A NONSELFADJOINT DYNAMICAL SYSTEM

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1. Introduction

This paper concerns criteria for assuring that every solution of a real fourth order nonselfadjoint differential equation

$$lu \equiv (p_2(x)u''-q_2(x)u')''-(p_1(x)u'-q_1(x)u)'+p_0(x)u = 0 \quad (p_2(x)>0) \tag{1.1}$$

is oscillatory at $x = \infty$. Our technique is a generalisation of that used by Whyburn (**1**) for the study of the selfadjoint equation,

$$Lv \equiv (P_2(x)v'')''-(P_1(x)v')'+P_0(x)v = 0 \quad (P_2(x)>0) \tag{1.2}$$

combined with the theory of H-oscillation of vector equations as introduced by Domšlak (**2**) and studied by Noussair and Swanson (**3**). Whyburn's technique consists of representing (1.2) as a dynamical system of the form

$$\begin{aligned} y'' &= a(x)y+b(x)z \quad (b(x)>0) \\ z'' &= c(x)y+a(x)z \end{aligned} \tag{1.3}$$

and then studying (1.3) in terms of polar coordinates in the y, z-plane. In Section 2 below we show how to represent (1.1) as a dynamical system of the form

$$\begin{aligned} y'' &= a(x)y+b(x)z \quad (b(x)>0) \\ z'' &= c(x)y+d(x)z. \end{aligned} \tag{1.4}$$

This system is studied in terms of polar coordinates in Section 3, and in Section 4 a comparison theorem for the H-oscillation of solutions of (1.4) is established. This comparison theorem yields oscillation criteria for (1.1) which are presented in Section 5, and non-oscillation criteria are discussed briefly in Section 6.

It is assumed throughout that the coefficients $p_k(x)$ and $q_k(x)$ are real, of class C^k, and that $p_2(x)>0$ in an interval $\mathscr{I} = [\alpha, \infty)$.

2. A representation of nonselfadjoint equations

The representation of (1.1) in the form (1.4) will be accomplished in two steps.

Lemma 2.1. *If l is defined by* (1.1) *and*

$$U(x) = u(x)\exp\left[-\int_\alpha^x \frac{q_2(t)}{2p_2(t)}\,dt\right] \tag{2.1}$$

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then

$$lu = \left(p_2(x)\exp\left[\int_\alpha^x \frac{q_2(t)}{2p_2(t)}\,dt\right]U''\right)'' + (\text{terms of order} \leqq 2).$$

Proof. If $u(x) = R(x)U(x)$, then

$$p_2u'' - q_2u' = p_2RU'' + (2p_2R' - q_2R)U' + (p_2R' - q_2R')U.$$

Choosing

$$R(x) = \exp\left[\int_\alpha^x \frac{q_2(t)}{2p_2(t)}\,dt\right]$$

yields

$$lu = (p_2RU'')'' + (\text{terms of order} \leqq 2)$$

as was to be shown.

Since the transformation (2.1) is oscillation preserving, we are justified in restricting our attention to nonselfadjoint equations of the form

$$lu \equiv (p_2(x)u'')'' - (p_1(x)u')' + q_1(x)u' + p_0(x)u = 0. \tag{2.2}$$

Theorem 2.2. *The equation* (2.2) *can be represented in the form*

$$\begin{aligned} y'' &= a(x)y + b(x)z \\ z'' &= c(x)y + d(x)z \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} a &= \frac{p_1 - \int_\alpha^x q_1}{2p_2} \\ b &= \frac{1}{p_2} \\ c &= \frac{p_1^2 - \left(\int_\alpha^x q_1\right)^2}{4p_2} - \frac{p_1'' - q_1'}{2} - p_0 \\ d &= \frac{p_1 + \int_\alpha^x q_1}{2p_2}. \end{aligned} \tag{2.4}$$

Proof. Setting $y(x) = u(x)$, $y'(x) = u'(x)$ it follows that $z = \frac{1}{b}u'' - \frac{a}{b}u$ and that

$$z'' = \left(\frac{1}{b}u''\right)'' - \frac{a}{b}u'' - 2\left(\frac{a}{b}\right)' u' - \left(\frac{a}{b}\right)'' u.$$

The choice $b = 1/p_2$ implies that

$$\left(\frac{1}{b}u''\right)'' = (p_1u')' - q_1u' - p_0u$$

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so that

$$z'' = \left(p_1 - \frac{a}{b}\right)u'' + \left(p_1' - q_1 - 2\left(\frac{a}{b}\right)'\right)u' - \left(p_0 + \left(\frac{a}{b}\right)''\right)u.$$

