Glasgow Math. J. **42** (2000) 455–468.

GRADED RINGS OF COHOMOLOGICAL DIMENSION 2

Q. S. WU

Institute of Mathematics, Fudan University, Shanghai, 200433, China
e-mail:cwu@math.washington.edu, qswu@fudan.edu.cn

and J. J. ZHANG

Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
e-mail:zhang@math.washington.edu

(Received 18 January, 1999)

Abstract. Let A be a noetherian connected graded ring with a balanced dualizing complex R. If A has cohomological dimension and Krull dimension 2, then

(1) R is Auslander;

(2) $\operatorname{Cdim} M = \operatorname{Kdim} M$ for all noetherian graded A-modules M.

In particular, if A is AS-Gorenstein of injective and Krull dimension 2, then

(3) A is Auslander-Gorenstein;

(4) A is 2-pure with a self-injective artinian quotient ring;

(5) A has a residue complex.

(1,3,4) generalize a result of Levasseur [**7**, 5.13] and (5) generalizes a result of Ajitabh [**1**, 3.12].

1991 *Mathematics Subject Classification*. 16E10, 16W50.

0. Introduction. Auslander property is closely related to other properties such as catenarity, localizability, and existence of nice dimension function (see [**16**]). The Auslander property is proved for some classes of rings such as AS-regular rings of global dimension no more than 3 [**2**, **3**, **7**] and the Sklyanin algebras [**11**, **12**]. It is an open question whether or not every noetherian AS-Gorenstein ring is Auslander-Gorenstein. Levasseur [**7**, 5.13] proved that a noetherian AS-Gorenstein ring A of injective dimension 2 is Auslander-Gorenstein if A has Gelfand-Kirillov dimension 2 and A is 2-homogeneous (i.e., 2-pure) with a self-injective artinian quotient ring. The main object of this note is to generalize Levasseur's result. The following is proved in Section 3.

THEOREM 0.1. *Let A be a noetherian connected graded ring with a balanced dualizing complex R. Suppose A has cohomological dimension 2. Then the following are equivalent:*

(1) *R is Auslander.*

(2) $\operatorname{Kdim} A = 2$ *where* Kdim *denotes the Krull dimension.*

(3) *there is a graded dimension function ∂ such that $\partial A < 3$.*

(4) *for any chain of graded primes $P' \subsetneq P \subset A$,* $\operatorname{Kdim} A/P \leq 1$.

(5) *the canonical dimension* Cdim *is a graded dimension function.*

(6) Cdim *is exact.*

(7) $\operatorname{Cdim} M = \operatorname{Kdim} M$ *for all noetherian graded A-modules M.*

The definitions of the Auslander property and Gorenstein property are given next.

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

DEFINITION 0.2. A balanced dualizing complex R over A is *Auslander* if for every noetherian graded A-module M and for every graded A-submodule $N \subset \mathrm{Ext}^i(M, R)$, $\mathrm{Ext}^j(N, R) = 0$ for all $j < i$.

DEFINITION 0.3. (1) A noetherian connected graded ring A is called *AS-Gorenstein* (where AS stands for Artin and Schelter) if A has injective dimension $d < \infty$ and $\mathrm{Ext}^i(k, A) = 0$ for $i \neq d$ and $\mathrm{Ext}^d(k, A) = k(e)$ for some $e \in \mathbb{Z}$, where k is viewed as either left or right trivial A-module.

(2) A noetherian (graded) ring A is called *(graded) Auslander-Gorenstein* if A has finite injective dimension and for every noetherian (graded) A-module M and for every (graded) A-submodule $N \subset \mathrm{Ext}^i(M, A)$, $\mathrm{Ext}^j(N, A) = 0$ for all $j < i$.

We refer to [**13**, **14**, **15**, **16**] for other definitions and notations.

By [**7**, 6.3] every connected graded Auslander-Gorenstein ring is AS-Gorenstein. By [**5**, 0.1], a noetherian graded ring is Auslander-Gorenstein if and only if it is graded Auslander-Gorenstein. By [**14**, 4.14], AS-Gorenstein rings have balanced dualizing complexes and in this case the injective dimension of A is its cohomological dimension. By [**15**, Section 0], a connected graded ring is Auslander-Gorenstein if and only if it is AS-Gorenstein and its balanced dualizing complex is Auslander.

In addition to Theorem 0.1, we prove the following for AS-Gorenstein rings in Section 4.

THEOREM 0.4. *Let A be a noetherian AS-Gorenstein ring of injective dimension 2. Then the following are equivalent (and equivalent to the conditions in Theorem 0.1):*

(1) *A is Auslander-Gorenstein.*
(2) *A has a self-injective artinian quotient ring.*
(3) *A has an artinian quotient ring.*
(4) *for every minimal prime ideal $P \subset A$,* $\mathrm{Kdim}\, A/P = 2$.
(5) *A has a residue complex.*
(6) *A is 2-pure, i.e., for every nonzero left (right) ideal $I \subset A$,* $\mathrm{Kdim}\, I = 2$.

We are unable to show that every noetherian connected graded ring of cohomological dimension 2 has Krull dimension 2. The following partial result is proved in Section 2.

PROPOSITION 0.5. *Let A be a noetherian connected graded ring with a balanced dualizing complex. If A has cohomological dimension 2, then*

(1) $\mathrm{Kdim}_{A^e} A \leq 2$, *where* $A^e = A \otimes A^{op}$,
(2) *there are only finitely many graded prime ideals P with* $\mathrm{Kdim}\, A/P \geq 2$.

1. Balanced dualizing complexes. Let k be a base field and A a connected graded ring over k, i.e., $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. Unless otherwise stated we are working on connected graded rings over k and their graded modules. If A is left (or right) noetherian, then each A_i is finite dimensional over k. Let $\mathfrak{m}$ denote the maximal graded ideal $A_{\geq 1}$ of A. The $\mathfrak{m}$-torsion submodule of M, denoted by $\Gamma_{\mathfrak{m}}(M)$, is the union of all finite dimensional submodules of M. The left (right) cohomological dimension of A is defined to be the cohomological dimension of $\Gamma_{\mathfrak{m}}$ applying to the left (right) graded A-modules.

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

Our main tool is the balanced dualizing complex introduced by Yekutieli [**14**]. A theorem of Van den Bergh [**13**, 6.3] states as follows: A noetherian connected graded ring A has a balanced dualizing complex if and only if A satisfies left and right χ-condition and has finite left and right cohomological dimension. Both χ-condition and having finite cohomological dimension are checked for a large class of noetherian graded rings [**4**, **16**].

