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Doubly transitive permutation groups involving the one-dimensional projective special linear group

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Let G be a doubly transitive permutation group on a finite set Ω, and for α in Ω suppose that G_α has a set Σ of non-trivial blocks of imprimitivity in $\Omega - \{\alpha\}$. If G_α is 2-transitive but not faithful on Σ, when is it true that the stabiliser in G_α of a block of Σ does not act faithfully on that block (that is, there is a nontrivial element in G_α which fixes every point of the block)? In a previous paper this question was answered when G_α^Σ is the alternating or symmetric group, or a Mathieu group in its usual representation. In this paper we answer the question when $PSL(2, q) \le G_\alpha^\Sigma \le P\Gamma L(2, q)$, permuting the $q + 1$ points of the projective line, for some prime power q. We show that the only groups which arise satisfy either

(i) $PSL(3, q) \le G \le P\Gamma L(3, q)$ in its natural representation, or

(ii) G is a group of collineations of an affine translation plane of order q, and contains the translation group.

This paper is a continuation of an investigation of permutation

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groups G, doubly transitive on a finite set Ω, such that the stabiliser G_α of a point α of Ω is multiply transitive on a set of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$. In [6] it was shown that apart from a few small cases, the setwise stabiliser in G_α of one of the blocks acts faithfully on that block whenever G_α permutes the set of blocks as the alternating or symmetric group or as one of the Mathieu groups in its natural representation. Our aim is to prove the following theorem.

THEOREM. *Let G be a doubly transitive permutation group on a finite set Ω, and for α in Ω suppose that the stabiliser G_α has a set $\Sigma = \{B_1, \ldots, B_t\}$ of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$; that is, $|B_i| > 1$, $|\Sigma| = t > 1$. Further suppose that $\mathrm{PSL}(2, q) \le G_\alpha^\Sigma \le \mathrm{P\Gamma L}(2, q)$ in its natural representation for some prime power q, and that G_α is not faithful on Σ. Then one of the following is true:*

(a) the setwise stabiliser in G_α of B_1 is faithful on B_1;

(b) $\mathrm{PSL}(3, q) \le G \le \mathrm{P\Gamma L}(3, q)$ in its natural representation;

(c) Ω with the translates under G of $B_1 \cup \{\alpha\}$ as lines is an affine translation plane of order q and G contains the group of translations.

In the first section some preliminary results are proved which give information about groups satisfying more general conditions than those of the theorem (Lemma 1.4 and Corollary 1.5). The theorem is proved in the second section.

NOTATION. Most of the notation used follows the conventions of [2] and [7]. If a group G has a permutation representation on a set Σ then the constituent of G on Σ is denoted by G^Σ; the set of fixed points of G in Σ is denoted by $\mathrm{fix}_\Sigma G$ and simply by $\mathrm{fix}\, G$ if the set Σ is clear from the context; and orbits of G containing more than one point are called long G-orbits.

DEFINITION. A *block design* consists of a set of v points and a set

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of b blocks with a relation called incidence between points and blocks, such that any block is incident with k points and any two points with λ blocks, where $\lambda > 0$ and $2 \le k < v-1$. The number r of blocks incident with a given point is also constant. If $k > 2$ it is called a *proper* design. By easy counting arguments we have

$$vr = bk \quad \text{and} \quad v(v-1)\lambda = bk(k-1) .$$

Also it is well known that $b \ge v$, or equivalently, $r \ge k$.

1. Preliminary results

Throughout the paper we shall assume the following hypothesis.

HYPOTHESIS (*). (a) G is a 2-transitive permutation group on a set Ω of n points. For α in Ω , the stabiliser G_α has a set $\Sigma = \{B_1, \ldots, B_t\}$ of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$, where $|\Sigma| = t > 1$, $|B_i| = b > 1$, and $n = 1 + tb$.

(b) We denote by K_i the subgroup of G_α fixing the block B_i setwise, by $\overline{K}_i$ the subgroup of G_α fixing B_i pointwise, and by H the subgroup of G_α fixing all blocks of Σ setwise.

