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THE HILBERT SERIES OF RINGS OF MATRIX CONCOMITANTS

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Introduction

Throughout this paper, $K$ will be a field of characteristic zero. Let $K\langle x_1, \ldots, x_m \rangle$ be the $K$-algebra in $m$ variables $x_1, \ldots, x_m$ and $I_{m,n}$ the $T$-ideal consisting of all polynomial identities satisfied by $m \times n$ matrices. The ring $R(n, m) = K\langle x_1, \ldots, x_m \rangle/I_{m,n}$ is called the ring of $m$ generic $n \times n$ matrices.

This ring can be described as follows. Let $X_1, \ldots, X_m$ be $m$ generic $n \times n$ matrices over the field $K$. That is $X_k = (x_{ij}(k))$, $1 \leq i, j \leq n$, $1 \leq k \leq m$, where the $x_{ij}(k)$ are independent commutative variables over $K$. Then $R(n, m)$ is the $K$-algebra generated by $X_1, \ldots, X_m$. We denote by $K[x_{ij}(k)]$, $1 \leq i, k \leq n$, $1 \leq k \leq m$, the commutative polynomial ring generated by the entries of generic $n \times n$ matrices $X_1, \ldots, X_m$. The subring of $K[x_{ij}(k)]$ generated by all the traces of monomials in $R(n, m)$ is called the ring of invariants of $m$ generic $n \times n$ matrices and will be denoted by $C(n, m)$. The subring of $M_n(K[x_{ij}(k)])$ generated by $R(n, m)$ and $C(n, m)$ is called the trace ring of $m$ generic $n \times n$ matrices and will be denoted by $T(n, m)$.

The functional equation of the Hilbert series of the ring $T(n, m)$ is proved by Le Bruyn [L1] for $n = 2$ and by Formanek [F2] for $m \geq n^2$. We prove the functional equation in a more general situation (4.3. Theorem).

Our method is as follows. The trace ring $T(n, m)$ is a fixed ring of $GL(n, K)$ and hence the Hilbert series has an integral expression by a classical result of Molien-Weyl. This formula reduces the problem to a problem of relative invariants for a torus group. By using a theorem of Stanley [S], we can prove the desired functional equation.

The rest of this paper was motivated by a result of Le Bruyn [L2], who treats trace ring of 2 by 2 generic matrices and proved, among other things, that the trace ring $T(2, m)$ is a Cohen-Macaulay module over its
center $C(n, m)$. Giving an explicit form of a homogeneous system of parameters for $C(2, m)$, we show that $T(2, m)$ is a free module of rank \[ \frac{1}{m-1}\left(\frac{2m-2}{m-1}\right)^2 \] over the polynomial ring $B(2, m)$ generated by elements of the homogeneous system of parameters for $C(2, m)$ (8.2. Theorem). As an example we give an explicit description of $C(2, 4)$ and $T(2, 4)$ (9.1. Theorem).

Procesi [P2] gave an explicit presentation of the Hilbert series of $T(2, m)$ and observed a close relation between the Hilbert series of $T(2, m)$ and that of the homogeneous coordinate ring of the Grassmannian $Gr(2, m)$ (see [L2]). Then 8.2. Theorem together with Procesi’s observation above suggest that there is a canonical free basis of $T(2, m)$ over the polynomial ring $B(2, m)$.

§ 1. Matrix invariants and concomitants

Let $G$ be a classical group in $GL(n, K)$. That is one of the groups, $SL(n, K), SO(n, K), Sp(n, K)$.

Let $V(G, m)$ be the vector space $\oplus^n \text{Lie}(G)$, where $\text{Lie}(G)$ denotes the Lie algebra of $G$. The group $G$ acts rationally on $V(G, m)$ according to the formula:

If $g \in G$, $(A_1, \ldots, A_m) \in V(G, m)$,
then $g(A_1, \ldots, A_m) = (\text{Ad}(g)A_1, \ldots, \text{Ad}(g)A_m),$
where $\text{Ad}(g)$ denotes the adjoint representation of $G$.