Choosing $2\left(\frac{a}{b}\right)' = p_1' - q_1$ yields

$$a = \frac{p_1 - \int q_1}{2p_2}$$

so that

$$z'' = \left(p_1 - \frac{p_1 - \int q_1}{2}\right)u'' - \left(p_0 + \frac{p_1'' - q_1'}{2}\right)u.$$

Making use of the fact that $u'' = y'' = ay + bz$ we get

$$\begin{aligned} z'' &= \tfrac{1}{2}\left(\int q_1 + p_1\right)\left(\frac{p_1 - \int q_1}{2p_2}\,y + \frac{1}{p_2}\,z\right) - \left(p_0 + \frac{p_1'' - q_1'}{2}\right)y \\ &= \left(\frac{p_1^2 - (\int q_1)^2}{4p_2} - \frac{p_1'' - q_1'}{2} - p_0\right)y + \frac{p_1 + \int q_1}{2p_2}\,z \end{aligned}$$

which completes the proof.

Since the equations (2.4) can be inverted to yield

$$\begin{aligned} p_2 &= \frac{1}{b} \\ p_1 &= \frac{a+d}{b} \\ q_1 &= \left(\frac{d-a}{b}\right)' \\ p_0 &= \frac{ad}{b} - \left(\frac{a}{b}\right)' - c \end{aligned} \tag{2.5}$$

it also follows that every dynamical system of the form (1.4), with $b(x) > 0$, can be represented in the form (2.2) with $p_2(x) > 0$.

3. The dynamical system

In this section the system

$$\begin{aligned} y'' &= a(x)y + b(x)z \\ z'' &= c(x)y + d(x)z \end{aligned} \tag{3.1}$$

will be studied in terms of the polar coordinates

$$\begin{aligned} r &= \sqrt{y^2 + z^2} \\ \theta &= \arctan y/z. \end{aligned} \tag{3.2}$$

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It is assumed that the coefficients of (3.1) are continuous and that $b(x) > 0$ in $\mathscr{I} = [\alpha, \infty)$. The polar representation of (3.1) is easily shown to be

$$r'' = r(\theta')^2 + \frac{1}{r} Q_1(y, z) \tag{3.3}$$

$$(r^2\theta')' = Q_2(y, z) \tag{3.4}$$

where Q_1 and Q_2 are quadratic forms defined by

$$Q_1(y, z) = ay^2 + (b+c)yz + dz^2$$
$$Q_2(y, z) = -cy^2 + (a-d)yz + bz^2.$$

Equations (3.3), (3.4) are equivalent to (3.1) on any interval $\mathscr{I}$ on which $y^2(x) + z^2(x) > 0$. The singularities which occur at the zeros of $y^2 + z^2$ can be eliminated by means of the following device used by Taam (**4**) in the special case $a(x) \equiv d(x)$, $c(x) \equiv -b(x)$. Let $\{x_k\}$ denote the zeros of $y^2 + z^2$ in $\mathscr{I}$, and define $\tilde{r}(x)$ by

$$\begin{aligned} \tilde{r}(x) &= r(x) && \text{if} \quad x_{2k} \leqq x \leqq x_{2k+1} \\ &= -r(x) && \text{if} \quad x_{2k-1} \leqq x \leqq x_{2k} \end{aligned}$$

and $\phi(x)$ by

$$\begin{aligned} \phi(x) &= \theta'(x) && \text{if} \quad x \neq x_k \\ &= 0 && \text{if} \quad x = x_k. \end{aligned}$$

Then by means of several applications of L'Hospital's rule at the $\{x_k\}$ it can be shown that $\tilde{r}(x)$ and $\phi(x)$ satisfy

$$\tilde{r}'' = \tilde{r}\phi^2 + \frac{1}{\tilde{r}} Q_1(y, z) \tag{3.3$'$}$$

$$(\tilde{r}^2\phi)' = Q_2(y, z) \tag{3.4$'$}$$

in $\mathscr{I}$.

Another means of dealing with the zeros of $y^2(x) + z^2(x)$ is to make further assumptions on the coefficients of (3.1) which assure the existence of an interval $\mathscr{I} = [\tilde{\alpha}, \infty)$ in which $r^2(x) = y^2(x) + z^2(x) > 0$.

Theorem 3.1. *If the quadratic form Q_1 is nonnegative definite in $[\alpha, \infty)$, then for any nontrivial solution y, z of* (3.1), *rr' can have at most one zero in* $[\alpha, \infty)$.