(c) $H \ne 1$, $\overline{K}_i \ne 1$.

(d) G_α^Σ is 2-transitive.

The first result is an easy consequence of [6], Lemma 1.1.

LEMMA 1.1. *If Hypothesis (*) is true then* G *is a group of automorphisms of a block design with* $\lambda = 1$ *, the blocks of which are the translates under* G *of the set* $B_1 \cup \{\alpha\}$ *. Moreover, either*

(a) $\mathrm{PSL}(m, q) \le G \le \mathrm{P\Gamma L}(m, q)$ *in its natural representation for some* $m \ge 3$ *and prime power* q *, or*

(b) H *is semiregular on* $\Omega - \{\alpha\}$.

Proof. If *(a)* holds then clearly G is a group of automorphisms of the design which has the points and lines of the projective geometry as points and blocks respectively.

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By O'Nan [4], Proposition 4, if *(a)* does not hold then H acts faithfully on each of its orbits in $\Omega - \{\alpha\}$. In particular $H \cap \overline{K}_1$ is trivial. Hence $\overline{K}_1^{\Sigma} = \overline{K}_1 H/H \simeq \overline{K}_1$ is a nontrivial normal subgroup of K_1^{Σ} which is transitive on $\Sigma - \{B_1\}$. It follows that $\mathrm{fix}_{\Sigma}\overline{K}_1 = \{B_1\}$ and hence that $\mathrm{fix}_{\Omega}\overline{K}_1 = B_1 \cup \{\alpha\}$. The rest follows from [6], Lemma 1.1, which in turn is an application of various results of O'Nan.

COROLLARY 1.2. *If Hypothesis* (*) *is true then* $1 + b \le t$ *and* $1 + b$ *divides* $t(t-1)$.

Proof. From the definition of a block design we have $1 + b \le t$, and $1 + b$ divides $nt = (1+tb)t = t^2(b+1) - t(t-1)$, so that $1 + b$ divides $t(t-1)$.

The rest of this section is concerned with the case where H is semiregular.

LEMMA 1.3. *Assume Hypothesis* (*) *and assume that* H *is semiregular on* $\Omega - \{\alpha\}$. *Then*

(a) $\overline{K}_1$ *is a weakly closed subgroup of* $G_{\alpha\beta}$ *with respect to* G , *where* $\beta \in B_1$; *that is, if* $\overline{K}_1^g \le G_{\alpha\beta}$ *for some* g *in* G *then* $\overline{K}_1^g = \overline{K}_1$,

(b) $N = N_G(\overline{K}_1)$ *is the setwise stabiliser of* $B_1 \cup \{\alpha\}$ *in* G , *and is* 2-*transitive on* $B_1 \cup \{\alpha\}$,

(c) $C = C_G(\overline{K}_1)$ *is transitive on* $B_1 \cup \{\alpha\}$.

Proof. *(a)* Let $g \in G$ be such that $\overline{K}_1^g \le G_{\alpha\beta}$. Then $\mathrm{fix}_{\Omega}\overline{K}_1$ and $\mathrm{fix}_{\Omega}\overline{K}_1^g$ are both blocks of a design with $\lambda = 1$, by Lemma 1.1, and both contain α and β . Hence $\mathrm{fix}_{\Omega}\overline{K}_1^g = \mathrm{fix}_{\Omega}\overline{K}_1 = B_1 \cup \{\alpha\}$, and so $\overline{K}_1^g$ is contained in the pointwise stabiliser $\overline{K}_1$ of $B_1 \cup \{\alpha\}$. Thus $\overline{K}_1^g = \overline{K}_1$.