We denote by $K[V(G, m)]$ the ring of polynomial functions on $V(G, m)$ and by $C(G, m)$ the ring of polynomial $G$-invariants of $K[V(G, m)]$. Let $K[V(G, m)]_d$ be the $K$-subspace of $K[V(G, m)]$ consisting of polynomials of multi-degree $d = (d_1, \ldots, d_m) \in \mathbb{N}^m$. The rings $K[V(G, m)]$ and $C(n, m)$ are graded rings:

\[ K[V(G, m)] = \bigoplus_{d \in \mathbb{N}^m} K[V(G, m)]_d, \]

and

\[ C(G, m) = \bigoplus_{d \in \mathbb{N}^m} C(G, m)_d \]

where

\[ C(G, m) = K[V(G, m)]_d \cap C(G, m). \]
A polynomial map \( f : V(G, m) \to \text{Lie}(G) \) is called a polynomial concomitant if that is compatible with the action of \( G \) i.e., \( f(g \cdot v) = \text{Ad}(g)f(v) \) for any \( g \in G \) and \( v \in V(G, m) \).

With \( T(G, m) \) we will denote the set of polynomial concomitants. Then \( T(G, m) \) is a \( C(G, m) \)-module. Let \( P(G, m) \) denote the set of polynomial maps from \( V(G, m) \) to \( \text{Lie}(G) \) and define the action of \( G \) on \( P(G, m) \) by
\[
(g \cdot f)(v) = \text{Ad}(g)f(g^{-1}v), \quad \text{if } g \in G, f \in P(G, m).
\]
Then \( T(G, m) \) is the fixed space of \( P(G, m) \) under the action of \( G \).

Let \( X_1, \ldots, X_m \) be generic matrices in \( \text{Lie}(G) \). Then, for each \( i \), \( X_i \) is identified with the \( i \)-coordinate map
\[
(A_1, \ldots, A_m) \mapsto A_i, \quad (A_1, \ldots, A_m) \in V(G, m).
\]

The following theorem is a direct consequence from some result of Procesi [P1].

1.1. Theorem. The ring \( C(G, m) \) is generated by factors of polynomials of the form \( \text{Tr}(X_{i_1}X_{i_2}\cdots X_{i_t}) \), where \( X_{i_1}\cdots X_{i_t} \) runs over all possible (non-commutative) monomials in \( m \) generic matrices \( X_1, \ldots, X_m \) in \( \text{Lie}(G) \).

§ 2. Molien-Weyl formula

Let \( G \) be a semi-simple linear algebraic group over the complex number field \( C \) and \( V \) a \( G \)-module. We denote by \( K[V] \) the polynomial ring on \( V \). The action of \( G \) on the vector space \( V \) can be extended on \( K[V] \) by a canonical way. Let \( K[V]^G \) be the subring of \( K[V] \) consisting of \( G \)-invariant polynomials. Then \( K[V]^G \) is a graded ring:
\[
K[V]^G = \bigoplus_{d \in \mathbb{N}} K[v]^d
\]
where \( K[v]^d \) is the \( K \)-vector space of \( G \)-invariant polynomials of degree \( d \).

The Hilbert series for the graded ring \( K[V]^G \) is the formal power series defined by
\[
\chi(K[V]^G, t) = \sum_{d \in \mathbb{N}} \dim K[V]^d t^d.
\]
The Molien-Weyl formula gives an integral expression for the Hilbert series \( \chi(K[V]^G, t) \).

2.1. Proposition. Let \( T \) be a maximal torus of a maximal compact subgroup \( K \) of \( G \). If \(|t| < 1\), then
where $W$ is the Weyl group of $G$ and $\alpha_1, \cdots, \alpha_N$ is the set of roots of $G$ with respect to $T$ and $dg$ is the normalized Haar-measure on $T$.

Let $V_1, \cdots, V_m$ be $G$-modules and set $V = \bigoplus_{i=1}^m V_i$. Then by defining $\deg t_i$, $1 \leq i \leq m$, is to be $(0, \cdots, 1, \cdots, 0)$, where $i$-th coordinate is $1$, $K[V]$ is an $N^m$-graded ring

\[ K[V] = \bigoplus_{d \in N^m} K[V]_d. \]

Corresponding to this decomposition of $K[V]$, we have

\[ K[V]^o = \bigoplus K[V]^o_d, \quad K[V]^o_d = K[V]^o \cap K[V]_d. \]

The multi-valued Hilbert series in $m$ variables $t = (t_1, \cdots, t_m)$ is defined by

\[ \chi(K[V]^o, t) = \sum d \dim K[V]^o_d t^d, \]

where if $d = (d_1, \cdots, d_m) \in N^m$, $t^d = \prod t_i^{d_i}$.

The Molien-Weyl formula in this case is only a slight modification of 2.1. Proposition.