Proof. For $r(x) > 0$ we have

$$\begin{aligned} \frac{d}{dx}(rr') &= \frac{d}{dx}(yy' + zz') \\ &= y'^2 + z'^2 + Q_1(y, z). \end{aligned}$$

If $\lim_{x \downarrow \gamma} (rr')(x) = 0$ and $\lim_{x \uparrow \delta} (rr')(x) = 0$, then

$$0 = rr' \big|_\gamma^\delta \geqq \int_\gamma^\delta Q_1(y, z)dx$$

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with equality if and only if $y(x) \equiv 0$ and $z(x) \equiv 0$ in $[\gamma, \delta]$. The nonnegative definiteness of Q_1 provides the desired contradiction.

Theorem 3.2. *If the quadratic form $Q_2(y, z)$ is positive definite (negative definite) in $[\alpha, \infty)$, then $r^2\theta'$ can have at most one zero in $[\alpha, \infty)$.*

Proof. Starting with (3.4), the proof is analogous to that of Theorem 3.1.

In the special case where $a(x) \equiv d(x)$ and $b(x) \equiv -c(x)$, (3.1) is equivalent to the single complex equation

$$w'' = (a - ib)w \tag{3.5}$$

and has a dynamical interpretation in terms of a particle of unit mass moving in the y, z-plane under the influence of a central force ar and a transverse force br acting perpendicular to the radius vector. In this case Theorems 3.1 and 3.2 follow from the Green's transform used by Hille (**5**) to study the behaviour of solutions of (3.5) in the complex plane. In terms of (3.5) they reduce to the statement that

(i) if $a(x) \geqq 0$ in $[\alpha, \infty)$ then for any nontrivial solution $w(x)$ of (3.5) $\mathrm{Re}\,[\bar{w}w']$ has at most one zero in $[\alpha, \infty)$.

(ii) if $b(x) > 0$ $(b(x) < 0)$ in $[\alpha, \infty)$ then for any nontrivial solution $w(x)$ of (3.5) $\mathrm{Im}\,[\bar{w}w']$ has at most one zero in $[\alpha, \infty)$.

In (**1**) Whyburn studies special cases of (3.1) such as

$$a \equiv d \equiv 0 \quad \text{and} \quad bc < 0$$

or

$$a \equiv d > 0,\ a' > 0 \quad \text{and} \quad bc < 0$$

which assure that Q_2 is positive definite. In order to apply these techniques to nonselfadjoint problems, it is important to be able to replace these conditions with the more general condition on Q_2.

Using Taam's transformation and the equation (3.3)′, Theorem 3.1 can be strengthened. Let $\lambda(x)$ be the smallest eigenvalue of the matrix

$$\begin{pmatrix} a & \dfrac{b+c}{2} \\ \dfrac{b+c}{2} & d \end{pmatrix}$$

so that

$$Q_1(y, z) = r^2 Q_1\left(\frac{y}{r}, \frac{z}{r}\right) \geqq \lambda(x) r^2.$$

Then (3.3)′ becomes

$$\tilde{r}\tilde{r}'' \geqq \lambda(x)\tilde{r}^2 \tag{3.3$''$}$$

and leads to the following

Theorem 3.3. *If the equation $u'' = \lambda(x)u$ is disconjugate in $[\alpha, \infty)$ then for any nontrivial solution y, z of* (3.1) *$y^2(x) + z^2(x)$ has at most one zero in $[\alpha, \infty)$.*

Proof. Suppose to the contrary that $\tilde{r}(\gamma) = \tilde{r}(\delta) = 0$ for $\alpha \leqq \gamma < \delta < \infty$ and some nontrivial solution $\tilde{r}(x)$ of (3.3)″. Since $u'' = \lambda(x)u$ is disconjugate on $[\alpha, \infty)$, we can choose a solution $u(x)$ which is positive in $(\alpha, \delta]$. Then for $\gamma < x \leqq \delta$

$$\frac{d}{dx}\left[\tilde{r}\tilde{r}' - \frac{\tilde{r}^2 u'}{u}\right] \geqq \left[\tilde{r}' - \frac{\tilde{r}u'}{u}\right]^2,$$

the right side being identically zero if and only if $\tilde{r}$ is a constant multiple of u. Using an appropriate limiting procedure at $x = \gamma$ (in case $\gamma = \alpha$), an integration from γ to δ yields the desired contradiction,

$$0 = \left[\tilde{r}\tilde{r}' - \frac{\tilde{r}^2 u'}{u}\right]_{x=\gamma}^{x=\delta} \geqq \int_\gamma^\delta \left[\tilde{r}' - \frac{\tilde{r}u'}{u}\right]^2 dx.$$

We have shown that there is a variety of criteria for assuring that

$$y^2(x) + z^2(x) > 0$$

in an interval of the form $[\alpha, \infty)$. However, the criteria of Theorem 3.2 will be especially important in the present paper because of the following additional consequence of the definiteness of the quadratic form Q_2.

