V_n$. Then we have
\[
J \cdot (V_1 \times \cdots \times V_n) = (-1)^n [J, V_1, \cdots, V_n],
\]
where $\cdot$ denotes the inner product of $E^{n+1}$, from which it follows that
\[
e_1 \times \cdots \times \epsilon_n \times x \times \cdots \times e_{n+1} = (-1)^{n+1} e_n,
\]
where the roof means the omitted term. In the following, we denote the combined operation of inner product and the exterior product by $(,)$, and the combined operation of the vector product and the exterior product by $[,]$. We list a few formulas for easy reference. For the relevant details, we refer to Amur [1], Chern [2] and Flanders [4].
\[
[e, \overbrace{dX, \cdots, dX}^{n-1}] = -(n-1)! * dX,
\]
where $*$ denotes the star operator.
\[
p = X \cdot e, \quad (de, *dX) = nK_idV, \quad (dX, *dX) = ndV,
\]
where $dV = \omega_1 \wedge \cdots \wedge \omega_n$ is the area element of $M$.
\[
[\underbrace{de, \cdots, de}_{r}, \overbrace{dX, \cdots, dX}^{n-r}] = r!(n-r)! \binom{n}{r} K_r dV,
\]
If \( f \) is a smooth function defined on \( M \). By \( \text{grad} f \) or \( \nabla f \) we mean \( \nabla f = f_i e_i \), where \( f_i \) are given by \( df = f_i \omega_i \), we have

\[
d f \wedge *dX = (\nabla f) dV.
\]

A self adjoint linear transformation \( A \) of the tangent space of \( M \) at \( X \) into itself is defined by

\[
A e_i = a_{ij} e_j,
\]

where the symmetric matrix \( \{a_{ij}\} \) is given by (11). It follows that

\[
AdX = A_0 e_i = \omega_i A e_i = \omega_i a_{ij} e_j = de
\]

We look for other intrinsic tangent vectors which are obtained as the result of repeated application of the transformation \( A \) to \( dX \). Let \( A^j dX \) denote the intrinsic tangent vector obtained from \( dX \) by applying \( A \) repeatedly \( j \) times. For convenience we write

\[
U_0 = dX, \quad U_j = A^j dX, \quad j = 1, 2, \ldots, n.
\]

As in [1], we have

\[
\sigma_r = -r!(n - r - 1)! \sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} n \\ i \end{array} \right) K_{r-i} U_i,
\]

where

\[
\sigma_r = [e, de, \ldots, de, dX, \ldots, dX],
\]

\[
r = 0, 1, \ldots, n - 1,
\]

\[
\sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} n \\ i \end{array} \right) K_{r-i} a_{j} a_{j} (\Pi_{k=1}^{i} a_{h_{k-1} h_{k}}) dV
\]

3. **Lemmas**

**Lemma 1.** Let

\[
\pi_r = (-1)^n d p \wedge (X \cdot \sigma_r) = (-1)^n (X \cdot de) \wedge (X \cdot \sigma_r),
\]

then we have

\[
\pi_r = (-1)^n r!(n - r - 1)! \sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} n \\ i \end{array} \right) K_{r-i} a_{j} a_{j} (\Pi_{k=1}^{i} a_{h_{k-1} h_{k}}) dV
\]

and
\[ (26) \quad (dp) \wedge \sigma_r = (-1)^r \frac{n-r}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-1} x_j a_i h_i (\prod_{k=1}^{i} a_{h_{k-1} h_k}) e_h dV \]

where \( r = 0, 1, \ldots, n-1 \).

**Proof.** By (19), (20) and (21) we have

\[ (27) \quad U_r = (\prod_{h=1}^{r} a_{h_{k-1} h_k}) \omega_h e_{h}. \]

Hence we get

\[ (28) \quad ^*U_r = (-1)^{h+1} (\prod_{h=1}^{r} a_{h_{k-1} h_k}) \omega_{1} \wedge \cdots \wedge \omega_{h} \wedge \cdots \wedge \omega_{n} e_{h}. \]

Thus by (22), we get

\[
\pi_r = (-1)^{n+1} \frac{n-r-1}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-1} x_j a_i h_i (\prod_{h=1}^{i} a_{h_{k-1} h_k}) e_h dV
\]

\[
= (-1)^{n} \frac{n-r}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-1} x_j a_i h_i (\prod_{h=1}^{i} a_{h_{k-1} h_k}) dV
\]

This proves (25). Formula (26) follow immediately from (22) and (28).