2.2. PROPOSITION. Notations being as above, if $|t_1| < 1$, \ldots, $|t_m| < 1$, \begin{equation*} \chi(K[V]^o, t) = \frac{1}{|W|} \int_{T} \frac{(1 - \alpha_i(g)) \cdots (1 - \alpha_N(g))}{\prod_i \det(1 - t_i g)} \, dg. \end{equation*} 

2.3. COROLLARY.

\[ \chi(C(G, m), t^\Psi) = \frac{1}{|W|} \prod_{i=1}^m (1 - t_i)^{-r} \int_{T} \frac{(1 - \alpha_i(g)) \cdots (1 - \alpha_N(g))}{\prod_i \prod_j (1 - t_i \alpha_j(g))} \, dg, \]

where $r = \text{rank of } G$.

§3. Linear diophantine equation

Let $a_1, \cdots, a_m$ and $b$ be fixed column vectors in $V$, and set

\[ E(A, b) = \{ x = (x_1, \cdots, x_m) \in N^m, a_1 x_1 + \cdots + a_m x_m = b \}, \]

where $A$ is the $r$ by $m$ matrix defined by

\[ A = [a_1, \cdots, a_m]. \]
Let $F(A, b, t)$ be the formal power series in $m$ variables $t = (t_1, \cdots, t_m)$ defined by

$$F(A, b, t) = \sum_{\alpha \in E(A, b)} t^\alpha,$$

where if $\alpha = (\alpha_1, \cdots, \alpha_m)$ then $t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$.

R. Stanley proved the following

**3.1. Theorem ([SI]).** Suppose that the system of linear equations $a_i x_i + \cdots + a_m x_m = b$ has a rational solution $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Q}^m$ with $-1 < \alpha_i \leq 0$ for all $i$ and $1 \in E(A, 0)$. Then $F(A, b, t)$ is a rational function in $t = (t_1, \cdots, t_m)$ which satisfies the functional equation

$$F(A, b, t^{-1}) = (-1)^d t_1 \cdots t_m F(A, -b, t),$$

where $t^{-1} = (t_1^{-1}, \cdots, t_m^{-1})$.

The next lemma will be used to prove the functional equation of the ring of polynomial concomitants.

**3.2. Lemma ([T1] Lemma 1.1).** If $|t_i| < 1$, $\ldots$, $|t_m| < 1$,

$$F(A, b, t) = \left(\frac{1}{2\pi \sqrt{-1}}\right)^r \int_T \prod_{i=1}^m (1 - e^{a_i t_i}) d\xi_1 \cdots d\xi_r,$$

where the integral is taken over the $r$-dimensional torus $T$ and, if $a = (a_1, \cdots, a_r) \in \mathbb{Z}^r$, $E^a = [1_i \xi_i^{a_i}$.

**§ 4. The functional equation of the Hilbert series $(T(G, m), t)$**

We return to the situation in section 1. Let $X_1, \cdots, X_m$ be generic matrices of Lie $(G)$. Define $\deg X_i$ to be the $i$-th unit vector $(0, \cdots, 1, \cdots, 0) \in \mathbb{N}^m$.

The Hilbert series $\chi(T(G, m), t)$ for the $N^m$-graded module $T(G, m)$ is defined by

$$\chi(T(G, m), t) = \sum_{d \in \mathbb{N}^m} \dim T(G, m)_d t^d.$$

Let $X_{m+1}$ be a new generic matrix in Lie $(G)$. Since the trace $\text{Tr}(X, Y)$, $X, Y \in \text{Lie}(G)$, is a nondegenerate bilinear form on $\text{Lie}(G) \times \text{Lie}(G)$, it follows that $\text{Tr}(X X_{m+1})$, $X \in T(G, m)$ defines an injection from $T(G, m)$ onto the subspace of $C(G, m + 1)$ consisting of invariants of degree one in $X_{m+1}$. Then by 2.3. Corollary we have
4.1. **Proposition.** With notations as 2.3. Corollary, the Hilbert series $\chi(T(G, m), t)$ has the following expression

$$\chi(T(G, m), t) = \frac{1}{|W|} \prod (1 - t_i)^{-r} \int \frac{\left( r + \sum \alpha_j(g) \right) \prod (1 - \alpha_j(g))}{\prod (1 - t_i \cdot \alpha_j(g))} \, dg.$$ 

By the theorem of Hochster-Roverts [H-R], $C(G, m)$ is a Cohen-Macaulay domain which is Gorenstein. It follows from a theorem of Stanley [S] that the Hilbert series satisfies a functional equation of the form

$$\chi(C(G, m), t^{-1}) = \pm (t_1, \ldots, t_m)^a \chi(C(G, m), t),$$

for some $a \in \mathbb{Z}$. Here $t^{-1} = (t_1^{-1}, \ldots, t_m^{-1})$.

