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Structural stability for fibrewise Anosov diffeomorphisms on principal torus bundles

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Abstract. We show that a fibre-preserving self-diffeomorphism which has hyperbolic splittings along the fibres on a compact principal torus bundle is topologically conjugate to a map that is linear in the fibres.

Key words: fibrewise Anosov diffeomorphisms, topological conjugacy
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1. *Introduction*

Let M be a closed Riemannian manifold. Recall that a diffeomorphism $f : M \to M$ is called *Anosov* if it satisfies the following conditions.

(1) There is a splitting of the tangent bundle $TM = E^s \oplus E^u$ which is preserved by the derivative df.

(2) There exist constants $C > 0$ and $\lambda \in (0, 1)$, such that for all $n > 0$, we have

$$\|df^n v\| \le C\lambda^n \|v\| \quad \text{for all } v \in E^s$$

and

$$\|df^{-n} v\| \le C\lambda^n \|v\| \quad \text{for all } v \in E^u.$$

Classification of Anosov diffeomorphisms is a well-known open problem. Examples are only known on tori, nilmanifolds and infranilmanifolds. The very first examples of Anosov diffeomorphisms are hyperbolic automorphisms which are given by hyperbolic matrices in $GL(d, \mathbb{Z})$ whose action on $\mathbb{R}^d$ descends to the torus $\mathbb{T}^d$. Franks [3] proved that every Anosov diffeomorphism which is homotopic to a hyperbolic automorphism is, in fact, conjugate to this automorphism. Manning [9] then completed the classification on tori (and more generally on infranilmanifolds) by proving that every Anosov diffeomorphism on the torus is homotopic to a hyperbolic automorphism. We would usually refer to such a property as the *global structural stability*. More precisely, there is a linear model for every Anosov diffeomorphism. The word 'global' is in contrast to the local version of structural stability, where we consider whether two diffeomorphisms that are close in C^r-topology for some $r \ge 0$ are conjugate.

Now suppose $p : E \to B$ is a C^1 principal torus bundle where E and B are smooth compact manifolds. Roughly speaking, E is a topological space endowed with a free action of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We will include a more detailed definition and properties of this object in §2. Fix a Riemannian metric $\| \cdot \|$ on E. Then the tangent space splits $TE = V \oplus H$, where $V = \ker(dp)$ denotes the vertical bundle tangent to the fibres and H the horizontal bundle which is its orthogonal complement. We call a C^1 diffeomorphism $F : E \to E$ a *fibrewise Anosov diffeomorphism* if it satisfies the following conditions.

(1) There exists a splitting of the vertical bundle $V = V^s \oplus V^u$ which is preserved under the derivative dF. (In particular, V is dF-invariant.)

(2) There exist constants C and $\lambda \in (0, 1)$, such that for all $n > 0$, we have

$$\|dF^n(v^s)\| \le C\lambda^n \|v^s\| \quad \text{for all } v \in V^s,$$

and

$$\|dF^{-n}(v^u)\| \le C\lambda^n \|v^u\| \quad \text{for all } v \in V^u.$$

If $G : E \to E$ is a fibre-preserving map on a principal $\mathbb{T}^d$ bundle that satisfies $G(e.g) = G(e).Ag$, for some $A \in GL(d, \mathbb{Z})$ and all $e \in E$, $g \in \mathbb{T}^d$ in the structure group, then we call G an *A-map*. In particular, when A is hyperbolic, we call G a *fibrewise affine Anosov diffeomorphism*.

We took this notion of fibrewise hyperbolicity from [2] where Farrell and Gogolev mainly studied the bundles that support such dynamics. There is also a more general notion of 'foliated hyperbolicity' by Bonatti, Gómez-Mont and Martínez that appeared in [1] where the authors studied diffeomorphisms that are hyperbolic along the leaves of a foliated manifold and proved elementary dynamical properties, e.g. the strong stable and unstable distributions are integrable.

We are interested in this class of fibrewise Anosov examples also partially because of the new class of partially hyperbolic systems constructed by Gogolev, Ontaneda and Rodriguez-Hertz in [6] with simply connected total spaces, although note that here a fibrewise Anosov diffeomorphism is not necessarily partially hyperbolic unless the base dynamics are 'dominated' by the dynamics in the fibres.

In this paper, we prove the global structural stability for fibrewise Anosov diffeomorphisms, generalizing the results of Franks and Manning. Namely, we prove the following theorem.