From now on A is a noetherian connected graded ring with a balanced dualizing complex R. It follows from [**13**, 4.8] that the left cohomological dimension of A is equal to the right cohomological dimension of A, which is denoted by $\text{cd}A$.

Let M be a graded A-module. The graded vector space dual of M is $M' = \oplus_n \text{Hom}_k(M_{-n}, k)$. The local duality theorem states the following: for any $X \in D(\text{Gr}\, A)$,

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X)' = \mathrm{RHom}_A(X, R) \qquad \text{(E1.1)}$$

(see [**14**, 4.18] and [**13**, 5.1]).

A dualizing complex R induces a convergent spectral sequence: for any noetherian module M

$$E_2^{p,q} := \text{Ext}^p(\text{Ext}^q(M, R), R) \Longrightarrow \mathbb{H}^{p-q}(M) \qquad \text{(E1.2)}$$

where $\mathbb{H}^{p-q}(M) = 0$ if $p \neq q$ and $\mathbb{H}^0(M) = M$ [**16**, 1.7].

Suppose A has cohomological dimension d with a balanced dualizing complex R. It follows from the local duality theorem (E1.1) that, for all $i > 0$ and $i < -d$,

$$\text{Ext}^i(M, R) = 0$$

for all left (or right) A-modules A. Hence we may choose R to be a complex in $D^b_{fg}(\text{Gr}\, A^e)$ of the form

$$R = \cdots 0 \to R^{-d} \to \cdots \to R^{-1} \to R^0 \to 0 \cdots$$

where each R^q is injective as left and as right A-module.

LEMMA 1.3. *Let M be a noetherian graded A-module.*
(1) $\text{Ext}^0(M, R) = \Gamma_{\mathfrak{m}}(M)'$.
(2) *If* $\dim_k M < \infty$, *then* $\text{Ext}^q(M, R) = 0$ *for all* $q \neq 0$ *and* $\text{Ext}^0(M, R) = M'$.
(3) *If* $\dim_k M = \infty$, *then* $\dim_k \text{Ext}^q(M, R) = \infty$ *for some* $q < 0$.

Proof. (1,2) Follow from the local duality (E1.1).

(3) Suppose that $\text{Ext}^q(M, R)$ is finite dimensional for all $q < 0$. By (1) $\text{Ext}^0(M, R)$ is finite dimensional. Hence, by (2), $\text{Ext}^p(\text{Ext}^q(M, R), R)$ is finite dimensional for all p, q. It follows from the spectral sequence (E1.2) that M is finite dimensional. □

If $\text{cd}A = 0$, then the short exact sequence $0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow A/\mathfrak{m} \longrightarrow 0$ induces an exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(\mathfrak{m}) \longrightarrow \Gamma_{\mathfrak{m}}(A) \longrightarrow A/\mathfrak{m} \longrightarrow \mathrm{R}^1\Gamma_{\mathfrak{m}}(\mathfrak{m}) = 0.$$

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

Hence $1 \in \Gamma_{\mathfrak{m}}(A)$, so $A = \Gamma_{\mathfrak{m}}(A)$ is finite dimensional. Next we consider the case $\mathrm{cd}A = 1$. Let GKdim denote the Gelfand-Kirillov dimension.

Lemma 1.4. *Let A and B be noetherian graded rings.*

(1) *If* $\mathrm{GKdim} A = 1$*, then A is PI (PI=satisfying a polynomial identity) and* $\mathrm{Kdim}\, A = 1$.

(2) *If* $\mathrm{Kdim}\, A = 1$*, then* $\mathrm{GKdim} A = 1$.

(3) *If M is a graded (A, B)-module noetherian on both sides, then the following are equivalent:*
(a) $\mathrm{Kdim}\, M_B = 1$, *(b)* $\mathrm{GKdim} M_B = 1$, *(c)* $1 \le \mathrm{GKdim} M_B < 2$, *(d)* $\mathrm{GKdim}_A M = 1$.

Proof. (1) By [**10**], if $\mathrm{GKdim} A = 1$, then A is PI. By [**8**, 6.4.8 and 13.10.6] $\mathrm{Kdim}\, A = \mathrm{GKdim} A = 1$.

(2) We may assume A is prime. By a graded version of Goldie's theorem [**9**, I.1.6], there is a homogeneous regular element $x \in A$ of degree $d > 0$. Hence $\mathrm{Kdim}\, A/xA = 0$, which implies that A/xA is finite dimensional. Hence $\mathrm{GKdim} A = 1$.

(3) By noetherian induction we may assume M is critical and $ann({}_AM) =: Q$ and $ann(M_B) =: P$ are graded prime ideals such that ${}_AM$ is A/Q-torsionfree and M_B is B/P-torsionfree. Hence $\mathrm{Kdim}\, M_B = \mathrm{Kdim}\, B/P$ and $\mathrm{GKdim} M_B = \mathrm{GKdim} B/P$. Now (a) and (b) are equivalent by (1,2), and (b) and (c) are equivalent by Bergman's gap theorem [**6**, 2.5]. (b) and (d) are equivalent because GKdim is symmetric [**6**, 5.4]. □

Lemma 1.5. *If* $\mathrm{cd}A = 1$*, then A is PI of* Kdim 1. *More generally, if M is a noetherian graded A-module such that (a)* $\mathrm{Ext}^{-i}(M, R) = 0$ *for all $i > 1$ and (b) $f : M(-d) \to M$ is an injective map for some $d > 0$, then $M/f(M)$ is finite dimensional.*

Proof. We may assume that A is prime [**4**, 8.5] and not k. By [**9**, I.1.6], there is a homogeneous regular element $x \in A$ of degree $d > 0$. Then there is a short exact sequence

$$0 \to A(-d) \to A \to A/xA \to 0.$$

By Lemma 1.4, it suffices to show $\mathrm{GKdim} A = 1$. To prove $\mathrm{GKdim} A = 1$, it suffices to prove that A/xA is finite dimensional over k.