In our case, we can determine the integer $a$.

4.2. **Theorem ([T]).** If $m \geq 2$, the Hilbert series for the ring $C(G, m)$ satisfies the functional equation

$$\chi(C(G, m), t^{-1}) = (-1)^d (t_1, \ldots, t_m)^a \chi(C(G, m), t),$$

where $d = (m - 1) \dim G$ and $a = \dim G$.

We prove the same functional equation for $T(G, m)$.

4.3. **Theorem.** With notations as before, if $m \geq 3$ then the Hilbert series for $T(G, m)$ satisfies the functional equation

$$\chi(T(G, m), t^{-1}) = (-1)^d (t_1, \ldots, t_m)^a \chi(T(G, m), t),$$

where $d = (m - 1) \dim G$ and $a = \dim G$.

**Proof.** The maximal torus of $G$ is isomorphic to the group

$$\left\{ \begin{array}{cccc} \varepsilon_1 \\ \vdots \\ \varepsilon_r \end{array} \right\}, \quad |\varepsilon_i| = 1, \quad r = \text{rank of } T$$

and every root $\alpha_j$ of $G$ with respect to $T$ can be written as $\alpha_j = \varepsilon^a_j$ for some $a_j = (a_{j_1}, \ldots, a_{j_r}) \in \mathbb{Z}^r$, where $\varepsilon^a_j = \varepsilon_1^{a_{j_1}} \cdots \varepsilon_r^{a_{j_r}}$.

By 4.1. Theorem, the Hilbert series $\chi(T(G, m), t)$ has the integral expression. We write the numerator

$$\sum_j \varepsilon^{a_j} \prod_j (1 - \varepsilon^{a_j})$$
in the integral as a linear combination of terms of the form $\varepsilon^{-b}$, where $-b$ is a vector in $\mathbb{Z}^r$ of the form

$$-b = a_{j_1} + \cdots + a_{j_k} + a_j \quad (j_1 < j_2 < \cdots < j_k).$$

Then the integral is a linear combination of terms of the form

$$F(b, t) = (2\pi\sqrt{-1})^{-1} \int \frac{\varepsilon^{-b}}{(1 - t\varepsilon^b)(1 - t^r)} d\varepsilon_1 \cdots d\varepsilon_r.$$

By 3.2. Lemma, $F(b, t)$ is a Hilbert series associated with a system of linear diophantine equations. If $m \geq 3$, this system of linear equations satisfies the condition of Stenley's theorem (3.1. Theorem) because the vector $b$ is a linear combination of roots $a_j$ with nonnegative integer coefficients $c$ such that $0 \leq c \leq 2$ for all $j$. Therefore we obtain the desired result because $a$ is a root if and only if $-a$ is a root.

§ 5. The functional equation of trace rings

Let $X_1, \cdots, X_m$ be $m$ generic $n \times n$ matrices. According to the decomposition of each matrix variable

$$X_i = \frac{1}{n} \text{Tr}(X_i) + X_i^0,$$

where $X_i^0$ is a an $n \times n$ generic matrix in Lie($SL(n, K)$), we have

$$T(n, m) = T(SL(n, K), m)[\text{Tr}(X_1), \cdots, \text{Tr}(X_m)] \oplus C(SL(n, K), m).$$

This remark, due to Procesi [P1], enables us translate the structure of the trace ring $T(n, m)$ into that of $T(SL(n, K), m)$.

5.1. Theorem. If $n \geq 3$, $m \geq 2$ or $n = 2$, $m \geq 3$, the Hilbert series of the trace ring of $m$ generic $n \times n$ matrices satisfies the functional equation

$$\gamma(T(n, m), t^{-1}) = (-1)^d(t_1, \cdots, t_m)^{-\gamma}(T(n, m), t),$$

where $d = (m - 1)n^2 + 1$.

Proof. If $m \geq 3$, this is a direct consequence from 4.3. Theorem. If $m = 2$, $n \geq 3$, it is easy to see that the proof of 4.3. Theorem holds good, and we obtain the desired result.
§ 6. Homogeneous coordinate rings of the Grassmannian $\text{Gr} (2, m)$

First we recall the definition of the homogeneous coordinate ring of the Grassmannian. Recall that if $\Omega$ denotes the set of all one dimensional linear subspaces in the $m - 1$ dimensional complex projective space $P^{m-1}$, we have an explicit embedding $\Omega \to P^N$, where $N = \binom{m}{2} - 1$. $\Omega$ is called the Grassmannian and denoted by $\text{Gr} (2, m)$. It is well known that dimension and degree of $\text{Gr}(2, m)$, $m \geq 2$, as a projective variety are $2m - 4$ and $\frac{1}{m-1} \binom{2m-4}{m-2}$ respectively.