Theorem 3.4. *If the quadratic form Q_2 is positive definite (negative definite) in $[\alpha, \infty)$, then for any nontrivial solution (y, z) of (3.1), $\theta(x)$ is monotone in an interval of the form $[\tilde{\alpha}, \infty)$.*

Proof. By Theorem 3.2 $y^2(x) + z^2(x)$ has at most one zero in $[\alpha, \infty)$, and we may therefore assume that $r^2(x)$ is positive. If $\theta'(x)$ were oscillatory near ∞, then $r^2\theta'$ would be oscillatory, contradicting (3.4). Therefore $\theta'(x)$ is of constant sign in an interval of the form $[\tilde{\alpha}, \infty)$ as was to be shown.

The quantity $r^2\theta'$ may be interpreted as the rate at which area is being swept out by the radius vector in the y, z-plane in a counterclockwise direction. The definiteness of Q_2 assures that this quantity is eventually nonzero and of constant sign; for (3.1) Q_2 is positive definite if

$$\begin{gathered} b > 0;\ c < 0 \\ 4bc + (a-d)^2 < 0. \end{gathered} \tag{3.6}$$

Clearly the condition of definiteness in Theorem 3.4 may be relaxed to semi-definiteness in part of $\mathscr{I}$.

4. A rotation criterion

A solution y, z of (3.1) will be called *rotary* at $x = \infty$ if $r(x) > 0$ for sufficiently large x and $\lim_{x\to\infty} |\theta(x)| = \infty$. If the quadratic form Q_2 is definite near $x = \infty$, then according to Theorems 3.2 and 3.4, $\lim_{x\to\infty} \theta(x)$ exists for any non-rotary solution y, z of (3.1). For this reason it will be assumed throughout this section that Q_2 is positive definite in $\mathscr{I} = [\alpha, \infty)$.

In order to establish specific rotation criteria we shall employ the concept of H-oscillation introduced by Domšlak (2) and studied by Swanson and Noussair (3). Our technique is closely related to that of (3), but the application of this technique is different.

Writing (3.1) in the form

$$Y'' = A(x)Y \tag{4.1}$$

where

$$Y(x) = \begin{pmatrix} y(x) \\ z\ x) \end{pmatrix}; \quad A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

we shall say that a nontrivial solution $Y(x)$ of (4.1) is *H-oscillatory* if $\langle H, Y(x)\rangle$ is oscillatory at $x = \infty$. Here H is to be a real nonzero constant vector, $H = (h_1, h_2)$, and $\langle H, Y(x))\rangle = h_1 y(x) + h_2 z(x)$. In view of Theorem 3.4, if a solution $Y(x)$ of (4.1) is not H-oscillatory for some (every) H, then $\lim_{x\to\infty} \theta(x)$ exists. Denoting this limit by θ_0, we shall obtain rotation criteria for (4.1) by choosing $H = (\cos\theta_0, \sin\theta_0)$, i.e. H will be a unit vector in the direction of θ_0.

Our rotation criteria will be formulated in terms of a continuous function $\lambda(x)$ satisfying

$$\lambda(x) \geqq \sup_{\langle H, H\rangle = 1} \langle H, A(x)H\rangle.$$

Theorem 4.1. *If $A(x)$ is continuous and there exists an $\varepsilon > 0$ such that $u'' = (\lambda(x)+\varepsilon)u$ is oscillatory at $x = \infty$, then every solution of* (4.1) *is rotary at $x = \infty$.*

