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OPTIMAL RESULTS IN LOCAL BIFURCATION THEORY

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Let us consider the abstract equation

(0.1) $$L(\varepsilon)u + F(\varepsilon,u) = 0 ,$$

where $F(\varepsilon,u) = O(|u|^2)$ for ε near zero. In this paper we define a multiplicity depending only on $L(\varepsilon)$ allowing us to obtain the following result: "Odd multiplicity entails bifurcation and, if the multiplicity is even, it is possible to find $F(\varepsilon,u)$ such that the only solution to (0.1) near the origin are the trivial ones".

1. Introduction.

Let U, V be two real Banach spaces and $N: R \times U \to V$ a nonlinear operator such that

(1.1) $$N(\varepsilon,0) = 0$$

for ε in a neighbourhood of zero. We seek nontrivial solutions to

(1.2) $$N(\varepsilon,u) = 0$$

bifurcating from $(\varepsilon,u) = (0,0)$, where we assume

(1.3) $$N(\varepsilon,u) = L(\varepsilon)u + F(\varepsilon,u)$$

and $L(\varepsilon)$ and $F(\varepsilon,u)$ satisfy the following conditions:

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HL1. - $L(\varepsilon) : U \to V$ is a linear continuous operator from U to V such that the mapping $\varepsilon \to L(\varepsilon)$, from R to $L(U,V)$, is of class three. Here we denote by $L(U,V)$ the space of linear continuous operators between U and V .

HL2. - $L(0)$ is a Fredholm operator of index zero.

HF . - $F(\varepsilon,u)$ is a C^2-mapping from a neighbourhood of zero in $R \times U$ to V such that

(1.4) $$F(\varepsilon,0) = 0, \; D_u F(\varepsilon,0) = 0$$

for ε sufficiently small.

By the implicit function theorem, a necessary condition for the origin to be a bifurcation point of (1.2) is

(1.5) $$\dim N(L(0)) = m \geqq 1 \; .$$

In the literature concerning this topic, it is usual to define a generalised algebraic multiplicity for $L(\varepsilon)$ at the critical value of the parameter, $\varepsilon = 0$. In all cases, an odd multiplicity entails bifurcation from $(\varepsilon,u) = (0,0)$ and there are "particular" counterexamples when the multiplicity is an even number.

Roughly speaking odd multiplicity implies an odd number of eigenvalues of $L(\varepsilon)$ (counted with their algebraic multiplicities) crossing the imaginary axis at $\varepsilon = 0$. Thus, odd multiplicity entails a change in the stability of the trivial solution $(\varepsilon,u) = (\varepsilon,0)$ at $\varepsilon = 0$. So, we obtain bifurcation from $(\varepsilon,u) = (0,0)$. See Chow-Hale [1] and Kielhöfer [4] for a more extensive information.

Not all notions of generalised multiplicities are sufficiently transparent since they do not show which intrinsic properties of $L(\varepsilon)$ yield an odd or an even multiplicity (Kielhofer [4]).

In this direction, we shall give here an "optimal result" involving $L(0)$, $L'(0)$ and $L''(0)$ (primes denotes derivation with respect to the parameter) allowing dim $N(L(0))$ to be even or odd.

More specifically, if $L(0)$, $L'(0)$, $L''(0)$ satisfy a suitable nondegeneracy condition (see (2.5)), we define a concept of multiplicity (see (2.7)) depending only on $L(0)$, $L'(0)$, $L''(0)$ and we obtain the following result (theorem in Section two):

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"Odd multiplicity entails bifurcation and, if the multiplicity is even, it is possible to find $F(\varepsilon,u)$ such that the only solution to (1.2) in a neighbourhood of $(\varepsilon,u) = (0,0)$ are the trivial ones".

Our nondegeneracy condition is a natural extension of the conditions of Crandall-Rabinowitz [2] and Westreich [7]. Our result generalises the above ones allowing dim $N(L(0))$ to be even. In [3] we gave a version of our result without proof.

In Section two we give the main result, in Section three the proof of the result in Section two and in Section four we give an example.