We now consider the general case when $0 \to M(-d) \to M \to M/f(M) \to 0$ is exact and show that $M/f(M)$ is finite dimensional. By the long exact sequence we see that

$$\mathrm{Ext}^{-i}(M/f(M), R) = 0 \quad \text{for all} \quad i > 1, \quad \text{and}$$

$$0 \to \mathrm{Ext}^{-1}(M/f(M), R) \to \mathrm{Ext}^{-1}(M, R) \to \mathrm{Ext}^{-1}(M(-d), R) \to \mathrm{Ext}^{0}(M/f(M), R)$$

is exact. Let $f^n(M)$ be the image of the map $f^n : M(-dn) \to M$. Then $f^n(M)/f^{n+1}(M) \cong (M/f(M))(-dn)$. The short exact sequence

$$0 \to f^n(M)/f^{n+1}(M) \to M/f^{n+1}(M) \to M/f^n(M) \to 0$$

yields a long exact sequence

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

$$0 \to \mathrm{Ext}^{-1}(M/f^n(M), R) \to \mathrm{Ext}^{-1}(M/f^{n+1}(M), R) \to$$
$$\mathrm{Ext}^{-1}(f^n(M)/f^{n+1}(M), R) \to \mathrm{Ext}^0(M/f^n(M), R).$$

Suppose $M/f(M)$ is infinite dimensional. By Lemma 1.3(3), $\mathrm{Ext}^{-1}(M/f(M), R)$, and hence $\mathrm{Ext}^{-1}(f^n(M)/f^{n+1}(M), R)$, is infinite dimensional. Since $\mathrm{Ext}^0(M/f^n(M), R)$ is finite dimensional, the injective map

$$\mathrm{Ext}^{-1}(M/f^n(M), R) \to \mathrm{Ext}^{-1}(M/f^{n+1}(M), R)$$

is not surjective. This shows that $\{\mathrm{Ext}^{-1}(M/f^n(M), R) \mid n \geq 0\}$ is an ascending chain of submodules of $\mathrm{Ext}^{-1}(M, R)$, which contradicts to the fact that $\mathrm{Ext}^{-1}(M, R)$ is noetherian. □

Let H^iR be the i-th cohomology of R. By definition, H^iR is noetherian on both sides for all i.

Lemma 1.6. *Suppose* $\mathrm{cd}A = d \geq 2$ *and M is a noetherian graded A-module.*
(1) *$H^{-d}R$ is noetherian on both sides and* $\mathrm{Ext}^{-d}(M, R) = \mathrm{Hom}(M, H^{-d}R)$.
(2) *Every nonzero left (right) submodule of $H^{-d}R$ has* Kdim ≥ 2.
(3) *If A is prime such that, for every nonzero graded prime P,* $\mathrm{Kdim}\, A/P \leq 1$, *then $H^{-d}R$ is A-torsionfree on both sides. Namely,* $\mathrm{Ext}^{-d}(M, R) = 0$ *for all A-torsion modules M.*

Proof. (1) Clear.

(2) For any α, the maximal submodule of $H^{-d}R$ of Krull dimension at most α is a graded submodule. Hence it suffices to consider graded submodules. Suppose N is a right graded submodule of $H^{-d}R$ of Kdim < 2. Since $H^{-d}R$ is noetherian on both sides, we may further assume that (a) N is a critical subbimodule of $H^{-d}R$, (b) $P := ann(N_A)$ is prime, and (c) N is A/P-torsionfree [**8**, 4.3.5(iii)]. Hence $\mathrm{Kdim}\, A/P = \mathrm{Kdim}\, N_A < 2$, and then $\mathrm{Kdim}\, A/P \leq 1$. By (1) and Lemma 1.3(2), $H^{-d}R$ contains no finite dimensional submodule. Thus $\mathrm{Kdim}\, A/P = 1$. By Lemma 1.4, A/P is PI, whence there is a central regular elements $x \in A/P$ of degree $d > 0$. The short exact sequence

$$0 \to N(-d) \to N \to N/Nx \to 0$$

yields a long exact sequence

$$\mathrm{Ext}^{-d}(N/Nx, R) \to \mathrm{Ext}^{-d}(N, R) \to \mathrm{Ext}^{-d}(N(-d), R) \to \mathrm{Ext}^{-d+1}(N/Nx, R).$$

Since N/Nx is finite dimensional, two ends are zero [Lemma 1.3(2)]. Since $\mathrm{Ext}^{-d}(N, R)$ is left bounded, a graded version of Nakayama lemma implies that $\mathrm{Ext}^{-d}(N, R)$ is zero. This contradicts to the fact $N \subset H^{-d}R$.

(3) Suppose $H^{-d}R$ is not A-torsionfree. Since $H^{-d}R$ is noetherian on both sides, there is a critical subbimodule $N \subset H^{-d}R$ such that (a) N_A is A-torsion, (b) $P := ann(N_A)$ is a nonzero graded prime ideal, and (c) N is A/P-torsionfree. By the hypothesis, $\mathrm{Kdim}\, N_A = \mathrm{Kdim}\, A/P \leq 1$. This contradicts to (2). □

If P is a graded prime ideal of A, let $E(A/P)$ denote the graded injective hull of A/P and $E_{A/P}$ denote the graded injective hull of a graded uniform right ideal of A/P.

LEMMA 1.7. *Let I be the minimal injective resolution of R as right A-module complex.*

(1) $I^0 = A'$.
(2) *Let B^{-i} be the image of $I^{-i-1} \to I^{-i}$. Then I^{-1}/B^{-1} is locally finite.*
(3) *If A is prime and* $\operatorname{Kdim} A \geq 2$*, then I^{-1} does not contain any shift of the injective module E_A.*
(4) *If P is a graded prime ideal of A such that* $\operatorname{Kdim} A/P \geq 2$*, then I^{-1} does not contain any shift of the injective module $E_{A/P}$.*

Proof. (1) By local duality (E1.1), $\operatorname{Ext}^0(k, R) = \Gamma_{\mathfrak{m}}(k)' = k$. Thus $I^0 = A' \oplus J$ where J is $\mathfrak{m}$-torsionfree. Suppose $I^0 \neq A'$, namely, $J \neq 0$. Then the inclusion $J \to I^0$ induces a nonzero element in $\operatorname{Ext}^0(J, R)$. Since J is $\mathfrak{m}$-torsionfree, by local duality $\operatorname{Ext}^0(J, R) = \Gamma_{\mathfrak{m}}(J)' = 0$, a contradiction.

(2) We have an exact sequence

$$0 \to H^{-1}R \to I^{-1}/B^{-1} \to I^0.$$

By (1) $I^0 = A'$ is locally finite. By the definition of dualizing complex, $H^{-1}R$ is noetherian and hence locally finite. The statement follows.