Let $C[p_{ij}]$, $1 \leq i < j \leq m$, be the polynomial ring in the $\binom{m}{2}$ variables $p_{ij}$, which coordinatize $P^N$. Let $I$ be the ideal of $C[p_{ij}]$ generated by all the polynomials of the form

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}, \quad 1 \leq i < j < k < l \leq m.$$ 

The quotient ring $C[p_{ij}]/I$ is called the homogeneous coordinate ring of $\text{Gr}(2, m)$ and will be denoted by $C[\text{Gr}(2, m)]$. It is convenient to define degree of $p_{ij}$ to be 2. Let $R_{2d}$ ($d \in N$) denote the vector space of $C[\text{Gr}(2, m)]$ generated by all homogeneous polynomials of degree $2d$:

$$C[\text{Gr}(2, m)] = \bigoplus_{d \in N} R_{2d}.$$ 

The Hilbert series for the graded ring $C[\text{Gr}(2, m)]$ is calculated by Hilbert [H]:

$$\chi(C[\text{Gr}(2, m)], t) = \sum_{d \in N} \frac{(d+1)(d+m-1)}{m-1}(m-2)! \prod_{i=2}^{m-1} (d+1)^{i} t^d.$$ 

Set, for $k = 3, 4, \ldots, 2m - 1$, $\theta_k = \sum_{i+j=k} p_{ij}$. Then it is well known and can be easily proved that $\theta_3, \ldots, \theta_{2m-1}$ is a homogeneous system of parameters of $C[\text{Gr}(2, m)]$. Since $C[\text{Gr}(2, m)]$ is a Cohen-Macaulay ring and degree of $\text{Gr}(2, m)$ is $\frac{1}{m-1} \binom{2m-4}{m-2}$, we have

6.1. **Lemma.** The homogeneous coordinate ring $C[\text{Gr}(2, m)]$ is a free module of rank $\frac{1}{m-1} \binom{2m-4}{m-2}$ over the polynomial ring $C[\theta_3, \ldots, \theta_{2m-1}]$.

We give an integral expression for the Hilbert series of $C[\text{Gr}(2, m)]$.

6.2. **Lemma.** The Hilbert series for the ring $C[\text{Gr}(2, m)]$ has the fol-
lowing integral expression

\[ \chi(C[\text{Gr}(2, m)], t) = \frac{1}{4\pi\sqrt{-1}} \int_{|z|=1} \frac{(1 - \varepsilon^2)(1 - \varepsilon^{-2})}{(1 - \varepsilon t)^m(1 - \varepsilon^{-1} t)^m} \frac{d\varepsilon}{\varepsilon}. \]

**Proof.** Let us consider the polynomial ring \( C[x_1, \ldots, x_m, y_1, \ldots, y_m] \) in \( 2m \) independent variables \( x_1, \ldots, x_m, y_1, \ldots, y_m \). The group action of the special linear group \( SL(2, \mathbb{C}) \) on the polynomial ring is defined by

\[
\left( \begin{array}{cc}
  x_i \\
  y_i
\end{array} \right) \rightarrow g \left( \begin{array}{cc}
  x_i \\
  y_i
\end{array} \right), \quad g \in SL(2, \mathbb{C}), \quad 1 \leq i \leq m.
\]

Let \( R \) be the ring of invariant polynomials under the action of \( SL(2, \mathbb{C}) \). Then \( R \) is generated by all invariant polynomials of the form

\[ a_{ij} = \det \left( \begin{array}{cc}
  x_i \\
  y_i
\end{array} \right), \quad 1 \leq i \leq j \leq m,
\]

and the map \( \theta_{ij} \rightarrow a_{ij} \) defines a degree preserving ring isomorphism

\[ C[\text{Gr}(2, m)] \xrightarrow{\sim} R. \]

Then, by the Molien-Weyl formula, we have

\[ \chi(R_m, t) = \frac{1}{4\pi\sqrt{-1}} \int_{|z|=1} \frac{(1 - \varepsilon^2)(1 - \varepsilon^{-2})}{(1 - \varepsilon t)^m(1 - \varepsilon^{-1} t)^m} \frac{d\varepsilon}{\varepsilon}, \]

which proves the lemma.