THEOREM 1.1. *Let $p : E \to B$ be a compact C^1 principal torus bundle and let $F : E \to E$ be a fibrewise Anosov diffeomorphism. Then there exists a fibrewise affine Anosov diffeomorphism $G : E \to E$ and a homeomorphism $h : E \to E$, which is homotopic to id_E and fibres over id_B, such that $h \circ F = G \circ h$.*

Remark 1.2. Note that because the structure group of a compact principal torus bundle only contains translations, a fibre-preserving map induces the same automorphism on the homology of the torus fibre over each point, provided that the base is connected. This is because the restrictions of the map to nearby fibres are homotopic, and then we can conclude by going from neighbourhood to neighbourhood.

We will get the semiconjugacy from the following proposition.

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PROPOSITION 1.3. *Suppose F is a continuous fibre-preserving map on a compact principal torus bundle whose base is connected, which induces an automorphism $A : H_1(\mathbb{T}^d) \to H_1(\mathbb{T}^d)$ on the first homology of the torus fibre. Then it is fibrewise homotopic to an A-map.*

If, in addition, we assume the bundle is C^1 and A is hyperbolic, then F is semi-conjugate to a fibrewise affine Anosov diffeomorphism.

We give some definitions in §2 and examples of fibrewise Anosov diffeomorphisms in §3. We will first prove in Proposition 4.1 that every fibrewise Anosov diffeomorphism is homotopic to a fibrewise affine Anosov diffeomorphism by topological arguments in §4. Then we prove in Proposition 5.1 that there is a semiconjugacy from each map covering the same map in the homotopy class of a fibrewise affine Anosov diffeomorphism, by applying a similar argument as in Franks' proof. Last, we establish the global structural stability at the end of §5, by showing that the fibrewise lift of each pair of stable and unstable leaves has a unique intersection.

2. *Preliminaries*

In this section, we give definitions and properties that will come into use.

Definition 2.1. [8] Let G be a Lie group and E a metric space where G acts continuously, freely and properly on the right. Let $B = E/G$. Then we call the projection $p : E \to B$ a principal fibre bundle over B with group G, and E the total space, B the base space, $p^{-1}b$ the fibre for each $b \in B$ and G the structure group.

In Theorem 1.1, by a C^1 principal torus bundle, we mean that $p : E \to B$ is a C^1 map.

Property 2.1. [10] The principal bundle E defined as above is locally trivial, that is, every point $x \in B$ has a neighbourhood U where there exists a homeomorphism $\varphi : U \times G \to p^{-1}(U)$ and a map $\psi : p^{-1}(U) \to G$ so $\varphi^{-1}(e) = (p(e), \psi(e))$ and $\psi(e.g) = \psi(e).g$, where $e \in E$ and $g \in G$.

From now on, we assume E is a smooth manifold and the chart φ is smooth.

Property 2.2. The fibres of a principal bundle with group G are diffeomorphic to G.

Property 2.3. We fix trivializations $\{(\varphi_i, U_i)\}$ for the fibre bundle $p : E \to B$ with structure group G and fibre F. Restricting φ_i and φ_j to $U_i \cap U_j$, there exists a unique map $g_{ji} : U_i \cap U_j \to G$ such that $\varphi_j^{-1}\varphi_i(b, x) = (b, g_{ji}(b)x)$ for $(b, x) \in (U_i \cap U_j) \times F$. The functions g_{ji} satisfy the following properties.

(1) For each $b \in U_i \cap U_j \cap U_k$, we have $g_{ki}(b) = g_{kj}(b)g_{ji}(b)$.
(2) For each $b \in U_i$, $g_{ii}(b) = \mathrm{id}_G$.
(3) For each $b \in U_i \cap U_j$, $g_{ij}(b) = g_{ji}^{-1}(b)$.

From now on, we will use additive notation '$+$' for the torus action.

Remark 2.4. Recall the definition of a fibrewise Anosov diffeomorphism $F : E \to E$ on a principal torus bundle from §1. The assumption that F preserves the vertical bundle implies that F is also fibre preserving. This can be seen as follows.