(3) Let $Q_{gr}(A)$ be the graded quotient ring of A. It is clear that $E(A)$ is isomorphic to $Q_{gr}(A)$ as left and as right graded A-module. If $E(A)$ is locally finite, then the Hilbert series of $E(A) = Q_{gr}(A)$ is periodic. Hence A has GKdim 1, which contradicts to the hypothesis. Therefore $E(A)$ is not locally finite. Since $E(A)$ is a finite direct sum of shifts of E_A, E_A is not locally finite.

Suppose E_A is a submodule of I^{-1}. Since A is prime and B^{-1} is A-torsion, E_A does not intersect with B^{-1}. Hence E_A is a submodule of I^{-1}/B^{-1}. By (2), E_A is locally finite, a contradiction.

(4) Let $R_{A/P}$ be the dualizing complex over A/P. By [**16**, 3.2 and 4.16], there is a morphism $\phi : R_{A/P} \to R$ in $D(\operatorname{Gr} A^e)$ such that

$$R_{A/P} \cong \operatorname{RHom}_A(A/P, R) \cong \operatorname{RHom}_{A^\circ}(A/P, R). \tag{E1.8}$$

Note that if M is an injective A-module, then $\operatorname{Hom}(A/P, M)$ is an injective A/P-module. Hence $I_{A/P} := \operatorname{Hom}(A/P, I)$ is a complex of injective A/P-modules. The minimality of the complex I implies that $I_{A/P}$ is a minimal complex of injective A/P-modules. By (E1.8), $I_{A/P}$ is a minimal injective resolution of $R_{A/P}$. Suppose I^{-1} contains injective A-module $E_{A/P}$. Then $I^{-1}_{A/P}$ contains injective A/P-module $E_{A/P}$, which contradicts with (3). □

2. Rings with cohomological dimension 2. By Lemma 1.5, if $\operatorname{cd} A = 1$ then A is PI of Kdim 1, which implies that R is Auslander. In this section we will discuss the case $\operatorname{cd} A = 2$. If $\operatorname{Kdim} A = 1$, then A is PI [Lemma 1.4]. By [**16**, 5.13 and 5.14], $\operatorname{cd} A = 1$. So $\operatorname{cd} A = 2$ implies $\operatorname{Kdim} A \geq 2$. Next is a consequence of Lemma 1.7(4).

PROPOSITION 2.1. *Suppose* $\operatorname{cd} A = 2$*. Then there are only finitely many graded primes $P \subset A$ with* $\operatorname{Kdim} A/P \geq 2$*. As a consequence A has a finite classical Krull dimension.*

Proof. Let I be the minimal injective resolution of R as right A-module complex. Then I^{-2} is the injective hull of $H^{-2}R$. Since $H^{-2}R$ is noetherian, its Goldie rank is finite and I^{-2} is a finite direct sum of indecomposable injectives. By Lemma 1.7(4) and [**16**, 1.11], if P is prime with $\operatorname{Kdim} A/P \geq 2$, then $E_{A/P}$ is a direct summand of I^{-2}. If $P \neq P'$, then $E_{A/P} \ncong E_{A/P'}$. Therefore the number of graded primes P with $\operatorname{Kdim} A/P \geq 2$ is no more than the Goldie rank of $H^{-2}R$.

The last assertion follows from [**9**, I.1.9]. □

LEMMA 2.2. *Let* $\operatorname{cd} A = 2$ *and let* M *be a noetherian graded right* A*-module.*

(1) *If* $\operatorname{Hom}(M, A/P) \neq 0$ *for a graded prime* P *with* $\operatorname{Kdim} A/P \geq 2$*, then* $\operatorname{Ext}^{-2}(M, R) \neq 0$.

(2) *Let* A *be prime. If* M *is* A*-torsion and every factor of* M *with* $\operatorname{Kdim} \geq 2$ *is* A*-faithful, then* $\operatorname{Ext}^{-2}(M, R) = 0$.

(3) *Let* A *be prime such that, for every nonzero graded prime* P*,* $\operatorname{Kdim} A/P \leq 1$. *If* M *is not* A*-torsion, then* $\operatorname{Ext}^{-2}(M, R)$ *is not* A*-torsion.*

Proof. (1) Since $\operatorname{Ext}^{-2}(M, R) = \operatorname{Hom}(M, H^{-2}R)$, it suffices to show that A/P as right A-module embeds to a finite direct sum of shifts of $H^{-2}R$. By Lemma 1.7(1,4), $E_{A/P}$ does not appear in I^0 and I^{-1}. Since $E_{A/P}$ appears in the minimal injective resolution of R [**16**, 1.11], it must appear in I^{-2}. Therefore $H^{-2}R$ contains a shift of a graded uniform right ideal of A/P, and hence A/P embeds to a finite direct sum of shifts of $H^{-2}R$.

(2) Let T be the A-torsion submodule of $H^{-2}R$ as right A-module. Then T is a subbimodule and hence noetherian on both sides. Since T_A is A-torsion, the right annihilator of T is not zero. Suppose $\operatorname{Ext}^{-2}(M, R) \neq 0$. Then there is a nonzero map $f: M \to T$. Thus the image of f has $\operatorname{Kdim} \geq 2$ and is not A-faithful.

(3) First we may replace M by a factor of M and assume M is a graded uniform right ideal of A. Then replace M by a finite direct sum of shifts of M we may assume M is an essential right ideal of A. The exact sequence

$$0 \to M \to A \to A/M \to 0$$

yields a long exact sequence

$$0 \to \operatorname{Ext}^{-2}(A/M, R) \to \operatorname{Ext}^{-2}(A, R) \to \operatorname{Ext}^{-2}(M, R) \to \cdots.$$

Since A/M is A-torsion, $\operatorname{Ext}^{-2}(A/M, R) = 0$ [Lemma 1.6(3)]. Since $\operatorname{Ext}^{-2}(A, R) = H^{-2}R$ is A-torsionfree [Lemma 1.6(3)], $\operatorname{Ext}^{-2}(M, R)$ is not A-torsion. □

LEMMA 2.3. *Suppose* $\operatorname{cd} A \leq 2$ *and* M *is a noetherian graded* A*-module.*

(1) *Let* $E^{p,q}(M) = \operatorname{Ext}^p(\operatorname{Ext}^q(M, R), R)$. *Then* $E^{p,q}(M) = 0$ *for* $(p, q) = (-2, 0)$*,* $(-1, 0)$*,* $(-1, -2)$*,* $(0, -2)$ *and* $(-2, -1)$.

(2) *If* $\operatorname{Ext}^{-2}(M, R) \neq 0$*, then* $\operatorname{Kdim} \operatorname{Ext}^{-2}(M, R) \geq 2$.