Proof. If $Y(x)$ is a non-rotary solution, then the vector $\operatorname{col}(y(x), z(x))$ tends to a limiting direction $\theta_0 = \lim_{x\to\infty} \theta(x)$. Choosing $H = (\cos\theta_0, \sin\theta_0)$ it follows that we can choose α sufficiently large so that $\langle H, Y(x)\rangle$ is positive and $\dfrac{\langle H, A(x)Y(x)\rangle}{\langle H, Y(x)\rangle} \leqq \lambda(x) + \frac{1}{2}\varepsilon$ in $[\alpha, \infty)$. Letting $u(x)$ be an oscillatory solution of $u'' + (\lambda(x)+\varepsilon)u = 0$, we can also choose $x_2 > x_1 \geqq \alpha$ such that

$$u(x_1) = u(x_2) = 0.$$

Then for $x_1 \leqq x \leqq x_2$ we have

$$\frac{d}{dx}\left[uu' - u^2 \frac{\langle H, Y'\rangle}{\langle H, Y\rangle}\right] = (\lambda+\varepsilon)u^2 - u^2 \frac{\langle H, AY\rangle}{\langle H, Y\rangle} + \left[u' - u\frac{\langle H, Y'\rangle}{\langle H, Y\rangle}\right]^2.$$

Integrating from x_1 to x_2 yields

$$0 = \left[uu' - u^2 \frac{\langle H, Y'\rangle}{\langle H, Y\rangle}\right]_{x = x_1}^{x = x_2} \geqq \int_{x_1}^{x_2} \left[\lambda + \varepsilon - \frac{\langle H, AY\rangle}{\langle H, Y\rangle}\right] u^2 dx,$$

so that the positivity of the integrand above provides the desired contradiction.

In the special case $a(x) \equiv d(x)$, a different technique can be used to establish Theorem 4.1 with $\varepsilon = 0$ (**1**, Theorem IV).

5. Oscillation criteria

It is a direct consequence of Theorem 2.2 that if every nontrivial solution y, z of

$$\begin{aligned} y'' &= a(x)y+b(x)z \\ z'' &= c(x)y+d(x)z \end{aligned} \tag{5.1}$$

is rotary at $x = \infty$, then every solution of

$$lu \equiv (p_2(x)u'')''-(p_1(x)u')'-q_1(x)u'+p_0(x)u \equiv 0 \tag{5.2}$$

has arbitrarily large zeros. Thus the transformation of Theorem 2.2 allows one to translate Theorem 4.1 into oscillation criteria for solutions of (5.2).

The requirement that $Q_2(x)$ be positive definite is satisfied if

$$b = \frac{1}{p_2} > 0$$

$$-c = p_0 + \frac{p_1''-q_1'}{2} + \frac{(\int q_1)^2-p_1^2}{4p_2} > 0$$

and

$$4bc+(a-d)^2 = \left(\frac{p_1}{p_2}\right)^2 - 2\frac{p_1''-q_1'}{p_2} - \frac{4p_0}{p_2} < 0.$$

These conditions are all satisfied if $p_2 > 0$ and

$$p_0 + \frac{p_1''-q_1'}{2} - \frac{p_1^2}{4p_2} > 0. \tag{5.3}$$

In the special case of constant coefficients and $q_1 = 0$, (5.3) implies that the roots of

$$p_2x^4 - p_1x^2 + p_0 = 0$$

are complex, and this is also sufficient to assure that all solutions of (5.2) be oscillatory at $x = \infty$.

To satisfy the other hypotheses of Theorem 4.1, we seek to assure that

$$\langle H, A(x)H\rangle < \lambda(x)\langle H, H\rangle$$

for all $H \neq 0$. This requires that

$$\langle H, (\lambda(x)I-A(x))H\rangle > 0$$

for all $H \neq 0$, which is equivalent to the positive definiteness of the matrix

$$\begin{pmatrix} \lambda - a & -\dfrac{b+c}{2} \\ -\dfrac{b+c}{2} & \lambda - d \end{pmatrix}$$

and requires that

$$\lambda - a = \lambda - \frac{p_1 - \int q_1}{2p_2} > 0, \tag{5.4}$$

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$$\lambda - d = \lambda - \frac{p_1 + \int q_1}{2p_2} > 0, \tag{5.5}$$

and

$$\lambda^2 - (a+d)\lambda + ad - \frac{(b+c)^2}{4} = \lambda^2 - \frac{p_1}{p_2}\lambda + \frac{p_1^2 - (\int q_1)^2}{4p_2^2} - \tfrac{1}{4}(b+c)^2 > 0. \tag{5.6}$$

These observations can be summarized as follows.

Theorem 5.1. *If there exists a continuous* $\lambda(x)$ *and* $\varepsilon > 0$ *such that*

$$u'' = (\lambda(x) + \varepsilon)u = 0$$

is oscillatory at $x = \infty$ *and* (5.3)-(5.6) *are satisfied, then every solution of* (5.2) *is oscillatory at* $x = \infty$.

