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REMARKS ON JAMES'S DISTORTION THEOREMS

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If a Banach space X contains a complemented subspace isomorphic to c_0 (respectively, ℓ^1), then X contains complemented almost isometric copies of c_0 (respectively, ℓ^1). If a Banach space X is such that X^* contains a subspace isomorphic to $L^1[0,1]$ (respectively, ℓ^∞), then X^* contains almost isometric copies of $L^1[0,1]$ (respectively, ℓ^∞).

In **[3]**, James proved that if a Banach space contains a subspace which is isomorphic to c_0 (respectively, ℓ^1), then it contains almost isometric copies of c_0 (respectively, ℓ^1). In this short note we shall prove complemented versions of these results and show that a dual Banach space containing a subspace isomorphic to $L^1[0,1]$ (respectively, ℓ^∞) must contain almost isometric copies of $L^1[0,1]$ (respectively, ℓ^∞). In particular, the $L^1[0,1]$ result is in sharp contrast to a result of Lindenstrauss and Pełczyński **[4]**, who show that $L^1[0,1]$ is arbitrarily distortable, and so $L^1[0,1]$ can be equivalently renormed so as not to contain almost isometric copies of $L^1[0,1]$ (with its usual norm). As for the ℓ^∞ result, it is known that if a Banach space contains a subspace isomorphic to ℓ^∞ then it must contain almost isometric copies of ℓ^∞. This result was proved by Partington in **[6]** . Unaware of Partington's result, Hudzik and Mastyło **[2]** reproved this result in the setting of function spaces.

The Results

Two Banach spaces X and Y are said to be λ-isometric (with $\lambda \geqslant 1$), if there exists a linear isomorphism $T : X \to Y$ so that $\|T\| \|T^{-1}\| \leqslant \lambda$. A Banach space X is said to contain almost isometric copies of the Banach space Y if, for each $\varepsilon > 0$, there exists a subspace Z of X so that Y and Z are $(1+\varepsilon)$-isometric.

Proposition 1. *If X is a Banach space which contains a complemented subspace isomorphic to c_0, then X contains complemented almost isometric copies of c_0.*

Proof: Let Y be a complemented subspace of X which is isomorphic to c_0. Let P be a bounded linear projection from X onto Y.

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Let $\varepsilon > 0$ be given. Since Y contains a subspace isomorphic to c_0, Y contains a subspace Z so that Z and c_0 are $(1+\varepsilon)$-isometric by the James Distortion Theorem [**1, 3, 5**]. Since Y is a separable Banach space and Z is a subspace of Y isomorphic to c_0, Z is complemented in Y by Sobczyk's Theorem [**1, 7**]. Let Q be a bounded linear projection of Y onto Z. Then QP is a bounded linear projection of X onto Z. This completes the proof. ∎

THEOREM 2. *If X is a Banach space which contains a complemented subspace isomorphic to ℓ^1, then X contains complemented almost isometric copies of ℓ^1.*

PROOF: Let Y_1 be a complemented subspace of X such that Y_1 is isomorphic to ℓ^1. Let P be a linear projection from X onto Y_1.

Let $\varepsilon > 0$ be given. Let $T : \ell^1 \to Y_1$ be a bounded linear isomorphism of ℓ^1 onto Y_1 and, for each $n \in \mathbb{N}$, let $x_n = T(e_n)$ where e_n is the usual n^{th} unit basis vector of ℓ^1. Then, for each element $(a_n)_n \in \ell^1$, we have

$$\frac{1}{\|T^{-1}\|}\sum_n |a_n| \leqslant \left\|\sum_n a_n x_n\right\| \leqslant \|T\| \sum_n |a_n| \ .$$

For each $n \in \mathbb{N}$, define

$$D_n = \left\{\sum_{k=n}^{l} a_k x_k : l \geqslant n \text{ and } \sum_{k=n}^{l} |a_k| = 1\right\} .$$

Let $K_n = \inf\{\|x\| : x \in D_n\}$. Then $K_n \leqslant K_{n+1}$ and $1/\|T^{-1}\| \leqslant K_n \leqslant \|T\|$ for all $n \in \mathbb{N}$. Therefore $\lim\limits_{n\to\infty} K_n$ exists. Let $K = \lim\limits_{n\to\infty} K_n$ and note that $K_n \leqslant K$ for all $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ so that $K_n > (1+\varepsilon)^{-1/2}K$ for all $n \geqslant N$. Choose $y_1 \in D_N$ so that $\|y_1\| < (1+\varepsilon)^{1/2}K$. Since $y_1 \in D_N$, y_1 can be written as

$$y_1 = \sum_{k=N}^{n_1} a_k^{(1)} x_k, \text{ where } n_1 \geqslant N \text{ and } \sum_{k=N}^{n_1} \left|a_k^{(1)}\right| = 1.$$