Proof. (1) We have a convergent spectral sequence (E1.2) and the E_2-page of (E1.2) is

$$\begin{matrix} E^{-2,0}(M) & E^{-1,0}(M) & E^{0,0}(M) \\ E^{-2,-1}(M) & E^{-1,-1}(M) & E^{0,-1}(M) \\ E^{-2,-2}(M) & E^{-1,-2}(M) & E^{0,-2}(M) \end{matrix}$$

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

with other terms being zero. The boundary maps in the E_r-page have degree $(r, r-1)$. For $(p, q) = (-2, 0), (-1, 0), (-1, -2), (0, -2)$,

$$E^{p,q}(M) = E_2^{p,q} = E_\infty^{p,q} = 0.$$

At $(p, q) = (-2, -1)$, we have $E_3^{-2,-1} = E_\infty^{-2,-1} = 0$. Therefore $E^{-2,-1}(M)$ is a submodule of $E^{0,0}(M)$ which is finite dimensional [Lemma 1.3(1)]. Let $N = \operatorname{Ext}^{-1}(M, R)$. Then (E1.2) for N shows that $E^{0,-2}(N) = 0$, which implies that $E^{-2}(N) = E^{-2,-1}(M)$ contains no nonzero $\mathfrak{m}$-torsion submodule. Therefore $E^{-2,-1}(M)$ must be zero.

(2) Let $N = \operatorname{Ext}^{-2}(M, R)$. By (1) $\operatorname{Ext}^p(N, R) = 0$ for $p = 0, -1$. If $\operatorname{Kdim} N < 2$, by Lemma 1.6(2) $\operatorname{Ext}^{-2}(N, R) = 0$. By (E1.2) $N = 0$. □

Proposition 2.4. *Suppose* $\operatorname{cd} A = 2$. *Then the following conditions are equivalent:*

(1) *for any chain of graded primes* $P' \subsetneq P \subset A$, $\operatorname{Kdim} A/P \leq 1$.

(2) *R is Auslander*.

(3) $\operatorname{Kdim} A = 2$.

(4) $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq 1$ *for all noetherian graded module M*.

Proof. (1) ⇒ (2) By [**16**, 4.18] we may assume A is prime. By Lemma 1.3(1), for any noetherian graded A-module M, $\operatorname{Ext}^0(M, R)$ is finite dimensional and by Lemma 1.3(2) $\operatorname{Ext}^q(N, R) = 0$ for all $N \subset \operatorname{Ext}^0(M, R)$ and for all $q < 0$. By the definition of cd, for all $N \subset \operatorname{Ext}^p(M, R)$ and for all $q < -\operatorname{cd} A$, $\operatorname{Ext}^q(N, R) = 0$. Hence it remains to show that $\operatorname{Ext}^{-2}(N, R) = 0$ for all $N \subset \operatorname{Ext}^{-1}(M, R)$. By Lemma 1.6(3), it suffices to show that $\operatorname{Ext}^{-1}(M, R)$ is A-torsion, which follows from Lemmas 2.2(3) and 2.3(1).

(2) ⇒ (3) By [**16**, 4.14], $\operatorname{Kdim} A \leq \operatorname{cd} A = 2$. So $\operatorname{Kdim} A = 2$.

(3) ⇒ (1) Clear, or see Lemma 3.3(2).

(3) ⇒ (4) Again we may assume A is prime. By the proof of (1) ⇒ (2), $\operatorname{Ext}^{-1}(M, R)$ is A-torsion. Hence $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq \operatorname{Kdim} A - 1 \leq 1$.

(4) ⇒ (2) By the prove of (1) ⇒ (2), it remains to show that $\operatorname{Ext}^{-2}(N, R) = 0$ for all $N \subset \operatorname{Ext}^{-1}(M, R)$. Since $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq 1$, $\operatorname{Kdim} N \leq 1$ and by Lemma 1.6(2), $\operatorname{Ext}^{-2}(N, R) = 0$. □

Remark 2.5. We can also prove the following, but tedious proofs are omitted. Suppose $\operatorname{cd} A = 2$. Then the following are equivalent:

(1) $\operatorname{Kdim} A = 2$.

(2) $\operatorname{Kdim} M \otimes_A P \leq 1$ for every graded primes P with $\operatorname{Kdim} A/P = 2$ and for all noetherian graded A-module M with $\operatorname{Kdim} M = 1$.

(3) $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq 1$ for every noetherian graded A-module M with $\operatorname{Kdim} M = 1$.

Lemma 2.6. *Suppose* $\operatorname{cd} A = 2$. *Let M be a graded A-bimodule noetherian on both sides*.

(1) *If* $\operatorname{Kdim} M = 1$, *then* $\operatorname{Ext}^{-2}(M, R) = 0$ *and* $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) = 1$.

(2) *If* $\operatorname{Kdim} M > 1$, *then* $\operatorname{Kdim} \operatorname{Ext}^{-2}(M, R) \geq 2$ *and* $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq 1$.

Proof. (1) By Lemma 1.6(2), $\operatorname{Ext}^{-2}(M, R) = 0$. By Lemma 1.3(3), $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) > 0$. So it remains to show $\operatorname{Kdim} \operatorname{Ext}^{-1}(M, R) \leq 1$. By noetherian

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

induction we may assume the bimodule M is right A/P-torsionfree for some graded ideal P and $\operatorname{Kdim} A/P = 1$. Hence $\operatorname{Ext}^{-1}(M.R)$ is a noetherian left A/P-module, which has Kdim ≤ 1.

(2) To prove $\operatorname{Kdim}\operatorname{Ext}^{-2}(M, R) \geq 2$ we may replace M by a factor of M and assume that every proper bimodule factor of M has Kdim no more than 1. Pick a nonzero subbimodule $M' \subset M$ such that M' is right A/P-torsionfree for some graded prime ideal P. Let $\overline{M} = M/M'$. Then $\operatorname{Kdim}\overline{M} \leq 1$. The short exact sequence

$$0 \to M' \to M \to \overline{M} \to 0$$

yields an exact sequence

$$0 \to \operatorname{Ext}^{-2}(\overline{M}, R) \to \operatorname{Ext}^{-2}(M, R) \to \operatorname{Ext}^{-2}(M', R) \to \operatorname{Ext}^{-1}(\overline{M}, R).$$

By (1) $\operatorname{Ext}^{-2}(\overline{M}, R) = 0$ and $\operatorname{Kdim}\operatorname{Ext}^{-1}(\overline{M}, R) \leq 1$. By Lemma 2.2(1), $\operatorname{Ext}^{-2}(M', R) \neq 0$ and by Lemma 2.3(2), $\operatorname{Kdim}\operatorname{Ext}^{-2}(M', R) > 1$. Hence $\operatorname{Kdim}\operatorname{Ext}^{-2}(M, R) \geq 2$. □

PROPOSITION 2.7. *Suppose* $\operatorname{cd} A = 2$.

