ON BOUNDED MATRICES WITH NON-NEGATIVE ELEMENTS

C. R. PUTNAM

1. Introduction. It is known (Perron **(10)**; Frobenius **(5, 6)**) that if $A = (a_{ik})$ is a finite matrix with elements $a_{ik} \geqslant 0$, then A has a real, non-negative eigenvalue μ, satisfying $\mu = \max|\lambda|$ where λ is in the spectrum of A, with a corresponding eigenvector $x = (x_1, \ldots, x_n)$ for which $x_i \geqslant 0$. Moreover if $a_{ik} > 0$, then μ is a simple point of the spectrum with an eigenvector x (unique, except for constant multiples) with components $x_i > 0$. Much has been written on this and related issues; cf., for example, the recent papers **(4, 12)** wherein are given several references. Rutman and Krein **(8, 11)** have placed the problem in the general setting of operators in a Banach space leaving invariant certain cones.

In the present paper, a Hilbert space consisting of real vectors $x = (x_1, x_2, \ldots)$, and bounded operators, represented by real matrices $A = (a_{ik})$, will be considered. Thus, for any such A, there exists a constant $M \geqslant 0$ such that $||Ax|| \leqslant M||x||$ whenever $||x||^2 = \sum x_k^2 < \infty$. For this case, the Rutman-Krein results lead to certain theorems on completely continuous operators. The object of the present note is to obtain certain analogous results for operators which are not necessarily completely continuous. In fact, a series of theorems will be given, where, in the beginning (cf. (I) below) only the assumption $a_{ik} \geqslant 0$ (and boundedness) will be made, and, as needed, additional restrictions will be imposed.

The author is indebted to the referee for pointing out some recent work of Bonsall **(1, 2, 3)** which includes, among other things, generalizations of certain results of Rutman and Krein. Theorem (I) below is contained in **(2**; cf. pp. 148 ff.**)**, also Theorem B of **(3**, p. 54**)**.

2. By $A \geqslant 0$ and $A > 0$ will be meant that $a_{ik} \geqslant 0$ and $a_{ik} > 0$ respectively. Similarly a vector $x = (x_1, x_2, \ldots,)$ will be written $x \geqslant 0$ or $x > 0$ according as all $x_i \geqslant 0$ or all $x_i > 0$ respectively. The spectrum of A will be denoted by $sp(A)$. The following will be proved:

(I) *If* $A \geqslant 0$, *then* $\mu = \sup|\lambda|$, *where* λ *is in* $sp(A)$, *also belongs to* $sp(A)$.

(II) *If* $A \geqslant 0$ *and if at least one diagonal element, say* d, *of* A^n (*for some* $n \geqslant 1$) *is positive, then* μ *of* (I) *above satisfies* $\mu \geqslant d^{1/n}$.

(III) *If* $A \geqslant 0$ *and if* μ *of* (I) *is positive and is a pole of the resolvent* $R(\lambda) = (A - \lambda I)^{-1}$ (*hence, in particular,* μ *is an isolated point of* $sp(A)$ *and is in the*

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point spectrum) then there exists at least one characteristic vector x $(Ax = \mu x, x \neq 0)$ *satisfying* $x \geqslant 0$.

(IV) *If* $A > 0$ (*or even, if for every pair* i, k *there exists an integer* $M = M(i, k) \geqslant 1$ *such that* $(A^M)_{ik} > 0$) *and if* μ *of* (I) (*where, by* (II), $\mu > 0$) *is a pole of* $R(\lambda)$, *then* μ *is* (a) *a simple pole of* $R(\lambda)$ *and* (b) *a simple characteristic number. Moreover,* (c) *there exists a characteristic vector* $x > 0$ *belonging to* μ.

Remarks. The above theorems are patterned after similar ones, in which A is supposed to be completely continuous, of (**8**, pp. 80–82). (Cf. also the last paragraph of §1 above.) Parts of some of the proofs (as noted below) are virtually identical with those in (**8**) but, in order to make the present paper self-contained, complete proofs of all the theorems will be given.