Choose $y_2 \in D_{n_1+1}$ so that $\|y_2\| < (1+\varepsilon)^{1/2}K$. Since $y_2 \in D_{n_1+1}$, y_2 can be written as

$$y_2 = \sum_{k=n_1+1}^{n_2} a_k^{(2)} x_k, \text{ where } n_2 \geqslant n_1 + 1 \text{ and } \sum_{k=n_1+1}^{n_2} \left|a_k^{(2)}\right| = 1.$$

Continuing in this manner, we can choose $y_j \in D_{n_{j-1}+1}$ so that $\|y_j\| < (1+\varepsilon)^{1/2}K$ and y_j can be written as

$$y_j = \sum_{k=n_{j-1}+1}^{n_j} a_k^{(j)} x_k, \text{ where } n_j \geqslant n_{j-1} + 1 \text{ and } \sum_{k=n_{j-1}+1}^{n_j} \left|a_k^{(j)}\right| = 1 \ ,$$

where $n_0 = N-1$.

For scalars $t_1, t_2, \ldots, t_p$ with $\sum\limits_{j=1}^{p} |t_j| = 1$, we have

$$\left\| \sum_{j=1}^{p} t_j y_j \right\| = \left\| \sum_{j=1}^{p} \sum_{k=n_{j-1}+1}^{n_j} t_j a_k^{(j)} x_k \right\| \geqslant K_N > (1+\varepsilon)^{-1/2} K ,$$

since $\sum\limits_{j=1}^{p} \sum\limits_{k=n_{j-1}+1}^{n_j} t_j a_k^{(j)} x_k \in D_N$.

On the other hand,

$$\left\| \sum_{j=1}^{p} t_j y_j \right\| \leqslant \sum_{j=1}^{p} |t_j| \, \|y_j\| < \sum_{j=1}^{p} |t_j| \, (1+\varepsilon)^{1/2} K = (1+\varepsilon)^{1/2} K.$$

Thus for any scalars $t_1, t_2, \ldots, t_p$, we have

$$K(1+\varepsilon)^{-1/2} \sum_{j=1}^{p} |t_j| \leqslant \left\| \sum_{j=1}^{p} t_j y_j \right\| \leqslant K(1+\varepsilon)^{1/2} \sum_{j=1}^{p} |t_j| .$$

Hence the Banach space $Y = \overline{\text{span}}\ \{y_j\}_{j=1}^{\infty}$ is a subspace of Y_1, and Y and ℓ^1 are $(1+\varepsilon)$-isometric.

Now let $z_j = \sum\limits_{k=n_{j-1}+1}^{n_j} a_k^{(j)} e_k$ for each $j \in \mathbb{N}$, where $n_0 = N-1$. Let $Z = \overline{\text{span}}\ \{z_j\}_{j=1}^{\infty}$. Since $(z_j)_j$ is a block basic subsequence of $(e_n)_n$, Z is a subspace of ℓ^1 which is isomorphic to ℓ^1 and complemented in ℓ^1 by a norm 1 projection Q. Note also that the restriction of T to Z, $T|_Z$, is an isomorphism of Z onto Y. Thus we have the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{P} & Y_1 & \underset{T}{\overset{T^{-1}}{\rightleftarrows}} & \ell^1 \\ & & & & \big\downarrow Q \\ & & Y & \xleftarrow[T|_Z]{} & Z \end{array}$$

Clearly, Y is complemented in X by the projection $T|_Z Q T^{-1} P$. This completes the proof. ∎

Remark. Much of the proof of Theorem 2 is similar to James's original proof [**3**], and the proof used in Theorem 2 could also be used to prove Proposition 1. The next result is a special case of a theorem of Partington [**6**].