(b) By Witt [8], N is 2-transitive on $B_1 \cup \{\alpha\}$. If L is the

set stabiliser of $B_1 \cup \{\alpha\}$ in G then $N \leq L$, and $N_\alpha \leq L_\alpha = K_1$. Since $\bar{K}_1$ is normal in K_1, then $N_\alpha = K_1$. So

$$1 + b = |N : N_\alpha| = |N : K_1| \leq |L : K_1| \leq 1 + b ,$$

that is, $N = L$.

(c) C is normal in N, and since C contains H, C acts non-trivially on $B_1 \cup \{\alpha\}$. Thus C is transitive on $B_1 \cup \{\alpha\}$ since N is 2-transitive.

LEMMA 1.4. *Assume that Hypothesis* (*) *is true and that* H *is semi-regular on* $\Omega - \{\alpha\}$. *Assume also that* $(\bar{K}_1)^\Sigma$ *contains its centraliser in* $G^\Sigma_{\alpha\beta}$, *where* $\beta \in B_1$. *Then* $C = C_G(\bar{K}_1)$ *acts as a Frobenius group on* $B_1 \cup \{\alpha\}$, $1 + b = r^c$ *for some prime* r *and positive integer* c, *and* C *has a normal subgroup* A *such that* $A^{B_1 \cup \{\alpha\}}$ *is elementary abelian and regular.*

Proof. Set $K = K_1$. Since $C_{\alpha\beta} \leq G_{\alpha\beta} = K_\beta$ and since $H \cap K_\beta = 1$ (for H is semiregular on $\Omega - \{\alpha\}$), then $C_{\alpha\beta} \simeq (C_{\alpha\beta})^\Sigma \leq (G_{\alpha\beta})^\Sigma$. Then as $C_{\alpha\beta}$ centralises $\bar{K}_1$ it follows from our assumptions that $(C_{\alpha\beta})^\Sigma \leq \bar{K}_1^\Sigma$, that is, $C_{\alpha\beta}H \leq \bar{K}_1 H$. Further, from

$$(C_{\alpha\beta}H)^{B_1} \leq (\bar{K}_1 H)^{B_1} = H^{B_1} ,$$

that is to say $C_{\alpha\beta}H\bar{K}_1 \leq H\bar{K}_1$, it follows that $C_{\alpha\beta} \leq H\bar{K}_1$ and hence that $C_{\alpha\beta} \leq \bar{K}_1$. Thus C acts on $B_1 \cup \{\alpha\}$ as a Frobenius group. By [7], 5.1, C has a normal subgroup A containing $C_{\alpha\beta}$ such that $A^{B_1 \cup \{\alpha\}}$ is a characteristic subgroup of $C^{B_1 \cup \{\alpha\}}$ and is regular. Then $A^{B_1 \cup \{\alpha\}}$ is normal in $N^{B_1 \cup \{\alpha\}}$, which is 2-transitive, and so by [7], 11.3, $|B_1 \cup \{\alpha\}| = 1 + b = r^c$ for some prime r and positive integer c, and

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$A_1^{B_1 \cup \{\alpha\}}$ is elementary abelian.

COROLLARY 1.5. *Assume that Hypothesis* (*) *is true with* $t \geq 3$ *and that* H *is semiregular on* $\Omega - \{\alpha\}$. *Then the conclusions of Lemma* 1.4 *hold if one of the following is true:*

(a) $\overline{K}_1$ *has an abelian subgroup which is transitive on* $\Sigma - \{B_1\}$;

(b) G_α^Σ *is* 2-*transitive and the only blocks of* Σ *fixed setwise by* $\overline{K}_1 \cap K_2$ *are* B_1 *and* B_2 ;

(c) G_α^Σ *is* 3-*transitive.*

Proof. *(a)* Assume first that $\overline{K}_1$ has an abelian subgroup Y which is transitive on $\Sigma - \{B_1\}$. Then by [7], 4.4, Y^Σ is self-centralising in $G_{\alpha\beta}^\Sigma$. Thus the centraliser of $\overline{K}_1^\Sigma$, which is contained in the centraliser of Y^Σ , lies in Y^Σ and hence in $\overline{K}_1^\Sigma$, and the assumptions of Lemma 1.4 hold.