2. Main result.

To simplify the notation, we shall write

(2.1) $$L_0 = L(0),\ L_1 = L'(0),\ L_2 = \tfrac{1}{2}L''(0)$$

Then equation (1.2) can be written as

(2.2) $$L_0u + \varepsilon L_1u + \varepsilon^2 L_2u + R(\varepsilon)u + F(\varepsilon,u) = 0\ ,$$

where

(2.3) $$R(\varepsilon) = O(\varepsilon^3)\ .$$

Now, we give the following definitions:

DEFINITION 1. *We say that zero is a generic eigenvalue of the chain* (L_0,L_1,L_2) *if the following conditions are satisfied*

(2.4) $$\dim N(L_0) = m \geqq 1\ ,$$

(2.5) $$L_1(N(L_0)) \oplus L_2(N(L_1) \cap N(L_0)) \oplus R(L_0) = V\ .$$

Remark 1. Crandall-Rabinowitz [2] and Westreich [7] use

(2.6) $$L_1(N(L_0)) \oplus R(L_0) = V\ ,$$

instead of (2.5). Since L_0 is a Fredholm operator of index zero, condition (2.6) entails

$$N(L_1) \cap N(L_0) = \mathrm{Span}[0]\ ,$$

hence, (2.5) is more general than (2.6).

DEFINITION 2. *If zero is a generic eigenvalue of* (L_0,L_1,L_2)*, we shall call the multiplicity of* (L_0,L_1,L_2) *at zero the number*

(2.7) $$\chi = n_1 + 2n_2\ ,$$

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where

$$(2.8) \qquad n_1 = \dim L_1(N(L_0)),\ n_2 = \dim L_2(N(L_1) \cap N(L_0)) .$$

Remark 2. Observe that χ is odd if and only if n_1 is odd. So, if (2.6) holds, χ is odd if and only if $\dim N(L_0)$ is odd. However, if $n_2 \neq 0$, it is possible for $\dim N(L_0)$ to be even and χ odd.

With this notation, we obtain the following result

THEOREM 1. *The following conditions are equivalent:*

C1. - χ *is an odd number.*

C2. - For all $F(\varepsilon,u)$ *satisfying HF the origin is a bifurcation point of the equation (2.2).*

Observe that, under condition (2.5), our result is optimal. That is, our multiplicity is optimal and it is given by intrinsic properties of $L(\varepsilon)$.

In particular our result implies the optimality of the result in Westreich [6].

Moreover, Theorem 1 tell us that, if χ is even, it is necessary to go to the full equation (2.2) in order to obtain conditions for bifurcation. This is what Lopez does in [5].

3. Proof of Theorem 1.

C1 ⟹ C2 .

Suppose χ is odd. By a Lyapunov-Schmidt reduction, we reduce our original problem, in general infinite-dimensional, to the one of solving a finite-dimensional equation.

Let X, Z be subspaces in U such that

$$N(L_0) = X \oplus [N(L_1) \cap N(L_0)] ,$$
$$U = N(L_0) \oplus Z .$$

Let now $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be continuous projections

$P_1 : U \to X$, along $[N(L_1) \cap N(L_0)] \oplus Z$,

$P_2 : U \to N(L_1) \cap N(L_0)$, along $X \oplus Z$,

$P_3 : U \to Z$, along $N(L_0)$,

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$$Q_1: V \to R(L_0), \quad \text{along} \quad L_1(X) \oplus L_2(N(L_1) \cap N(L_0)),$$

$$Q_2: V \to L_2(N(L_1) \cap N(L_0)), \quad \text{along} \quad L_1(X) \oplus R(L_0),$$

$$Q_3: V \to L_1(X) \quad \text{along} \quad L_2(N(L_1) \cap N(L_0)) \oplus R(L_0).$$

If, for each $u \in U$, we denote

$$x = P_1u, \ y = P_2u, \ z = P_3u,$$

then $u = x + y + z$ and the solutions to (2.2) are given by the solutions to the system

(3.1a) $$Q_1L_0z+\varepsilon Q_1L_1z+\varepsilon^2Q_1L_2(x+z)+Q_1R(\varepsilon)(x+y+z)+Q_1F(\varepsilon,x+y+z) = 0,$$