§ 7. Rings of invariants of generic 2 by 2 matrices

Let \( X_1, \ldots, X_m \) be \( m \) generic 2 by 2 matrices. Let \( p_3, \ldots, p_{2m-1} \) be elements of \( C(2, m) \) defined by

\[ p_k = \sum_{i < j \leq k} \text{Tr} (X_i X_j), \quad 3 \leq k \leq 2m - 1. \]

We denote by \( B(2, m) \) the subring of \( C(2, m) \) generated by invariants:

\[ \text{Tr} (X_i), \quad \text{Tr} (X_i^2), \quad 1 \leq i \leq m, \quad p_3, \ldots, p_{2m-1}. \]

**7.1. Theorem.** Let \( C(2, m) \) be the ring of invariants of \( m \) generic 2 by 2 matrices. If \( m \geq 2 \) then \( C(2, m) \) is a free module of rank

\[ \frac{1}{m-1} \binom{2m-4}{m-2} 2^{m-2}. \]
over the ring $B(2, m)$.

Proof. Let $(A_1, \cdots, A_m)$ be a tuple of 2 by 2 matrices such that any invariant in $\text{Tr}(X_1), \text{Tr}(X_2), p_3, \cdots, p_{r_2 - 1}, 1 \leq i \leq m$ vanishes at $(A_1, \cdots, A_m)$. We first prove by induction on $m$ that any invariant which is not constant vanishes at $(A_1, \cdots, A_m)$. If $A_i = 0$, then our assertion is obvious by assumption of induction and hence we can assume that $A_i$ is not zero matrix. Note that $A_1, \cdots, A_m$ are nilpotent matrices since $\text{Tr}(A_i) = \text{Tr}(A_i^2) = 0$ for $i = 1, 2, \cdots, m$. Then by a suitable componentwise adjoint action of the group $GL(2, K)$ on the matrices $A_1, \cdots, A_m$, we can assume that $A_i$ has the form

$$A_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \quad \text{for some } a_i = 0.$$ 

In general, let $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ be a nilpotent 2 by 2 matrix which satisfies the equation $\text{Tr}(A_i B) = 0$. Then we have

$$\text{Tr}\left( \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \right) = a_i b_i = 0,$$

and hence $b_i = 0$. Since $B$ is a nilpotent matrix, $B$ has the form

$$B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$ 

By using this fact and the equation $p_3 = \cdots = p_{r_2 - 1} = 0$ successively, one observe that each matrix $A_i$ has the form

$$A_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \quad 1 \leq i \leq m.$$ 

This implies that $\text{Tr}(A_{i_1}, \cdots, A_{i_\ell}) = 0$, for any monomial $A_{i_1}, \cdots, A_{i_\ell}$, and hence any invariant which is not constant vanishes at $(A_1, \cdots, A_m)$. Therefore it follows from a fundamental theorem of Hilbert [H] that $C(2, m)$ is integral over the polynomial ring $B(2, m)$. Since Krull dimension of $C(2, m)$ is $4m - 3$, it follows that $\text{Tr}(X_1), \text{Tr}(X_2), p_3, \cdots, p_{r_2 - 1}$ is a homogeneous system of parameters of the ring $C(2, m)$. Then the Cohen-Macaulay property of the ring $C(2, m)$ implies that $C(2, m)$ is a free module over the polynomial ring $B(2, m)$. Then by [T2], rank of
$C(2, m)$ over $B(2, m)$ is $\frac{1}{m-1} \left(\frac{2m - 4}{m - 2}\right) 2^{m-2}$.

§ 8. Trace rings of generic 2 by 2 matrices

We now turn to consideration of trace rings of generic 2 by 2 matrices. Procesi [P2] proved a one-to-one correspondence between a $K$-basis of the ring $T(SL(2, K), m)$ and standard Young tableaux of shape $\sigma = 3^a 2^b 1^c$ for all $a, b, c \in \mathbb{N}$.

Procesi’s theorem in particular gives an explicit presentation of the Hilbert series of the trace ring $T(SL(2, K), m)$

$$(T(SL(2, K), m) = \sum_{a, b, c \in \mathbb{N}} L_{a, b, c} t^{a+b+c}$$

where $L_{a, b, c}$ is the number of standard Young tableaux of shape $3^a 2^b 1^c$ filled with indices from 1 to $m$.

From this fact Procesi (see [L2]) observed the following proposition and gave an elegant combinatorial proof of the functional equation for the Hilbert series $\chi(T(2, m), t)$. We give here a simple direct proof of Procesi’s observation.