(b) If G_α^Σ is 2-transitive then $\mathrm{fix}_\Sigma \overline{K}_1 = \{B_1\}$ and the centraliser of $\overline{K}_1^\Sigma$ in G_α^Σ fixes B_1 . If also $\overline{K}_1 \cap K_2$ fixes only B_1 and B_2 setwise, then the centraliser of $\overline{K}_1^\Sigma$ in G_α^Σ , which also centralises $(\overline{K}_1 \cap K_2)^\Sigma$, fixes B_2 . Similarly the centraliser of $\overline{K}_1^\Sigma$ fixes each block of Σ setwise, since K_1 normalises $\overline{K}_1$ and is transitive on $\Sigma - \{B_1\}$. Hence the centraliser of $\overline{K}_1^\Sigma$ in $G_{\alpha\beta}^\Sigma$ is trivial.

(c) If G_α^Σ is 3-transitive, then K_1 is 2-transitive on $\Sigma - \{B_1\}$, and hence (by [7], 12.1, 11.3, 10.4, 5.1) $\overline{K}_1$ is either primitive and not regular on $\Sigma - \{B_1\}$ or contains an abelian subgroup transitive on

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$\Sigma - \{B_1\}$. In the former case *(b)* holds by [7], 8.6, and so in either case the assumptions of Lemma 1.4 are true.

2. Proof of the theorem

Assume Hypothesis (*) and in addition assume that

$$\mathrm{PSL}(2, q) \le G_\alpha^\Sigma \le \mathrm{P\Gamma L}(2, q)$$

in its natural representation on the $t = q + 1$ points of the projective line, where $q = p^a$ for some prime p and positive integer a . If $\mathrm{PSL}(m, r) \le G \le \mathrm{P\Gamma L}(m, r)$ in its natural representation for some $m \ge 3$ and prime power r , then G_α has a unique set of blocks of imprimitivity in $\Omega - \{\alpha\}$, namely $B_i \cup \{\alpha\}$ is a line of the projective space and $\mathrm{PGL}(m-1, r) \le G_\alpha^\Sigma \le \mathrm{P\Gamma L}(m-1, r)$ in its natural representation. Hence $m = 3$, $r = q$, and the theorem is true. So we assume that this is not the case, and from Lemma 1.1 it follows that H is semiregular on $\Omega - \{\alpha\}$ and the translates of $B_1 \cup \{\alpha\}$ under G form the blocks of a design with $\lambda = 1$. First we prove:

LEMMA 2.1. *Either*

(a) Ω *together with the translates under* G *of* $B_1 \cup \{\alpha\}$ *as lines is an affine translation plane of order* $q = 1 + b$, *and* G *contains the group of translations, or*

(b) $1 + b < q$.

Proof. By Corollary 1.2, $1 + b \le t = 1 + q$. If $1 + b = 1 + q$ then the design is a projective plane and by [5], Theorem 5, G contains $\mathrm{PSL}(3, q)$. We assumed above that this was not the case, so $1 + b \le q$. If $1 + b = q$ then (by [1], 2.2.6, p. 71) the design is an affine plane. By [5], Theorem 1, the plane is a translation plane and G contains the translation group. This completes the proof of Lemma 2.1.

Thus we may assume that $1 + b < q$. We shall obtain a contradiction. Now K_1^Σ has a unique minimal normal subgroup which is elementary abelian

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of order $q = p^a$ and is regular on $\Sigma - \{B_1\}$. Then, since $\overline{K}_1 \simeq \overline{K}_1^\Sigma$, $\overline{K}_1$ has a normal subgroup P which is elementary abelian of order q and transitive on $\Sigma - \{B_1\}$. It follows from Lemma 1.4 and Corollary 1.5 that $1 + b = r^c$ for some prime r and positive integer c, and that $C = C_G(\overline{K}_1)$ has a normal subgroup A such that $A^{B_1 \cup \{\alpha\}}$ is elementary abelian and regular.