(3.1b) $$\varepsilon Q_2L_1z+\varepsilon^2Q_2L_2(x+y+z)+Q_2R(\varepsilon)(x+y+z)+Q_2F(\varepsilon,x+y+z) = 0,$$

(3.1c) $$\varepsilon Q_3L_1(x+z)+\varepsilon^2Q_3L_2(x+z)+Q_3R(\varepsilon)(x+y+z)+Q_3F(\varepsilon,x+y+z) = 0.$$

The left hand side of (3.1a) defines a C^2-mapping (denoted by $G(\varepsilon,x,y,z)$) from a neighbourhood of zero in $R \times X \times [N(L_1) \cap N(L_0)] \times Z$ into $R(L_0)$ satisfying

$$G(0,0,0,0) = 0, \ D_zG(0,0,0,0) = Q_1L_0.$$

Hence, the implicit function theorem gives the existence of a neighbourhood $B_{\varepsilon xy}$ of the origin in $R \times X \times [N(L_1) \cap N(L_0)]$, a neighbourhood B_z of the origin in Z and an unique function of class two

$$\Xi: B_{\varepsilon xy} \to B_z,$$

such that $\Xi(0,0,0) = 0$ and for all $(\varepsilon,x,y) \in B_{\varepsilon xy}$,

(3.2) $$G(\varepsilon,x,y,\Xi(\varepsilon,x,y)) = 0.$$

Moreover, since $G(\varepsilon,0,0,0) = 0$ for sll ε in a neighbourhood of zero, we obtain

(3.3) $$\Xi(\varepsilon,0,0) = 0$$

for ε sufficiently small. Now, by differentiating in (3.1a), we obtain

(3.4) $$D_x\Xi(0,0,0) = 0, \ D_y\Xi(0,0,0) = 0$$
$$D_{\varepsilon x}\Xi(0,0,0) = 0, \ D_{\varepsilon y}\Xi(0,0,0) = 0.$$

Thus, we have reduced our general problem to solving the finite-dimensional system which we shall call *bifurcation equation*:

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(3.5a) $$\varepsilon^2 Q_2 L_2 x+\varepsilon^2 Q_2 L_2 y+Q_2 \overline{R}(\varepsilon)(x+y)+Q_2 \overline{F}(\varepsilon,x+y) = 0 ,$$

(3.5b) $$\varepsilon Q_3 L_1 x+\varepsilon^2 Q_3 L_2 x+Q_3 \overline{R}(\varepsilon)(x+y)+Q_3 \overline{F}(\varepsilon,x+y) = 0 ,$$

where

(3.6) $$\overline{R}(\varepsilon) = O(\varepsilon^3)$$

and $\overline{F}$ is of order two in (x,y) uniformly in ε .

Now since

$$Q_2 L_2 : N(L_1) \cap N(L_0) \to L_2(N(L_1) \cap N(L_0))$$

and

$$Q_3 L_1 : X \to L_1(X)$$

are both isomorphisms, solving (3.5) is equivalent to solve the system

(3.7a) $$\varepsilon^2 P_2 (Q_2 L_2)^{-1} Q_2 L_2 x+\varepsilon^2 y+P_2 (Q_2 L_2)^{-1} Q_2 \overline{R}(\varepsilon)(x+y)+P_2 (Q_2 L_2)^{-1} Q_2 \overline{F}(\varepsilon,x+y) = 0,$$

(3.7b) $$\varepsilon x+\varepsilon^2 P_1 (Q_3 L_1)^{-1} Q_3 L_2 x+P_1 (Q_3 L_1)^{-1} Q_3 \overline{R}(\varepsilon)(x+y)+P_1 (Q_3 L_1)^{-1} Q_3 \overline{F}(\varepsilon,x+y) = 0.$$

Now, if we choose bases in X and $N(L_1) \cap N(L_0)$, we can write (3.7) in coordinates as an equation of the form