In (I) and (II), where only $A \geqslant 0$ is assumed, it should be noted that μ may not be in the point spectrum and that A may not have any point spectrum whatever. In fact, if A is the Jacobi matrix belonging to $2 \sum x_n x_{n+1}$, then $A \geqslant 0$ and A has no point spectrum. Actually, A can be chosen so as to satisfy $A > 0$, for instance, a bounded Toeplitz matrix with positive elements; then necessarily the point spectrum is empty (cf. **2**, p. 149; **7**, p. 868).

In (III) and (IV), the assumption that μ (>0) be a pole of $R(\lambda)$ is surely fulfilled if A is completely continuous. In fact, in this case, the above results (except possibly (a) of (IV)) are contained in the results of (**8**, pp. 80–82). Part (a) of (IV) does not seem to be contained here, although something similar to it (when A is completely continuous) is contained in Theorems 5 (Rutman) and 5a (Krein) of (**11**, pp. 91–92). In these latter theorems however, it is assumed that the "invariant cone" has an interior point. In the present case however, what corresponds to this cone is the set of vectors $x \geqslant 0$ in Hilbert space, and this set has no interior points.

It can be remarked that the statement given in (**8**, p. 91), namely that *if* A *is completely continuous, if* $A \geqslant 0$, *and if all the diagonal elements of every power* A^n *are zero, then* 0 *is the only point of* $sp(A)$, may not be true without the assumption of complete continuity. (The proof given, *loc. cit.*, pp. 91–92, involves the approximation of A by its sections.)

3. *Proof of* (I). Since A is bounded its spectrum is contained in a finite portion of the complex plane, so that the resolvent $R(\lambda)$ is given by $R(\lambda) = -\sum A^n/\lambda^{n+1}$ for $|\lambda|$ sufficiently large. The elements of $-R(\lambda)$ are series of the form $\sum a_n z^n$ where $z = \lambda^{-1}$ and $a_n \geqslant 0$. If every one of these series is convergent for $|z|$ arbitrarily large, then, of course, 0 is the only point of the spectrum of A. Otherwise, there exists a real number $\alpha > 0$ such that every series is convergent for $|z| < \alpha$ (that is, for $|\lambda| > 1/\alpha$) but at least one series is divergent for $|z| > \alpha$. By the Vivanti-Pringsheim theorem (**9**, p. 72), $z = \alpha$ must be a singularity of such a series. Consequently the number $\lambda = 1/\alpha$ is in $sp(A)$ while every λ in $sp(A)$ satisfies $|\lambda| \leqslant \alpha$. This completes the proof of (I).*

*The above proof, using the Vivanti-Pringsheim theorem, is due to the late Professor Wintner, with whom the author had several valuable discussions concerning non-negative matrices.

4. *Proof of* (II). Let d denote the kth diagonal element of A^n. Then the kth diagonal element of A^{nm} is given by a series with non-negative terms, one of which is d^m. Hence, if λ is real and satisfies $\lambda > \mu$, then the kth diagonal element of $-R(\lambda)$, which is not less than the kth diagonal element of

$$\sum_{m=1}^{\infty} A^{nm}/\lambda^{nm+1},$$

is not less than

$$\sum_{m} d^m/\lambda^{nm+1}.$$

But, since this last series is divergent if $\lambda < d^{1/n}$, it follows that $\mu \geqslant d^{1/n}$. This completes the proof of (II). Cf. (**8**, pp. 68–69), wherein is given a similar proof for a completely continuous operator in a Banach space.