8.1. Proposition. Let $\chi(T(2, m), t)$ be the usual Hilbert series in one variable $t$ for the trace ring $T(2, m)$. Then we have

$$\chi(T(2, m), t) = (1 - t)^{-m}\chi(C[Gr(2, m)], t).$$

Proof. By the Molien-Weyl formula for the trace ring $T(2, m)$ we have

$$\chi(T(2, m), t) = \frac{1}{4\pi \sqrt{1-t}} \sum_{d} \frac{(d+1)(d+m-1)}{(m-1)!} \frac{1}{(m-2)!} \prod_{i=2}^{m-2} (d+i)t^i$$

by 6.1 Lemma.

8.2. Corollary.

$$\chi(T(2, m), t) = \frac{1}{(1-t)^{2m}} \sum_{d} \frac{(d+1)(d+m-1)}{(m-1)!} \frac{1}{(m-2)!} \prod_{i=2}^{m-2} (d+i)t^i.$$
the ring $C(2, m)$. Recall that $\text{Tr}(X_j), \text{Tr}(X^2), p_3, \cdots, p_{2m-1}$, $1 \leq i \leq m$, is a homogeneous system of parameters of the ring $C(2, m)$. Then the Cohen-Macaulay property of the trace ring $T(2, m)$ says that $T(2, m)$ is a free module over the polynomial ring $B(2, m)$. Therefore we obtain

8.3. **Theorem.** The trace ring $T(2, m)$ ($m \geq 2$) is a free module of rank $1 + (2m - 4)(m - 2)^2$ over the polynomial ring $B(2, m)$.

**Proof.** Note that the map $\theta_i \mapsto p_i$ ($3 \leq i \leq 2m - 1$) defines a degree preserving isomorphism

\[ K[\theta_3, \cdots, \theta_{2m-1}] \rightarrow K[p_3, \cdots, p_{2m-1}] . \]

Then the theorem follows from 6.1. Lemma and 8.1. Proposition.

The following proposition gives relations in the ring $T(SL(2, K), m)$ corresponding to the Plücker relations

8.4. **Proposition.** Let $X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}$ be 2 by 2 matrices whose traces are all zeros. Then the following relation holds.

\[
X_{i_4}X_{i_3}X_{i_2}X_{i_1} - (X_{i_4}X_{i_3})X_{i_2}X_{i_1} + (X_{i_4}X_{i_3})X_{i_1}X_{i_2}
- (X_{i_4}X_{i_3})X_{i_1}X_{i_2} + (X_{i_4}X_{i_3})X_{i_1}X_{i_2}
+ (X_{i_4}X_{i_3})X_{i_1}X_{i_2} + \frac{1}{2} (\text{Tr}(X_{i_1}X_{i_2}) \text{Tr}(X_{i_3}X_{i_4}))
- \text{Tr}(X_{i_4}X_{i_3}) \text{Tr}(X_{i_2}X_{i_1}) + \text{Tr}(X_{i_4}X_{i_3}) \text{Tr}(X_{i_2}X_{i_1}) = 0 .
\]

**Proof.** Recall the multi-linear Cayley-Hamilton theorem for 2 by 2 matrices $A$ and $B$:

\[ AB + BA - \text{Tr}(A)B - \text{Tr}(B)A + \text{Tr}(A) \text{Tr}(B) - \text{Tr}(AB) = 0 . \]

Applying the multi-linear Cayley-Hamilton theorem, we have

\[
X_{i_4}X_{i_3}X_{i_2}X_{i_1} + X_{i_4}X_{i_3}X_{i_1}X_{i_2} - (X_{i_4}X_{i_3})X_{i_2}X_{i_1} - (X_{i_4}X_{i_3})X_{i_1}X_{i_2}
+ (X_{i_4}X_{i_3}) \text{Tr}(X_{i_1}X_{i_2}) - \text{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4}) = 0 ,
\]

and

\[
X_{i_4}X_{i_3}X_{i_2}X_{i_1} = X_{i_4}X_{i_3}X_{i_2}X_{i_1} + (X_{i_4}X_{i_3})X_{i_2}X_{i_1} + (X_{i_4}X_{i_3})X_{i_1}X_{i_2}
- (X_{i_4}X_{i_3})X_{i_1}X_{i_2} - (X_{i_4}X_{i_3})X_{i_1}X_{i_2}
- \text{Tr}(X_{i_4}X_{i_3}) \text{Tr}(X_{i_2}X_{i_1}) + \text{Tr}(X_{i_4}X_{i_3}) \text{Tr}(X_{i_2}X_{i_1}) .
\]
Hence we have