**Lemma 2.** Let \( \sigma_r \) and \( \pi_r \) be given by (23) and (24). Then we have

\[ (29) \quad n! p^{m-K} dV - n! p^{m-1} K dV + (-1)^{n} \frac{n!}{n-r} p^{m-1} \pi_r = d(p^{n-1} X \cdot \sigma_r), \]

\[ r = 0, 1, \ldots, n-1. \]

**Proof.** Since

\[
d(p^{n-1} X \cdot \sigma_r) = (-1)^{n+1} \frac{n-r}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-1} x_j a_i h_i (\prod_{h=1}^{i} a_{h_{k-1} h_k}) dV
\]

\[
+ (m-1) p^{m-2} dp \wedge (X \cdot \sigma_r) + (-1)^{n} \frac{n!}{n-r-1} \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} [d(p^{n-1} K_{r-1} X \cdot \sigma_r)]
\]

\[
= (m-1)(-1)^{n} \pi_r - n! p^{m-1} K dV + n! p^{m-1} K_{r+1} dV.
\]

This gives (29).

**Lemma 3.** Let \( U_t \) and \( \sigma_r \) be given by (20) and (23). Then we have

\[ (30) \quad r! (n-r-1)! \binom{n}{r} \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} [d(p^{n-1} K_{r-1} X \cdot \sigma_r) + r! (n-r-1)! \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} [d(p^{n-1} K_{r-1} X \cdot U_t)]
\]

\[ = p^{n-1} X \cdot d(K_{r-t} U_t) \]

\[ r = 0, 1, \ldots, n-1. \]
Proof. Since by the identities of Newton for the elementary symmetric functions (see, for instance, [1]), we can easily verify that

\begin{equation}
\sum_{i=1}^{r} (-1)^{i-n} \binom{n}{r-i} K_{r-i}(dX, \ast U_i) = \binom{n}{r} K_r dV.
\end{equation}

Hence we have

\begin{align*}
\sum_{i=1}^{r} (-1)^i \binom{n}{r-i} &\left[d(p^{n-1}K_{r-i}X \ast U_i) - p^{n-1}K_r dK_{r-i}U_i\right] \\
= &\sum_{i=1}^{r} (-1)^i (m-1) p^{n-2}K_{r-i} \binom{n}{r-i} dp \wedge X \ast U_i \\
+ &\sum_{i=1}^{r} (-1)^i \binom{n}{r-i} p^{n-1}K_{r-i}(dX, \ast U_i) \\
= &\sum_{i=1}^{r} (-1)^i (m-1) p^{n-2}K_{r-i} \binom{n}{r-i} dp \wedge X \ast U_i - \binom{n}{r} p^{n-1}K_r dV \\
= &-\binom{n}{r} p^{n-1}K_r dV - \binom{n}{r} p^{n-1}K_{r+1} dV \\
= &-\binom{n}{r} p^{n-1}K_r dV - (1/r!(n-r-1)!)(p^{n-1}X \cdot \sigma_r)
\end{align*}

Hence, by Lemma 2, it equals to

\begin{align*}
= &-(m-1) p^{n-2}K_{r} \binom{n}{r} x_i x_j \alpha_i dV - \binom{n}{r} p^{n-1}K_r dV + (n-r) \binom{n}{r} p^{n-1}K_{r+1} dV \\
= &-\binom{n}{r} p^{n-1}K_r dV - (1/r!(n-r-1)!)(p^{n-1}X \cdot \sigma_r)
\end{align*}

From this formula we can easily get (30).

4. The Proofs of Theorems 1 and 2

Proof of Theorem 1. By (22), we have

\begin{equation}
\sigma_r = -r!(n-r-1)! \left[\binom{n}{r} K_r \ast dX + \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} K_{r-i} \ast U_i\right]
\end{equation}

By taking exterior differentiation, we get

\begin{align*}
(r+1) \binom{n}{r+1} K_{r+1} dV = &n \binom{n}{r} K_r dV - \binom{n}{r} X \cdot \nabla K_r dV \\
- &\sum_{i=1}^{r} (-1)^i \binom{n}{r-i} d(K_{r-i} \ast U_i)
\end{align*}

Taking scalar product with $X$ and multiplying by $p^{n-1}$, we get
Thus by Lemma 3,
\[
LHS = - \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} d(p^{n-i}K_{r-i} \cdot \mathbf{X} \cdot \mathbf{U}_i) - (1/r!(n - r - 1)!) d(p^{n-1} \cdot \mathbf{X} \cdot \sigma_r) + \binom{n}{r} [(n - r) p^{m} K_{r+1} - np^{m-1} K_r - (m - 1) p^{m-2} K_r x_i x_j a_{ij}] dV.
\]

On the other hand, we have
\[
\sum_{i=1}^{r} (-1)^{i+1} \binom{n}{r-i} d(p^{n-i}K_{r-i} \cdot \mathbf{X} \cdot \mathbf{U}_i) - (1/r!(n - r - 1)!) d(p^{n-1} \cdot \mathbf{X} \cdot \sigma_r) = \sum_{i=1}^{r} (-1)^{i+1} \binom{n}{r-i} d(p^{n-i}K_{r-i} \cdot \mathbf{X} \cdot \mathbf{U}_i) + \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} d(p^{n-1}K_{r-i} \cdot \mathbf{X} \cdot \mathbf{U}_i)
\]
\[
= \binom{n}{r} d(p^{m-1}K_r \cdot \mathbf{X} \cdot \mathbf{d}X).
\]