(1) *Let* M *be a graded* A*-bimodule noetherian on both sides. If* $M \supset M_1 \supset M_2 \supset \cdots$ *is a descending chain of subbimodules, then* $\operatorname{Kdim} M_i/M_{i+1} \leq 1$ *for* $i \gg 0$. *As a consequence* $\operatorname{Kdim}_{A^e} M \leq 2$.

(2) $\operatorname{Kdim}_{A^e} A \leq 2$.

Proof. (2) is a special case of (1).

(1) Let $N_i = M/M_i$ and let $M_i' = M_i/M_{i+1}$. Suppose $\operatorname{Kdim} M_i' > 1$ for all i. The exact sequence

$$0 \to M_i' \to N_{i+1} \to N_i \to 0$$

yields an exact sequence

$$0 \to \operatorname{Ext}^{-2}(N_i, R) \to \operatorname{Ext}^{-2}(N_{i+1}, R) \to \operatorname{Ext}^{-2}(M_i', R) \to \operatorname{Ext}^{-1}(N_i, R) \to \cdots.$$

By Lemma 2.6, $\operatorname{Kdim}\operatorname{Ext}^{-1}(N_i, R) \leq 1$ and $\operatorname{Kdim}\operatorname{Ext}^{-2}(M_i', R) \geq 2$. Hence the injective map $\operatorname{Ext}^{-2}(N_i, R) \to \operatorname{Ext}^{-2}(N_{i+1}, R)$ is not surjective for all i. Thus $\{\operatorname{Ext}^{-2}(N_i, R) \mid i \geq 0\}$ is an ascending chain of submodules of $\operatorname{Ext}^{-2}(M, R)$, which contradicts to the fact $\operatorname{Ext}^{-2}(M, R)$ is noetherian. □

Proposition 0.5 follows from Corollary 2.1 and Proposition 2.7.

REMARK 2.8. (1) We don't know whether or not $\operatorname{Kdim} A = 2$ implies $\operatorname{GKdim} A = 2$.

(2) If A is PI or FBN or more generally satisfies similar submodule condition, then R is Auslander and $\operatorname{Kdim} A = \operatorname{GKdim} A = \operatorname{cd} A$ [**16**, 5.13 and 5.14].

3. Graded dimension function.

DEFINITION 3.1. A **graded dimension function** is a map ∂ sending finitely generated right (and left) graded modules over a connected graded ring to the set of $-\infty$, all real numbers, and all ordinals $\geq \omega$ satisfying the following properties:

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

(d0) $\partial 0 = -\infty$, and $\partial M = 0$ if and only if M is finite dimensional;
(d1) If M is infinite dimensional, then $\partial M \geq 1$;
(d2) $\partial M = \max\{\partial(M/N), \partial N\}$ for any $N \subset M$ (this property is called **exactness**);
(d3) If x is a regular element of $M = A/P$ for some prime ideal $P \subset A$, then $\partial M \geq \partial(M/xM) + 1$;
(d4) $\partial M = \partial(M(1))$ where $M(1)$ is the degree shift of M.

The graded dimension function is slightly different from the dimension function in the ungraded case (see [**8**, Sect. 6.8.4] or [**16**, 2.4]) because we add some natural conditions (d0), (d1) and (d4) concerning graded modules. Given a finitely generated graded module M, Kdim M in the graded module category is equal to Kdim M in the ungraded module category. Using this fact one can easily check that Krull dimension is also a graded dimension function. For Gelfand-Kirillov dimension, (d0,d1,d3,d4) are clear and the next lemma shows that (d2) is true.

Lemma 3.2. *Let A be a right noetherian connected ring and let M be a finitely generated right graded A-module. Then*
(1) $\text{GKdim} M = \lim \log_n(\sum_{i \leq n} \dim_k M_i)$.
(2) $\text{GKdim} M = \max\{\text{GKdim} M/N, \text{GKdim} N\}$ *for any graded submodule* $N \subset M$.
(3) *If* $f : M(-d) \to M$ *is an injective map for some integer* $d > 0$, *then* $\text{GKdim} M/f(M) = \text{GKdim} M - 1$.

Proof. (1) This follows from the facts that A is a finitely generated algebra and that M is a finitely generated right A-module.

(2) By [**8**, 8.3.2(ii)], $\text{GKdim} M \geq \max\{\text{GKdim} M/N, \text{GKdim} N\}$. Let α be any number bigger than $\max\{\text{GKdim} M/N, \text{GKdim} N\}$. Then there is a constant C such that

$$\sum_{i \leq n} \dim_k (M/N)_i \leq Cn^\alpha \qquad \text{and} \qquad \sum_{i \leq n} \dim_k N_i \leq Cn^\alpha$$

for $n \gg 0$. Thus $\sum_{i \leq n} \dim_k M_i \leq 2Cn^\alpha$ for $n \gg 0$. By (1) $\text{GKdim} M \leq \alpha$.

(3) By [**8**, 8.3.5], $\text{GKdim} M/f(M) \leq \text{GKdim} M - 1$. To prove the opposite inequality let α be any number bigger than $\text{GKdim} M/f(M)$. Let $\dim_k (M/f(M))_i = h(i)$. Then there is a constant C such that $\sum_{j \leq i} h(j) \leq Ci^\alpha$ for all $i > 0$. Let $\dim_k M_i = g(i)$. Then

$$g(i) = h(i) + h(i-d) + h(i-2d) + \cdots \leq \sum_{j \leq i} h(j) \leq Ci^\alpha,$$

whence $\sum_{j \leq i} g(j) \leq C' i^{\alpha+1}$ for some constant C' and for $i > 0$. Therefore $\text{GKdim} M \leq \alpha + 1$. □

Lemma 3.3. *Let ∂ be a graded dimension function.*
(1) *If* $\partial(A_A) < 2$, *then A is PI of* Kdim ≤ 1.
(2) *If* $\partial(A_A) < 3$, *then, for any chain of graded primes* $P' \subsetneq P \subset A$, Kdim $A/P \leq 1$.