(3.8) $$\begin{bmatrix} A(\varepsilon) & B(\varepsilon) \\ C(\varepsilon) & D(\varepsilon) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} P_2 (Q_2 L_2)^{-1} Q_2 \overline{F}(\varepsilon,x+y) \\ P_1 (Q_3 L_1)^{-1} Q_3 \overline{F}(\varepsilon,x+y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $A(\varepsilon)$ is a $n_2 \times n_1$-matrix such that $A(\varepsilon) = O(\varepsilon^2)$, $B(\varepsilon)$ is a $n_2 \times n_2$-matrix such that $B(\varepsilon) = \varepsilon^2 I + O(\varepsilon^3)$, $C(\varepsilon)$ is a $n_1 \times n_1$-matrix such that $C(\varepsilon) = \varepsilon I + O(\varepsilon^2)$ and $D(\varepsilon)$ is a $n_1 \times n_2$-matrix such that $D(\varepsilon) = O(\varepsilon^3)$. Thus, we have

(3.9) $$\det \begin{bmatrix} A(\varepsilon) & B(\varepsilon) \\ C(\varepsilon) & D(\varepsilon) \end{bmatrix} = \pm\varepsilon^{n_1+2n_2} + O(\varepsilon^{n_1+2n_2+1}),$$

and, since $\chi = n_1 + 2n_2$ is odd, the following Lemma 1 (Theorem 7.1. in page 201 of Chow-Hale [1]) forces the origin to be a bifurcation point for (2.2).

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LEMMA 1. *Suppose* $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ *is an open neighbourhood of* $(\varepsilon_0, 0)$,

$$F: \Omega \to \mathbb{R}^d$$

$$F(\varepsilon, v) = B_0(\varepsilon)v + F_1(\varepsilon, v)$$

where $v \in \mathbb{R}^d$, $B_0(\varepsilon)$ *is a* $d \times d$, C^m , $m \geqq 2$, *matrix function of* ε , F_1 *is a* C^m *vector function of* ε, v

$$F_1(\varepsilon, 0) = 0, \; D_v F_1(\varepsilon, 0) = 0 \; .$$

If $\varepsilon_0 \in \mathbb{R}$ *is such that*

$$\sigma(B_0(\varepsilon_0)) = \{0\} \; ,$$

$$\det B_0(\varepsilon) \text{ changes sign at } \varepsilon = \varepsilon_0 \; ,$$

then $(\varepsilon_0, 0)$ *is a bifurcation point for the equation*

$$F(\varepsilon, v) = 0 \; .$$

Also, there is a connected set $C \subset \mathbb{R} \times (\mathbb{R}^d - \{0\})$ *of zeros of* F *with* $(\varepsilon_0, 0) \in \overline{C}$, *the closure of* C .

C2 $\Longrightarrow$ C1 .

Suppose now $\chi = n_1 + 2n_2$ is even; that is, n_1 is even. We shall find then F_1, F_2, F_3 with values in $R(L_0)$, $L_2(N(L_1) \cap N(L_0))$, $L_1(X)$, respectively, such that the unique solutions to the following system in a neighbourhood of $(\varepsilon, x, y, z) = (0, 0, 0, 0)$ are the trivial ones.

(3.10a) $$Q_1L_0z + \varepsilon Q_1L_1z + \varepsilon^2 Q_1L_2(x+z) + Q_1R(\varepsilon)(x+y+z) + F_1(\varepsilon, x+y+z) = 0 \; ,$$

(3.10b) $$\varepsilon Q_2L_1z + \varepsilon^2 Q_2L_2(x+y+z) + Q_2R(\varepsilon)(x+y+z) + F_2(\varepsilon, x+y+z) = 0 \; ,$$

(3.10c) $$\varepsilon Q_3L_1(x+z) + \varepsilon^2 Q_3L_2(x+z) + Q_3R(\varepsilon)(x+y+z) + F_3(\varepsilon, x+y+z) = 0 \; .$$

First, we shall prove the following result

LEMMA 2. *There exists a linear continuous operator*

$$M(\varepsilon): L_1(X) \to R(L_0)$$

such that

$$M(\varepsilon)(\varepsilon Q_3L_1x + \varepsilon^2 Q_3L_2x + Q_3R(\varepsilon)x) = \varepsilon^2 Q_1L_2x + Q_1R(\varepsilon)x$$

for ε *in a neighbourhood of zero.*

Proof. Since the operator

$$Q_3L_1+\varepsilon Q_3L_2+ Q_3R(\varepsilon)\varepsilon^{-1}: X \to L_1(X)$$

is invertible for ε sufficiently small, if we define

$$M(\varepsilon) = (\varepsilon Q_1L_2+Q_1R(\varepsilon)\varepsilon^{-1})P_1(Q_3L_1+\varepsilon Q_3L_2+ Q_3R(\varepsilon)\varepsilon^{-1})^{-1}, \tag{3.12}$$