Therefore we get
\[
p^{m-1}(pK_r K_r - K_{r+1}) dV = p^{m-1} \cdot \nabla K_r dV - (m - 1)p^{m-2} K_r x_i x_j a_{ij} dV
\]
\[
= d(p^{m-1}K_{r-1} \cdot \mathbf{X} \cdot \mathbf{d}X).
\]

By applying the Stokes theorem to this formula, we get formula (2). This completes the proof of Theorem 1.

Proof of Theorem 2. Since we have
\[
d(p^{m} \sigma_r) = p^{m} \left[ \frac{d e_r \cdots d e_1 d X_1 \cdots d X_r}{r+1} \right] + mp^{m-1} d p \wedge \sigma_r
\]
\[
= (r+1)! (n - r - 1)! (\binom{n}{r+1}) K_{r+1} edV + mp^{m-1} d p \wedge \sigma_r
\]

Hence, by the Stokes theorem and Lemma 1, we get (3). Furthermore, by setting \( r = n - 1 \) or 0 and applying formula (31), we get (4) and (5). This completes the proof of Theorem 2.

5. Some Applications

Corollary 1. Under the same assumption of Theorem 1, we have
\[
(n - r) \binom{n}{r} K_{r+1} p^{m} dV - n \binom{n}{r} K_r p^{m} dV + \binom{n}{r} p^{m-1} \cdot \nabla K_r dV
\]
\[
= - \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} p^{m-1} \cdot \mathbf{d}(K_{r-i} \cdot \mathbf{U}_i).
\]
\[ r!(n-r-1)!(m-1) \sum_{i=0}^{r} (-1)^{r+i} \left( \begin{array}{c} n \\ r-i \end{array} \right) \int_{M} p^{m-2}K_{r-i}x_{i}x_{h_{i}} \left( \prod_{k=1}^{i} a_{h_{k}} \right) p^{m-2}dV, \]
\[ r = 0, 1, \cdots, n-1. \]

In particular, by setting \( m = 1 \), we have the Minkowski formulas:
\[(33) \int_{M} pK_{r+1}dV = \int_{M} K_{r}dV + \int_{\partial M} X \cdot \sigma_{r}/n! \quad r = 0, 1, \cdots, n-1.\]

This Corollary follows immediately from (25), Lemma 2 and the Stokes theorem. This Corollary was obtained by Shahin [8] for \( r = 0, n-1 \), and by Yano and Tani [9] for the closed case.

**Corollary 2.** Under the same assumption of Theorem 1, we have
\[(34) n! \int_{M} K_{r+1}dV = \int_{\partial M} \sigma_{r}, \quad r = 0, 1, \cdots, n-1.\]

Two applications of Corollary 1 for the case \( m = 1 \), one to \( M \) and the other to \( M + c, c \) in \( E^{n+1} \), gives us (34).

**Corollary 3.** Under the same assumption of Theorem 1, we have
\[(35) \int_{M} X \cdot \nabla K_{r} + n \int_{M} p(K_{r+1} - K_{r}K_{r})dV = \int_{\partial M} K_{r}X \cdot \sigma_{r}dX - \int_{\partial M} X \cdot \sigma_{r}/n!, \]
\[ r = 0, 1, \cdots, n-1. \]

This Corollary follows immediately from Theorem 1 and Corollary 1.

**Corollary 4.** There is no minimal closed hypersurface in \( E^{n+1} \).

**Proof.** Set \( r = 0 \), then by (32), we know that if \( M \) is closed, then the volume \( v(M) \) of \( M \) is given by
\[(36) v(M) = \int_{M} pK_{1}dV. \]

Hence, if \( M \) is a minimal hypersurface of \( E^{n+1} \), then \( K_{1} = 0 \), hence \( v(M) = 0 \). But this is impossible. This Corollary was proved by Chern and Hsiung.

**Corollary 5.** Under the same assumption of Theorem 1, if \( M \) is closed, then
\[(37) \int_{M} \nabla K_{r}dV = n \int_{M} K_{r}K_{r}dV, \quad r = 0, 1, \cdots, n-1. \]

In particular, if the mean curvature \( K_{1} \) is constant, then we have
\[ \int_M \nabla K_r dV = 0, \quad r = 0, 1, \ldots, n - 1. \]

Two applications of Corollary 3, one to \(M\) and the other to \(M + c\), gives us (37). Formula (38) follows immediately from (37) if \(K_i\) is constant. This Corollary was obtained by Amur [1].

References


University of Notre Dame, Notre Dame, Indiana,
Michigan State University, East Lansing, Michigan, U.S.A.