Proof. (1) We may assume A is prime and not k. Since A is noetherian, by [**9**, I.1.6], there is a homogeneous regular element $x \in A$ of positive degree. Then $\partial(A/xA) \leq \partial A - 1 < 1$. Hence $\partial(A/xA) = 0$ and A/xA is finite dimensional. Therefore $\text{GKdim} A = 1$ and A is PI of Kdim 1 [Lemma 1.4].

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

(2) Since P/P' is a nonzero prime ideal of A/P', $\partial(A/P') < 3$ implies $\partial(A/P) < 2$. Then the statement follows from (1) . □

Let R be a balanced dualizing complex. The canonical dimension is defined by

$$\mathrm{Cdim}\, M = -\min\{i \mid \mathrm{Ext}^i(M, R) \neq 0\}.$$

Note that $\mathrm{cd} A = \mathrm{Cdim}\, A$. If R is Auslander, then Cdim is a dimension function for ungraded modules [**16**, 2.10]. To verify Cdim is a graded dimension function, we need to check (d0), (d1) and (d4). But (d0) and (d1) follow from Lemma 1.3(2,3) and (d4) is clear for Cdim. Therefore we have:

PROPOSITION 3.4. *If R is Auslander, then* Cdim *is a graded dimension function.*

Proof of Theorem 0.1. By Proposition 2.4, (1), (2) and (4) are equivalent.

(2) ⇒ (3) is clear and (3) ⇒ (4) is Lemma 3.3(2).

(1) ⇒ (5) is Proposition 3.4.

(5) ⇒ (7) By [**16**, 4.14], $\mathrm{Kdim}\, M \leq \mathrm{Cdim}\, M$. It remains to show $\mathrm{Cdim}\, M \leq \mathrm{Kdim}\, M$. This is clear when $\mathrm{Kdim}\, M = 0$ or 2. So we only consider the case when $\mathrm{Kdim}\, M = 1$. Since both dimension functions are exact, we may use noetherian induction on M. So we may assume M is critical faithful over A/P for some graded prime P. There are two cases. Case 1 is when $\mathrm{Kdim}\, A/P = 2$. Since $\mathrm{Kdim}\, A = 2$, P is minimal. Hence M is A/P-torsion. Thus $\mathrm{Cdim}\, M \leq \mathrm{Cdim}\, A/P - 1 \leq 1$. Case 2 is when $\mathrm{Kdim}\, A/P \leq 1$. Then by Lemma 3.3(1), A/P is PI and of $\mathrm{GKdim} \leq 1$. Hence $\mathrm{Cdim}\, A/P \leq 1$ and hence $\mathrm{Cdim}\, M \leq 1$.

(7) ⇒ (6) Clear.

(6) ⇒ (1) By the proof of Proposition 2.4, it suffices to show $\mathrm{Ext}^{-2}(N, R) = 0$ for all submodules $N \subset \mathrm{Ext}^{-1}(M, R)$ and for all noetherian graded module M. By Lemma 2.3(1) (for $(p, q) = (-2, -1)$), $\mathrm{Cdim}\, \mathrm{Ext}^{-1}(M, R) < 2$. Since Cdim is exact, $\mathrm{Cdim}\, N < 2$, i.e., $\mathrm{Ext}^{-2}(N, R) = 0$. □

4. AS-Gorenstein rings of injective dimension 2. By [**14**, 4.14] a noetherian AS-Gorenstein ring A of injective dimension d has a balanced dualizing complex $R = A^{\sigma}[-d](-e)$ for some graded automorphism σ of A and some integer e in the Definition 0.3(1).

PROPOSITION 4.1. *If A is AS-Gorenstein of injective dimension 2 and has an artinian quotient ring (with respect to the set of the regular elements of A), then A is Auslander-Gorenstein.*

Note that Proposition 4.1 is [**7**, 5.13] without the hypothesis that $\mathrm{GKdim} A = 2$. It also follows from Theorem 0.4 that the artinian quotient ring of A is in fact self-injective.

Proof. By Theorem 0.1, it suffices to show that, for any chain of graded primes $P' \subsetneq P \subset A$, $\mathrm{Kdim}\, A/P \leq 1$. Suppose this is not true, namely, there are graded primes $P' \subsetneq P \subset A$ such that $\mathrm{Kdim}\, A/P > 1$. By Lemma 2.2(1),

$$\mathrm{Hom}(A/P, A) \cong \mathrm{Ext}^{-2}(A/P, R(e)) \cong \mathrm{Ext}^{-2}(A/P, R)(e) \neq 0$$

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

because $R = A^{\sigma}[-2](-e)$. Let Q be the artinian quotient ring of A. Then $\mathrm{Hom}(A/P, A) \neq 0$ implies that $A/P \otimes_A Q \neq 0$. Next we show that this contradicts to the fact P is not a minimal prime. By [**8**, 4.1.3(iv)], $A/N(A) \otimes_A Q = Q/N(Q) := Q'$ where $N(-)$ is the prime radical of $-$, and $Q/Q(N)$ is the artinian quotient ring of $A/N(A)$. Since P is not a minimal prime, $P/N(A)$ is an essential (left and right) ideal of $A/N(A)$. By Goldie's theorem, $Q'/PQ' = 0$. Hence

$$A/P \otimes_A Q = Q/PQ = Q'/PQ' = 0.$$

This contradicts to $A/P \otimes_A Q \neq 0$. □

Let ∂ be a (graded) dimension function. A nonzero (graded) module N is called *p-pure* with respect to ∂ if $\partial M = p$ for all nonzero noetherian (graded) submodule $M \subset N$. In most cases we will take $\partial = \mathrm{Cdim}$. Let R be the balanced dualizing complex over A and let ∂ be a graded dimension function. Following Yekutieli's definition [**15**, 2.3], R is called *residue complex* over A (with respect to ∂) if (i) each A-bimodule R^q is graded injective on both sides and (ii) each A-bimodule R^q is pure of ∂-dimension $-q$ on both sides. Note that in [**15**] this is called strong residue complex, and later in [**17**], this is called residue complex. By [**15**, 2.6], if R is a residue complex over A with respect to ∂, then R is Auslander and $\partial M = \mathrm{Cdim}\, M$ for all noetherian graded modules M.

We are ready to prove Theorem 0.4.

Proof of Theorem 0.4. First of all Theorem 0.4(1) is equivalent to Theorem 0.1(1) for AS-Gorenstein rings.