$M(\varepsilon)$ satisfies relation (3.11). □

Let us call now, H_1, H_2, H_3 the left hand sides of (3.10a), (3.10b), (3.10c), respectively. Then, by (3.11), we obtain

$$\begin{aligned}H_1(\varepsilon,x,y,z)-M(\varepsilon)H_3(\varepsilon,x,y,z)\\ &= Q_1L_0z+\varepsilon Q_1L_1z+\varepsilon^2Q_1L_2z-\varepsilon M(\varepsilon)Q_3L_1z+Q_1R(\varepsilon)(y+z)\\ &-\varepsilon^2M(\varepsilon)Q_3L_2z-M(\varepsilon)Q_3R(\varepsilon)(y+z)+F_1-M(\varepsilon)F_3\\ &= 0\ .\end{aligned}$$

Now, supposed F_2, F_3 have been given, then we can define

$$F_1 = M(\varepsilon)F_3\ . \tag{3.14}$$

The choice of F_2, F_3 will be made below. So, we have

$$F_1-M(\varepsilon)F_3 = 0\ . \tag{3.15}$$

Thus, for this choice, (3.13) is written as

$$\begin{aligned}&Q_1L_0z+\varepsilon Q_1L_1z+\varepsilon^2Q_1L_2z-\varepsilon M(\varepsilon)Q_3L_1z+Q_1R(\varepsilon)(y+z)\\ &-\varepsilon^2M(\varepsilon)Q_3L_2z-M(\varepsilon)Q_3R(\varepsilon)(y+z)\\ &= 0\ .\end{aligned} \tag{3.16}$$

This equation can be written equivalently as

$$\begin{aligned}&(Q_1L_0+\varepsilon Q_1L_1+\varepsilon^2Q_1L_2-\varepsilon M(\varepsilon)Q_3L_1+Q_1R(\varepsilon)\ -\varepsilon^2M(\varepsilon)Q_3L_2-M(\varepsilon)Q_3R(\varepsilon))z\\ &= (M(\varepsilon)Q_3R(\varepsilon)-Q_1R(\varepsilon))y\ .\end{aligned} \tag{3.17}$$

Thus, since the operator of the left hand side of (3.17) is invertible for ε sufficiently small, we can solve (3.17) to obtain

$$Z(\varepsilon,y) = (Q_1L_0+O(\varepsilon))^{-1}(M(\varepsilon)Q_3R(\varepsilon)-Q_3R(\varepsilon))y\ . \tag{3.18}$$

Now, putting $Z(\varepsilon,y)$ given by (3.18) in (3.10b) and (3.10c), we obtain

(3.19a) $$\varepsilon^2 Q_2 L_2(x+y)+Q_2\overline{R}(\varepsilon)(x+y)+F_2(\varepsilon,x+y+Z(\varepsilon,y)) = 0\ ,$$

(3.19b) $$\varepsilon Q_3 L_1 x+\varepsilon^2 Q_3 L_2 x+Q_3\overline{R}(\varepsilon)(x+y)+F_3(\varepsilon,x+y+Z(\varepsilon,y)) = 0\ ,$$

where $\overline{R}(\varepsilon) = O(\varepsilon^3)$.

Now, we need the following result

LEMMA 3. *There exists a linear continuous operator*

$$N(\varepsilon):\ L_2(N(L_1)\ \cap\ N(L_0)) \to L_1(X)$$

such that

(3.20) $$N(\varepsilon)(\varepsilon^2 Q_2 L_2 y+Q_2\overline{R}(\varepsilon)y) = Q_3\overline{R}(\varepsilon)y$$

for ε *in a neighbourhood of zero.*

Proof. Since the operator

$$Q_2 L_2+\varepsilon^{-2}Q_2\overline{R}(\varepsilon):\ N(L_1)\ \cap\ N(L_0) \to\ L_2(N(L_1)\ \cap\ N(L_0))$$

is invertible for ε sufficiently small, if we define

(3.21) $$N(\varepsilon) = \varepsilon^{-2}Q_3\overline{R}(\varepsilon)\ P_2(Q_2L_2+\varepsilon^{-2}Q_2\overline{R}(\varepsilon))^{-1}\ ,$$