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Corollary 3. *If X is a Banach space such that X^* contains a subspace isomorphic to ℓ^∞, then X^* contains almost isometric copies of ℓ^∞.*

Proof: If X^* contains a subspace isomorphic to ℓ^∞, then X contains a complemented subspace isomorphic to ℓ^1 [**1**]. By Theorem 2, X contains complemented almost isometric copies of ℓ^1. Thus for each $\varepsilon > 0$, there is a complemented subspace Y of X so that Y and ℓ^1 are $(1+\varepsilon)$-isometric. Thus Y^* and ℓ^∞ are $(1+\varepsilon)$-isometric, and since Y is complemented in X, Y^* is isometric to a subspace of X^*. This completes the proof. ∎

Theorem 4. *If X is a Banach space such that X^* contains a subspace isomorphic to $L^1[0,1]$, then X^* contains almost isometric copies of $L^1[0,1]$.*

Proof: Let $\varepsilon > 0$ be given. Since X^* contains a subspace isomorphic to $L^1[0,1]$, X contains a subspace isomorphic to ℓ^1 [**1**]. Hence, by the James Distortion Theorem [**1, 3, 5**], X contains a subspace Y so that Y and ℓ^1 are $(1+\varepsilon)$-isometric. Hence Y^* and ℓ^∞ are $(1+\varepsilon)$-isometric. Since ℓ^∞ contains a subspace which is isometric to $L^1[0,1]$, there is an isomorphism $I : L^1[0,1] \to Y^*$ with $\|I\|\,\|I^{-1}\| \leqslant 1+\varepsilon$. Without loss of generality we can and do assume that $\|I\| = 1$ and $\|I^{-1}\| \leqslant 1+\varepsilon$.

Let $i : Y \to X$ be the natural inclusion map. Taking an adjoint gives the following diagram:

$$\begin{array}{ccc} L^1[0,1] & \xrightarrow{\ I\ } & Y^* \\ & & \uparrow{\scriptstyle i^*} \\ & & X^* \end{array}$$

Taking adjoints again and using the fact that $L^\infty[0,1]$ is an injective space, there exists a bounded linear mapping $S : X^{**} \to L^\infty[0,1]$ so that $\|S\| = \|I^*\| = 1$ and the following diagram commutes:

$$\begin{array}{ccc} & & X^{**} \\ & \swarrow{\scriptstyle S} & \uparrow{\scriptstyle i^{**}} \\ L^\infty[0,1] & \xleftarrow[\ I^*\ ]{} & Y^{**} \end{array}$$

Taking adjoints yet again, we get another commutative diagram:

$$\begin{array}{ccccccc} L^1[0,1] & \xrightarrow{\ J\ } & (L^\infty[0,1])^* & \xrightarrow{\ I^{**}\ } & Y^{***} & \xrightarrow{\ R^*\ } & Y^* \\ & & & \searrow{\scriptstyle S^*} & \uparrow{\scriptstyle i^{***}} & & \uparrow{\scriptstyle i^*} \\ & & & & X^{***} & \xrightarrow{\ Q^*\ } & X^* \end{array}$$

where Q and R denote the canonical mappings from X and Y to X^{**} and Y^{**} (respectively) and the mapping J is the canonical mapping from $L^1[0,1]$ into $\left(L^1[0,1]\right)^{**} = (L^\infty[0,1])^*$.

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Let $U = Q^*S^*J$. Then $U : L^1[0,1] \to X^*$ and since $\|i^*\| = 1$ and $I^{**} = I$ on $L^1[0,1]$, we have for each $f \in L^1[0,1]$

$$\begin{aligned}\|Uf\| &= \|Q^*S^*Jf\| \\ &\geqslant \|i^*Q^*S^*Jf\| \\ &= \|R^*I^{**}Jf\| \\ &= \|If\| \\ &\geqslant (1+\varepsilon)^{-1}\|f\|_1 .\end{aligned}$$

On the other hand,

$$\begin{aligned}\|Uf\| &= \|Q^*S^*Jf\| \\ &\leqslant \|Q^*\|\,\|S^*\|\,\|J\|\,\|f\|_1 \\ &= \|f\|_1 .\end{aligned}$$

Thus $U\big(L^1[0,1]\big)$ is a subspace of X^* so that $U\big(L^1[0,1]\big)$ and $L^1[0,1]$ are $(1+\varepsilon)$-isometric. This completes the proof. ∎

REMARK. The proof of Theorem 4 can also be used to show that if a Banach space X contains a subspace isomorphic to ℓ^1, then X^* contains almost isometric copies of $\ell^1(2^{\mathbb{N}})$.

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