Suppose first that $r \neq p$. Since A_α is a subgroup of $\overline{K}_1$ and centralises $P \leq \overline{K}_1$, then $A_\alpha^\Sigma \simeq A_\alpha$ centralises P^Σ. However, by [7], 4.4, P^Σ is self-centralising and so A_α is a p-group. Hence A has a unique Sylow r-subgroup X which is elementary abelian of order r^c, and acts regularly on B_1, and X is normal in C. In particular X is normalised by H. Now P has $b = r^c - 1$ orbits of length q in Ω. Since X is an r-group, X fixes one of these orbits, say Γ, setwise. Then since X^Γ is an r-group centralising P^Γ and since P^Γ is self-centralising by [7], 4.4, it follows that X fixes Γ pointwise. Now H acts semiregularly on the set of long P-orbits and since H normalises X, then X fixes at least $|H| \geq 2$ of the orbits pointwise. Hence for a nonidentity element x in X, $(B_2 \cup \{\alpha\})^x \cap (B_2 \cup \{\alpha\})$ contains at least $|H| \geq 2$ points, namely points of the long P-orbits fixed by X. Then since $B_2 \cup \{\alpha\}$ and $(B_2 \cup \{\alpha\})^x$ are both blocks of a design with $\lambda = 1$, it follows that $B_2 \cup \{\alpha\} = (B_2 \cup \{\alpha\})^x$. However since X acts regularly on $B_1 \cup \{\alpha\}$, then α^x is a point of B_1 and so does not lie in $B_2 \cup \{\alpha\}$, a contradiction.

Hence $r = p$. As above, $A_\alpha \leq P$ so that A, and hence AP, is a p-group. Since P has $b = p^c - 1$ orbits of length q in Ω and since AP is a p-group centralising P, then AP fixes some orbit Γ of P of length q setwise. Now $(AP)^\Gamma$ centralises P^Γ and by [7], 4.4, P^Γ

is self-centralising. Hence the kernel X of AP on Γ has index q in AP, that is $|X| = p^c$, and $X \cap P = 1$ since P is faithful on Γ. It follows that $AP = X \times P$. Since P fixes $B_1 \cup \{\alpha\}$ pointwise it follows that $A^{B_1 \cup \{\alpha\}} = (AP)^{B_1 \cup \{\alpha\}} = X^{B_1 \cup \{\alpha\}}$, and so X is elementary abelian of order p^c and is regular on $B_1 \cup \{\alpha\}$. Since G is 2-transitive there is a conjugate X' of X contained in $G_{\alpha\beta}$. Since $1 + b = p^c < p^a$, then $|\mathrm{fix}_\Omega X'| \geq p^a \geq p(1+b) > 1 + 2b$, so that X' fixes at least three blocks of Σ setwise.

Since the stabiliser of three points in $\mathrm{P\Gamma L}(2, p^a)$ is cyclic of order a and is induced from Galois automorphisms of the Galois field $\mathrm{GF}(p^a)$ of order p^a, it follows that $X' \simeq (X')^\Sigma$ is cyclic of order p and that $1 + b = p$ divides a. The fixed field of X' acting as a group of Galois automorphisms of $\mathrm{GF}(p^a)$ has order $p^{a/p}$ (see [3], Theorem 2, p. 194), and it follows that X' fixes $1 + p^{a/p}$ blocks of Σ setwise. Hence

$$p^a \leq |\mathrm{fix}_\Omega X'| \leq 1 + b\left(1+p^{a/p}\right) = p + (p-1)p^{a/p} \leq p + p^a - p^{a/p} \leq p^a ,$$

(since for any integers $x, y \geq 2$, it is true that $xy \leq y^x$). Thus equality holds at all stages, and in particular, $p.p^{a/p} = p^a$, that is $a = p/(p-1)$. Since a is an integer it follows that $p = 2$, and so $b = p - 1 = 1$, a contradiction. This completes the proof.

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