$N(\varepsilon)$ satisfies relation (3.20). □

Let us call now, $\overline{H}_2$, $\overline{H}_3$ the left hand sides of (3.19a), (3.19b), respectively. Then by (3.20), we obtain

(3.22)
$$\begin{aligned}\overline{H}_3(\varepsilon,x,y)&-N(\varepsilon)\overline{H}_2(\varepsilon,x,y)\\ &= \varepsilon Q_3L_1x+\varepsilon^2Q_3L_2x+Q_3\overline{R}(\varepsilon)x-\varepsilon^2N(\varepsilon)Q_2L_2x-N(\varepsilon)Q_2\overline{R}(\varepsilon)x\\ &\quad + F_3(\varepsilon,x+y+Z(\varepsilon,y))-N(\varepsilon)F_2(\varepsilon,x+y+Z(\varepsilon,y))\\ &= 0\ .\end{aligned}$$

This equation can be written as

(3.23) $$\varepsilon x = -(Q_3L_1+Q_3\hat{R}(\varepsilon))^{-1}(F_3-N(\varepsilon)F_2)(\varepsilon,x+y+Z))$$

where $\hat{R}(\varepsilon) \in L(U,V)$ satisfies $\hat{R}(\varepsilon) = O(\varepsilon)$.

Now, by (3.14), we obtain

(3.24)
$$\begin{aligned}F_3(\varepsilon,x+y+Z(\varepsilon,y))&-N(\varepsilon)F_2(\varepsilon,x+y+Z(\varepsilon,y))\\ &= Q_3(I-N(\varepsilon)Q_2)(F_2\ +\ F_3)(\varepsilon,x+y+Z(\varepsilon,y))\ .\end{aligned}$$

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Let us choose bases in X and $N(L_1) \cap N(L_0)$. It is easy to prove that there exist F_2, F_3 such that

(3.25) $$P_1(Q_3L_1+Q_3\hat{R}(\varepsilon))^{-1}Q_3(I-N(\varepsilon)Q_2)(F_2+F_3) = -(x_2^3,-x_1^3,\dots,x_{n1}^3,-x_{n1-1}^3)$$

and

(3.26) $$(Q_2L_2+Q_2\overline{R}(\varepsilon))^{-1}F_2 = (y_1^3,\dots,y_{n2}^3) .$$

That is, the system (3.25), (3.26) can be solved in F_i, $i = 2,3$.

By (3.25), equation (3.23) is equivalent to

$$\begin{aligned} \varepsilon x_1 &= x_2^3 \\ \varepsilon x_2 &= -x_1^3 \\ &\vdots \\ \varepsilon x_{n1-1} &= x_{n1}^3 \\ \varepsilon x_{n1} &= -x_{n1-1}^3 , \end{aligned}$$

whose unique solution is $x = 0$ for all values of ε (remember that n_1 is an even number).

Now, by substituting $x = 0$ in equation (3.19a), we obtain

(3.27) $$\varepsilon^2 Q_2L_2y+Q_2\overline{R}(\varepsilon)y+F_2(\varepsilon,y+Z(\varepsilon,y)) = 0 .$$

This equation can be written as

(3.28) $$\varepsilon^2 y = -(Q_2L_2+Q_2\overline{R}(\varepsilon))^{-1}F_2(\varepsilon,y+Z(\varepsilon,y)) ,$$

and, by (3.26), equation (3.28) is equivalent to

$$\begin{aligned} \varepsilon^2 y_1 &= -y_1^3 \\ &\vdots \\ \varepsilon^2 y_{n2} &= -y_{n2}^3 \end{aligned}$$

whose unique solution is $y = 0$ for all values of the parameter. Now, substituting $y = 0$ in (3.18) we obtain $z = 0$, too. So, for ε sufficiently small the unique solutions are the trivial ones.