\[ (*) \quad 2X_1X_2X_3X_4 - \text{Tr} (X_1X_2)X_3X_4 + \text{Tr} (X_1X_3)X_2X_4 \\
- \text{Tr} (X_1X_2)X_3X_4 + \text{Tr} (X_1X_3)X_2X_4 + \text{Tr} (X_1X_4)X_2X_3 \\
+ \text{Tr} (X_1X_2) \text{Tr} (X_3X_4) - \text{Tr} (X_1X_4) \text{Tr} (X_2X_3) \\
+ \text{Tr} (X_1X_2) \text{Tr} (X_3X_2) - \text{Tr} (X_1X_3X_2X_4) = 0. \]

We claim that

\[ 2 \text{Tr} (X_1X_2X_3X_4) = \text{Tr} (X_1X_2) \text{Tr} (X_3X_4) - \text{Tr} (X_1X_3) \text{Tr} (X_2X_4) \\
+ \text{Tr} (X_1X_2) \text{Tr} (X_3X_2). \]

Since both sides of the equation above are linear with respect to matrices \( X_1, \ldots, X_4 \), the claim is true if it is true when each \( X_i \) is replaced by one of matrices consisting of a basis of \( \text{Lie}(\text{SL}(2, m)) \). This can be easily verified. Then the lemma follows from the relation \((*)\) and the claim.

§9. An explicit description of \( C(2, 4) \) and \( T(2, 4) \)

Explicit description of the rings of invariants and the trace rings of two and three generic 2 by 2 matrices are given in [F-H-L], [F1] and [L-V]. They showed:

1. \( C(2, 2) = B(2, 2) \) and \( T(2, 2) \) is a free \( C(2, 2) \) module with basis \( 1, X_1, X_2, X_3X_4 \) (see [F-H-L]).

2. \( C(2, 3) \) is a free \( B(2, 3) \) module with basis \( 1, \text{Tr} (X_1X_2X_3) \) (see [F2]) and \( T(2, 3) \) is a free \( B(2, 3) \) module with basis \( 1, X_1, X_2, X_3, X_4, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4 \) (see [L-V]).

In this section we will give an explicit description of the ring of invariants and the trace ring of four generic 2 by 2 matrices.

9.1. **Theorem.** (1) \( C(2, 4) \) is a free module over the polynomial ring \( B(2, 4) \) with basis \( 1, \text{Tr} (X_1X_i), \text{Tr} (X_2X_i), \text{Tr} (X_3X_i), \text{Tr} (X_4X_i), \text{Tr} (X_1X_2X_i), \text{Tr} (X_1X_3X_i), \text{Tr} (X_1X_4X_i), \text{Tr} (X_2X_3X_i), \text{Tr} (X_2X_4X_i), \text{Tr} (X_3X_4X_i), 1 \leq i \leq 4, 1 \leq i < j \leq 4, 1 \leq i < j < k \leq 4. \)

(2) \( T(2, 4) \) is a free module over the ring \( B(2, 4) \) with basis \( 1, X_1, X_2X_3, X_1X_4X_5, X_2X_3X_4, X_2X_3X_5, X_1X_2X_3X_4, \text{Tr} (X_1X_i), \text{Tr} (X_2X_i), \text{Tr} (X_3X_i), \text{Tr} (X_4X_i), \text{Tr} (X_iX_jX_k), 1 \leq i \leq 4, 1 \leq i < j \leq 4, 1 \leq i < j < k \leq 4. \)

**Proof.** Formanek [F2] calculated the multi-valued Hilbert series:

\[ \chi(T(2, 4), t) = \frac{(1 + t)^6(1 + t^2)(1 + t^3)}{(1 - t)(1 - t^2)^3}. \]
It is easy to prove (2) by using 8.3. Theorem, 8.4. Proposition and the formula above. The trace map $T: T(2, 4) \to C(2, 4)$ is surjective and hence (1) follows from (2), 7.1. Theorem and the following relation

$$2 \text{Tr} (X_1X_2X_3X_4) = \text{Tr} (X_1X_2) \text{Tr} (X_3X_4) - \text{Tr} (X_1X_3) \text{Tr} (X_2X_4)$$

$$+ \text{Tr} (X_1X_4) \text{Tr} (X_2X_3),$$

where Tr $(X_i) = 0$, for $1 \leq i \leq 4$.

References


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