Canad. Math. Bull. Vol. **44** (1), 2001 pp. 19–21

# Multiplicities of Binary Recurrences

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*Abstract.* In this note the multiplicities of binary recurrences over algebraic number fields are investigated under some natural assumptions.

Let $\mathbb{K}$ be an algebraic number field of degree $d$ and $u_0$, $u_1$ algebraic integers in $\mathbb{K}$, and $\omega \in \mathbb{K}^*$. Furthermore, let $\{u_n\}_{n=0}^{\infty}$ be a non-degenerate binary recurrence sequence with companion polynomial $f(X) \in \mathbb{Z}[X]$. Denote by $\lambda$ and $\mu$ the zeros of $f(X)$. The $\omega$-multiplicity of the sequence $\{u_n\}_{n=0}^{\infty}$ is defined as the number of indices $m$ such that $u_m = \omega$ (*cf.* [ST] and the references given there). For an element $\alpha \in \mathbb{K}$ the (usual) height is denoted by $H(\alpha)$.

***Theorem*** *If* $\min(|\lambda|, |\mu|) > 1$ *and* $\max\bigl(H(u_0), H(u_1)\bigr) > c(d, f, \omega)$, *where* $c(d, f, \omega)$ *is an effectively computable constant depending only on* $d$, $f$ *and* $\omega$, *then the* $\omega$*-multiplicity of* $\{u_n\}_{n=0}^{\infty}$ *is at most one.*

Therefore if $\mathbb{K}$, $\omega$ and the companion polynomial (with $|\lambda| > 1$, $|\mu| > 1$) are given, then apart from some effectively determinable exceptional pairs $(u_0, u_1)$ the $\omega$-multiplicity of the sequence is at most one.

## Auxiliary Results

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be nonzero algebraic numbers. Write $\mathbb{L}$ for their splitting field and put $g = [\mathbb{L} : \mathbb{Q}]$. Denote by $A_1, \dots, A_n$ upper bounds for the respective heights of $\alpha_1, \dots, \alpha_n$, where we suppose that $A_j \geq 2$ for $1 \leq j \leq n$. Write

$$\Omega' = \prod_{j=1}^{n-1} \log A_j, \quad \Omega = \Omega' \log A_n.$$

Let $b_1, \dots, b_n$ be rational integers, not all zero, and set $B = \max\{|b_1|, \dots, |b_n|, 2\}$.

***Lemma 1*** *If* $\Lambda = |\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| \neq 0$, *then*

$$\Lambda > \exp\{-c(n, g)\Omega \log \Omega' \log B\},$$

*where* $c(n, g)$ *is an effectively computable constant depending only on* $g$ *and* $n$.

Received by the editors March 9, 1999.

Research supported in part by Grant T25371 from the Hungarian National Foundation for Scientific Research.

AMS subject classification: Primary: 11B37; secondary: 11J86.

**Proof** See [SPTS, p. 66].

It is well known that $(\mu-\lambda)u_n = \alpha\lambda^n + \beta\mu^n$ with $\alpha = u_0\mu - u_1$, $\beta = u_1 - u_0\lambda$. The $\omega$-multiplicity therefore is the number of $n \in \mathbb{N}$ with

$$\alpha\lambda^n + \beta\mu^n = \gamma, \tag{1}$$

where $\gamma = (\mu-\lambda)\omega$. For an upper bound to the number of solutions of (1) in a very general case we refer to the paper of Beukers and Schlickewei [BS]. The crucial point is to handle the algebraic number field case.

***Lemma 2*** *Suppose $\lambda$, $\mu$, $\alpha$, $\beta$ lie in $\mathbb{K}$, with $|\lambda| > 1$, $|\mu| > 1$, and $\lambda/\mu$ not a root of 1, $\alpha\beta \neq 0$. Then there is an effectively computable $c_0 = c_0(d,\lambda,\mu)$ such that there is at most one $n \in \mathbb{N}$ with*

$$0 < |\alpha\lambda^n + \beta\mu^n| < \max(|\alpha|,|\beta|)\bigl(2 + \log H(\alpha/\beta)\bigr)^{-c_0}. \tag{2}$$

**Proof** $c_1, c_2, \ldots$ will be effectively computable constants depending on $d$, $\lambda$, $\mu$. We may suppose that $|\alpha| \le |\beta|$ and set $h = 2 + \log H(\alpha/\beta)$. Then (2) may be rewritten as

$$0 < |(-\alpha/\beta)^1(\lambda/\mu)^n - 1| < |\mu|^{-n}h^{-c_0} < |\mu|^{-n}. \tag{3}$$