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4. An example.

Let us consider the problem

$$L_0w+\varepsilon L_1w+\varepsilon^2L_2w+a_{11}ww+a_{12}ww''+a_{22}w''w''+f(\varepsilon,w,w'') = 0 \,, \tag{4.1}$$

$$w'(0) = w'(1) = w'''(0) = w'''(1) = 0 \,, \tag{4.2}$$

where

$$L_0w = w''''+5\pi^2w''+4\pi^4w \,, \tag{4.3a}$$

$$L_1w = a_1w''+b_1w \,, \tag{4.3b}$$

$$L_2w = a_2w''+b_2w \,, \tag{4.3c}$$

and

$$f(\varepsilon,w,w'') = O(\varepsilon^3)w+O(\varepsilon^3)w''+[w,w'']_3+O(\varepsilon)[w,w'']_2 \,. \tag{4.4}$$

Here we denote by $[w,w'']_i$ the terms of order i in (w,w'') .

Then, $(\varepsilon,w) = (\varepsilon,0)$ is a solution of (4.1), (4.2) for all ε . We shall look for nontrivial solutions to (4.1), (4.2) bifurcating from the origin.

We consider the above operators defined on

$$U = \{w \in C^4(0,1): w'(0) = w'(1) = w'''(0) = w'''(1) = 0\}$$

with values in $V = C(0,1)$. Then

$$N(L_0) = \text{Span}\,[\cos\pi t, \cos 2\pi t] \,.$$

Moreover, L_0 is a selfadjoint operator. So, the Fredholm theory assures us that

$$R(L_0) = \{v \in V: \int_0^1 v(t)\cos\pi t\,dt = \int_0^1 v(t)\cos 2\pi t\,dt = 0\} \,.$$

We have

$$L_1\cos\pi t = (b_1-\pi^2a_1)\cos\pi t \,,$$

$$L_1\cos 2\pi t = (b_1-4\pi^2a_1)\cos 2\pi t \,,$$

$$L_2\cos\pi t = (b_2-\pi^2a_2)\cos\pi t \,,$$

$$L_2\cos 2\pi t = (b_2-4\pi^2a_2)\cos 2\pi t \,,$$

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and, by applying Theorem 1, we obtain the following result concerning (4.1), (4.2)

THEOREM 2. *Suppose* $a_1 \neq 0$. *Then, any of the following conditions*

C1. $b_1 = \pi^2 a_1$, $b_2 \neq \pi^2 a_2$;

C2. $b_1 = 4\pi^2 a_1$, $b_2 \neq 4\pi^2 a_2$;

is sufficient to have bifurcation from $(\varepsilon, w) = (0,0)$.

Proof. Suppose, for instance, C1 is satisfied. Then

$$L \cos\pi t = 0 ,$$

$$L_1 \cos 2\pi t = -3\pi^2 a_1 \cos 2\pi t ,$$

$$L_2 \cos\pi t = (b_2 - \pi^2 a_2)\cos\pi t .$$

So, we are able to apply theorem 1. □

Let us observe that the linear part of (4.1) does not give us any information if

(4.5) $$b_1 \neq \pi^2 a_1 \text{ and } b_1 \neq 4\pi^2 a_1 .$$

In fact, if (4.5) holds, then $\dim L_1(N(L_0)) = 2$ and theorem 1 forces us to go to the full equation (4.1) in order to obtain some positive answer concerning bifurcation of solutions from $(\varepsilon, w) = (0,0)$. Furthermore, the proof of theorem 1 tell us that the terms $[w, w'']_3$ in (4.1) are "bad terms" to obtain bifurcation. However, the second order terms are "good terms" to obtain bifurcation (see Lopez [5]). In fact, with the techniques in [5], it is possible to obtain the following result

THEOREM 3. *Suppose*

(4.6) $$(2a_{11} - 5\pi^2 a_{12} + 8\pi^4 a_{22})(b_1 - 4\pi^2 a_1) \neq 0 .$$

Then, $(\varepsilon, w) = (0,0)$ *is a bifurcation point of (4.1).*

Added note. When this work was finished, we found out that our multiplicity takes the same value as that of Magnus in [6] but in a different and more explicit way.

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