(1) ⇒ (2) follows from [**16**, 6.23.3], (2) ⇒ (3) is clear and (3) ⇒ (1) is Proposition 4.1.

(1) ⇒ (4) is [**16**, 6.23.2] and (4) ⇒ (1) follows from the implication (2) ⇒ (1) of Theorem 0.1.

(5) ⇒ (1) is [**15**, 2.6].

(1) ⇒ (5) By Theorem 0.1(6) $\mathrm{Cdim}\, M = \mathrm{Kdim}\, M$ for graded noetherian modules M. (Note that $\mathrm{Cdim}\, M$ and $\mathrm{Kdim}\, M$ could be different for ungraded module M though both dimensions are well-defined.) Hence purity with respect to Cdim is equivalent to purity with respect to Kdim.

Take the minimal injective resolution of the left A-module complex $R = A^{\sigma}[-2](-e)$, say

$$\cdots 0 \to I^{-2} \to I^{-1} \to I^{0} \to 0 \cdots. \tag{E4.2}$$

By Lemma 1.7(1), $I^0 = A'$ which is 0-pure. Let Q be the (ungraded) artinian quotient ring of A. Since I^{-2} (as ungraded module) is a submodule of Q and since Q is 2-pure as an ungraded module, I^{-2} is 2-pure. Since Q/A is A-torsion, so is $I^{-2}/A(-e)$. Thus $\mathrm{Cdim}\, N \leq 1$ for all noetherian submodule $N \subset I^{-2}/A(-e)$. Since A is AS-Gorenstein, $I^{-2}/A(-e) \subset I^{-1}$ contains no finite dimensional submodules. Therefore $I^{-2}/A(-e)$ is 1-pure. Since the complex (E4.2) is exact at I^{-1} and since I^0 is 0-pure, I^{-1} is 1-pure. Similarly, the minimal injective resolution of the right A-module complex R has a pure resolution. By [**17**, 3.8], the Cousin complex ER of R is a residue complex, namely, ER is a balanced dualizing complex in $D^b_{fg}(\mathrm{Gr}A^e)$ which is a pure minimal injective resolution on both sides.

https://doi.org/10.1017/S0017089500030123 Published online by Cambridge University Press

(5) $\Rightarrow$ (6) By (5) A is 2-pure with respect to Cdim and Kdim.

(6) $\Rightarrow$ (1) follows from Theorem 0.1 because Kdim $A = 2$. □

REMARK 4.3. Part (5) is a generalization of a result of Ajitabh [**1**, 3.12], which proves the existence of residue complexes for AS-regular algebras of global dimension 2.

COROLLARY 4.4. *Let A be AS-Gorenstein of injective dimension* 2. *If ∂ is a graded dimension function such that $\partial\, A \le 2$ then*

$$\partial\, M = \operatorname{Kdim} M = \operatorname{Cdim} M$$

for all noetherian graded left and right A-modules M.

Proof. First of all, by Theorem 0.1, A is Auslander-Gorenstein and $\operatorname{Cdim} M = \operatorname{Kdim} M$. So it remains to show that $\partial M = \operatorname{Kdim} M$.

Let P be a minimal prime of A. By Theorem 0.4(4) $\operatorname{Kdim} A/P = 2$, whence $A/(P + xA)$ is not finite dimensional over k for any x of positive degree. Pick $\bar{x} := x + P$ a homogeneous regular element of A/P of positive degree. Then $\partial A/P \ge \partial A/(P + xA) + 1 \ge 2$. Thus $\partial A = \partial(A/P) = 2$ for all minimal prime P. Since both Kdim and ∂ are exact we may use noetherian induction and may assume M is a critical module and faithful over A/P for a graded prime P. If $P = \mathfrak{m}$, then M is k and hence both dimensions are 0. If M is infinitely dimensional and either P is not minimal or P is minimal and M is A/P-torsion, then both dimensions are 1 (by (d1) and (d3)). The last possibility is when M is a right ideal of A/P when P is minimal. In this case, $\operatorname{Kdim} M = \operatorname{Kdim} A/P$ and $\partial M = \partial A/P$. We have already shown that both dimensions on A/P are equal to 2. That completes our proof. □

ACKNOWLEDGEMENTS. This research was finished during the first author's visit to the Department of Mathematics at the University of Washington supported by a research fellowship from the China Scholarship Council and he thanks these two institutions for the hospitality and the support. The second author was supported in part by the NSF and a Sloan Research Fellowship.

REFERENCES

1. K. Ajitabh, Residue complex for regular algebras of dimension 2, *J. Algebra* **179** (1996), No. 1, 241–260.

2. M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in *The Grothendieck Festschrift, Vol. I*, (ed. P. Cartier et al.), (Birkhäuser, 1990), 33–85.

3. M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), No. 2, 335–388.

4. M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994), 228–287.

5. E. K. Ekström, *The Auslander condition on graded and filtered Noetherian rings*. Lecture Notes in Mathematics, No. 1404, (Springer-Verlag, 1989), 220–245,

6. G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised Edition, Graduate Studies in Mathematics, Vol. 22 (AMS, Providence, Rhode Island, 1999).

7. T. Levasseur, Some properties of noncommutative regular rings, *Glasgow Math. J.* **34** (1992), 277–300.

8. J. C. McConnell and J. C . Robson, *Noncommutative Noetherian rings* (Wiley, 1987).

9. C. Năstăsescu and F. van Oystaeyen, *Graded ring theory* (North-Holland Publishing Co., 1982).

10. L. W. Small and R. B. Warfield Jr., Prime affine algebras of Gelfand-Kirillov dimension one, *J. Algebra* **91** (1984), No. 2, 386–389.

11. S. P. Smith and J. T. Stafford, Regularity of the four-dimensional Sklyanin algebra, *Compositio Math.* **83** (1992), No. 3, 259–289.

12. J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, *Invent. Math.* **124** (1996), No. 1–3, 619–647.

13. M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, *J. Algebra* **195** (1997), No. 2, 662–679.

14. A. Yekutieli, Dualizing complexes over noncommutative graded algebras, *J. Algebra* **153** (1992), 41–84.

15. A. Yekutieli, The residue complex of a noncommutative graded algebra, *J. Algebra* **186** (1996), 522–543.

16. A. Yekutieli and J. J. Zhang, Rings with Auslander dualizing complexes, *J. Algebra* **213** (1999), No. 1, 1–51.

17. A. Yekutieli and J. J. Zhang, Residual complex over noncommutative rings, preprint.