By Lemma 1,

$$|(-\alpha/\beta)^1(\lambda/\mu)^n - 1| > \exp(-c_1 h \log n).$$

Comparison with (3) and taking logarithms yields $-c_1 h \log n < -n \log|\mu|$, hence $n/\log n < c_2 h$, hence

$$n < c_3 h \log h. \tag{4}$$

Suppose $0 < n_1 < n_2$ were two solutions of (2), hence of (3). When $c_0 \ge 2$, we have $h^{-c_0} \le 1/4$, and we obtain

$$|(\lambda/\mu)^{n_2-n_1} - 1| < 4h^{-c_0}. \tag{5}$$

Since $\lambda/\mu$ is not a root of 1, the left hand side is $\neq 0$, so that by Lemma 1 it is

$$> \exp\bigl(-c_4 \log(n_2-n_1)\bigr) > \exp(-c_5 \log h) = h^{-c_5}$$

by (4). Comparison with (5) gives $h^{c_0-c_5} < 4$, whence $2^{c_0-c_5} < 4$, which is impossible if $c_0 \ge c_5 + 2$.

In what follows, $\sigma$ will denote embeddings $\mathbb{K} \hookrightarrow \mathbb{C}$, and for $\xi \in \mathbb{K}$ we set $\overline{|\xi|} = \max_\sigma |\sigma(\xi)|$.

***Lemma 3*** *Let $\gamma \in \mathbb{K}^*$, and $\alpha, \beta \in O_{\mathbb{K}}$. Suppose*

$$\min_\sigma \min\bigl(|\sigma(\lambda)|, |\sigma(\mu)|\bigr) > 1 \tag{6}$$

*and*

$$\max(\overline{|\alpha|}, \overline{|\beta|}) > c_6(d,\lambda,\mu,\gamma). \tag{7}$$

*Then the equation* (1) *possesses at most one solution $n \in \mathbb{N}$.*

https://doi.org/10.4153/CMB-2001-003-4 Published online by Cambridge University Press

**Proof** Set $m = \max(\overline{|\alpha|}, \overline{|\beta|})$, and suppose $m$ is so large that

$$\frac{m}{(2 + d\log m)^{c_0}} > \overline{|\gamma|}.$$

We may suppose that $\overline{|\alpha|} \leq \overline{|\beta|}$, and after an appropriate embedding we may further suppose that $\overline{|\beta|} = |\beta|$, so that $m = \max(|\alpha|, |\beta|)$. Then (1) yields

$$|\alpha\lambda^n + \beta\mu^n| = |\gamma| \leq \overline{|\gamma|} < \frac{m}{(2 + d\log m)^{c_0}} \leq \frac{\max(|\alpha|, |\beta|)}{\left(2 + \log H(\alpha/\beta)\right)^{c_0}},$$

because $\log H(\alpha/\beta) \leq d\log m$, since $\alpha, \beta$ are in $O_{\mathbb{K}}$. According to Lemma 2, there is at most one such $n$.

## Proof of the Theorem

Now $\lambda, \mu$ are rational or are conjugate quadratics, so that $\min(|\lambda|, |\mu|) > 1$ yields (6). As noted above, we are dealing with (1) where $\alpha = u_0\mu - u_1$, $\beta = u_1 - u_0\lambda$, $\gamma = (\mu - \lambda)\omega$, so that

$$u_0 = \frac{\alpha + \beta}{\lambda - \mu}, \quad u_1 = \frac{\lambda\alpha + \mu\beta}{\lambda - \mu}.$$

Since $\alpha, \beta \in O_{\mathbb{K}}$,

$$\max\left(H(u_0), H(u_1)\right) \leq c_7(\lambda, \mu)\left(\max(\overline{|\alpha|}, \overline{|\beta|})\right)^d$$

and therefore $\max\left(H(u_0), H(u_1)\right) > c(\lambda, \mu, d, \omega)$ implies (7).

## References

[1] F. Beukers and H. P. Schlickewei, *The equation $x + y = 1$ in finitely generated groups.* Acta Arith. **78**(1996), 189–199.

[2] T. N. Shorey, A. J. van der Poorten, R. Tijdeman and A. Schinzel, *Applications of the Gel'fond-Baker Method to Diophantine Equations.* In: Transcendence Theory: Advances and Applications (eds. A. Baker and D. W. Masser), Academic Press, London-New York-San Francisco, 1977, 59–77.

[3] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations.* Cambridge University Press, Cambridge, 1986.

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