Since the hypotheses of Theorem 5.1 are rather complicated, it seems appropriate to discuss their implications in general terms. The condition that $u'' = (\lambda(x) + \varepsilon)u$ be oscillatory at $x = \infty$ requires that $\lambda(x)$ be " sufficiently negative " for large values of x. The simplest admissible choice of $\lambda(x)$ is any negative constant, and the principal restrictions in choosing $\lambda(x)$ are (5.4) and (5.5). If $p_1(x)$ is sufficiently negative, then (5.4) and (5.5) allow for large negative choices of $\lambda(x)$ and this in turn makes (5.6) easier to satisfy. However (5.3) imposes limits on the negativeness of $p_1(x)$. The principal restriction imposed by (5.6) concerns the magnitude of $\frac{1}{4}(b+c)^2$. Since $b(x) = \dfrac{1}{p_2(x)} > 0$ and (5.3) assures that $c(x) < 0$, there is a range of possible values of the coefficients of (5.2) which lead to small values of $(b+c)^2$, and it is within this range that applications of Theorem 5.1 are most feasible. A special class of such equations are those for which $p_1' = q_1$ and $p_0 = \dfrac{1}{p_2}$. In this case Theorem 5.1 asserts that all solutions of (5.2) are oscillatory if $-2p_0 \leqq p_1 \leqq 0$.

6. Nonoscillation criteria

It is natural to try to apply the techniques of Section 4 to obtain nonrotation criteria for (4.1) and thereby to establish nonoscillation criteria for (5.2). To that end we consider a differential equation

$$u'' = \lambda(x)u \tag{6.1}$$

which has a solution $u(x)$ which is positive in $[\alpha, \infty)$. If $Y(x)$ satisfies $Y'' = AY$, then for any constant vector H,

$$\frac{d}{dx}\left[\langle H, Y\rangle\langle H, Y'\rangle - \langle H, Y\rangle^2 \frac{u'}{u}\right]$$
$$= \langle H, Y\rangle\langle H, AY\rangle - \lambda(x)\langle H, Y\rangle^2 + \left[\langle H, Y'\rangle - \langle H, Y\rangle \frac{u'}{u}\right]^2$$

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in $[\alpha, \infty)$. If for all $Y \neq 0$

$$\frac{\langle H, AY\rangle}{\langle H, Y\rangle} \geqq \lambda(x) \tag{6.2}$$

in $[\alpha, \infty)$, then by an argument analogous to that of Section 4, condition (6.2) precludes the existence of two zeros of $\langle H, Y(x)\rangle$ in $[\alpha, \infty)$.

The difficulty with this procedure is that an inequality such as (6.2) is very difficult to establish unless one can hypothesize a limiting direction for the vector $Y(x)$. While conditions such as (6.2) have been used by Noussair and Swanson (3) as the basis for H-oscillation criteria, it seems unlikely that they can be translated into manageable conditions on the coefficients of (5.2).

7. Concluding remarks

It is of interest to compare the oscillation theory studied above with that when oscillation is defined in terms of conjugate points. The first conjugate point of α with respect to (1.1) is defined as the smallest $\beta > \alpha$ such that

$$u(\alpha) = u'(\alpha) = 0 = u(\beta) = u'(\beta)$$

is satisfied nontrivially by a solution of (1.1), and it is denoted by $\eta_1(\alpha)$. If $\eta_1(\alpha)$ does not exist, then (1.1) is said to be disconjugate on $[\alpha, \infty)$.

In the case of selfadjoint equations such as (1.2) conjugate points can be identified with the singularities of conjoined matrix solutions of a related Hamiltonian system (see for example (6)) and in this way oscillation properties of (1.2) can readily be studied. However nonselfadjoint equations such as (1.1) and (2.2) lead to matrix systems which do not possess conjoined solutions, and this fact seems to preclude effective use of this method for establishing upper bounds for $\eta_1(\alpha)$ (although lower bounds can still be obtained, (7)-(9)). In the selfadjoint case $\eta_1(\alpha)$ varies monotonely with the $p_i(x)$ $(p_2(x) > 0)$ (6) in the sense that an increase in the $p_i(x)$ increases $\eta_1(\alpha)$.

When oscillation is defined in terms of oscillatory behaviour of all solutions of the selfadjoint equation (1.2), there is no simple monotonicity relation between the size of the coefficients and the rate of oscillation. And now it is the oscillation criteria which can be extended to the nonselfadjoint equation (1.1), while the techniques of Section 4 seem ineffective in establishing criteria for nonoscillation.

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