5. *Proof of* (III). If $\mu > 0$ is a pole of $R(\lambda)$ then $R(\lambda)$ is given by

$$R(\lambda) = \sum_{n=-N}^{\infty} c_n(\lambda - \mu)^n$$

for $|\lambda - \mu|$ sufficiently small and positive, where $N \geqslant 1$, $c_{-N} \leqslant 0$ and $c_{-N} \neq 0$. In fact, as was noted above, if $\mu > 0$, then some element of $R(\lambda)$ tends to $-\infty$ when $\lambda \to \mu + 0$ (λ real). The remainder of the proof is essentially identical with that of (**8**, p. 66). For

$$(\lambda - \mu)^N I = (\lambda - \mu)^N (A - \lambda I)\, R(\lambda) = (A - \lambda I) c_{-N} + B,$$

where B represents a term which tends to 0 as λ tends to μ. Consequently $(A - \mu I)c_{-N} = 0$ and a characteristic vector $x \geqslant 0$ of μ is given by any column of $-c_{-N}$ which does not consist entirely of zeros.

6. *Proof of* (a) *of* (IV). It follows from the functional equation of the resolvent that $dR/d\lambda = R^2(\lambda)$. Hence, if N (*see above*) satisfies $N > 1$, then, on equating coefficients in the series for $dR/d\lambda$ and $R^2(\lambda)$ one obtains $c_{-N}^2 = 0$. If $c_{-N} = (c_{ik})$, then this last result is that

$$\sum_{m} c_{im} c_{mk} = 0.$$

In particular, if $i = k$, it follows that $c_{ii} = 0$, for all $i = 1, 2, \ldots$. By hypothesis, for every pair (i, k) there exists a positive integer $M = M(i, k)$ such that $(A^M)_{ik} > 0$. But $A^M c_{-N} = \mu^M c_{-N}$, and hence

$$\sum_{m} (A^M)_{im} c_{mi} = \mu^M c_{ii} = 0.$$

Consequently, $c_{ki} = 0$ (i, k arbitrary) and so $c_{-N} = 0$, a contradiction. This completes the proof of (a).

Proof of (c) *of* (IV). The proof in §5 shows that if some element c_{im} of the mth column of C is zero, then in fact every element $c_{1m}, c_{2m}, \ldots,$ of this column is zero. For, suppose $c_{im} = 0$; then for this fixed i, choose an arbitrary k and then $M = M(i, k)$ as before. Then

$$\sum_p (A^M)_{ip} c_{pm} = \mu^M c_{im} = 0$$

and hence $c_{km} = 0$ $(k = 1, 2, \ldots,)$. Consequently, since $c_{-N} \leqslant 0$ and $c_{-N} \neq 0$, it follows that there exists at least one column of $-c_{-N}$ consisting of positive elements only. This column serves as a positive characteristic vector and the proof of (c) is complete.

Proof of (b) *of* (IV). The proof is essentially that given in (**8**, pp. 78–80, 82) for integral equations. First, let y be any characteristic vector of A^*. Then

$$\sum_k a_{ki} y_k = \mu y_i$$

and hence

$$\sum_k a_{ki} |y_k| \geqslant \mu |y_i|.$$

Let x be a positive characteristic vector of A belonging to μ (see above). Multiplication by x_i of both sides of the last inequality followed by a summation and an interchange of the orders of summations, yields

$$\sum_k x_k |y_k| \geqslant \sum_i x_i y_i$$

where, since all $x_i > 0$, the sign $>$ (and hence a contradiction) obtains only if the components of y fail to satisfy either all $y_i \geqslant 0$ or all $y_i \leqslant 0$. Thus if y is *any* characteristic vector of A belonging to μ, then either $y \geqslant 0$ or $y \leqslant 0$. Interchanging the roles of A and A^* (and noting that $R^*(\lambda)$ is the resolvent of A^* and that μ plays the same role for A^* as it does for A) it follows that any characteristic vector z of A belonging to μ satisfies either $z \geqslant 0$ or $z \leqslant 0$. Consequently, μ is a simple point of the spectrum of A. Otherwise, there would exist a characteristic vector, say z (necessarily $z \geqslant 0$ or $z \leqslant 0$) orthogonal to $x(> 0)$ and this is clearly impossible. This completes the proof of (b) of (IV).

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Purdue University