Let $e_0 \in E$, $p(e_0) = b$, and suppose $\psi(e_0) = 0$ with the notion in Property 2.1. The fibre $\mathbb{T}^d$ is a connected compact Lie group where the exponential map is globally surjective. We identify the Lie algebra of left invariant vector fields on $\mathbb{T}^d$ and the tangent space $T_0\mathbb{T}^d$. For any $e_1 \in p^{-1}b$, we have an integrable curve $\gamma(t) = \exp(tX)$ in $p^{-1}b$, where $X \in T_0\mathbb{T}^d$, such that $\gamma(0) = e_0$ and $\gamma(1) = e_1$. Since $dF(d/dt|_{t=t_0}(e_0 + \exp(tX)) = d/dt|_{t=t_0} F(e_0 + \exp(tX)) \in V_{F(e_0+\exp(t_0X))}$ for all $t_0 \in \mathbb{R}$, and V is integrable whose integrable manifold at a point e is just the fibre at $pF(e)$, we have $F(e_0 + \exp(X)) = F(e_1) \in p^{-1}(p(F(e_0)))$.

We denote the map covered by F as $f : B \to B$.

We would also like to recall the following lemma from general topology which we will refer to as the tube lemma.

LEMMA 2.5. (Tube lemma) *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$ about $x_0 \times Y$, where W is a neighbourhood of x_0 in X.*

THEOREM 2.6. (Pugh closing lemma, [11]) *Let f be a C^1 diffeomorphism of a compact manifold M and let $x \in M$ be a non-wandering point of f. Then any C^1-neighbourhood U of f contains a diffeomorphism $g \in U$ such that x is a periodic point of g.*

Remark 2.7. The stable and unstable distributions of a fibrewise Anosov diffeomorphism along the fibres are integrable following Theorem 2.6, [1].

*Remark 2.8. (*Local product structure) Consider a pair of transverse foliations. If the foliated manifold is compact, there exists a constant ε, such that any open neighbourhood of diameter $< \varepsilon$ can be thought of parametrized by the plaques of the leaves.

The fibrewise leaves of a fibrewise Anosov diffeomorphism also have the local product structure. More precisely, for $b \in B$ and any $e, e' \in \mathbb{T}^d_b$ (or $\tilde{e}, \tilde{e}' \in \mathbb{R}^d_b$) which denotes the fibre over b, there exists a constant $\varepsilon > 0$ such that if $d(e, e') < \varepsilon$ (or $d(\tilde{e}, \tilde{e}') < \varepsilon$), then $W^s_b(e) \cap W^u_b(e')$ and $W^u_b(e) \cap W^s_b(e')$ (or $\widetilde{W}^s_b(\tilde{e}) \cap \widetilde{W}^u_b(\tilde{e}')$ and $\widetilde{W}^u_b(\tilde{e}) \cap \widetilde{W}^s_b(\tilde{e}')$) each contains exactly one point.

Now there is also such a constant even if we lift the leaves to the covering space. Since the bundle is compact and a sufficiently small neighbourhood of $\mathbb{T}^d_b$ for each b is evenly covered, by definition of the covering map $\mathbb{R}^d_b \to \mathbb{T}^d_b$, there is a uniform ε for the bundle with which the local product structure is satisfied. We would refer to ε as 'the constant of the local product structure' just to save a letter.

3. *Examples of fibrewise Anosov diffeomorphisms*

Example 3.1. (Trivial example) Let $E = B \times \mathbb{T}^d$. Then $G : E \to E$, $G(b, t) = (f(b), A(t) + v(b))$, where $f : B \to B$ is a diffeomorphism and $v : B \to \mathbb{T}^d$ is a differentiable function, is a fibrewise affine Anosov diffeomorphism.

Example 3.2. (Nilmanifold automorphisms) Let $E = N/\Gamma$ be a nilmanifold where N is a simply connected nilpotent Lie group and $\Gamma \subset N$ a discrete subgroup that acts cocompactly on N. Let $Z(N)$ denote the centre of N. Then $Z(N)/(Z(N) \cap \Gamma)$ is a compact

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abelian Lie group and thus a torus group. It acts continuously, freely and properly on E. Thus, E is a principal bundle with torus fibres.

Suppose $A : N \to N$ is an automorphism and $A(\Gamma) = \Gamma$. If the restriction of A to $Z(N)$ is hyperbolic, it induces a hyperbolic toral automorphism on $Z(N)/(Z(N) \cap \Gamma)$, and thus a fibrewise affine Anosov diffeomorphism on E.

Remark 3.3. There are criteria given in [6] whether a bundle would support fibrewise Anosov diffeomorphisms. We want to point out that one is able to construct many examples, with some algebraic topology.

Example 3.4. ($K3$ surface, [6]) Let the base be a $K3$ surface by Kummer's construction [12, 3.3]. It is given by $X := \mathbb{T}^2_{\mathbb{C}} \# 16\overline{\mathbb{C}P}^2/\iota$, where $\overline{\mathbb{C}P}^2$ is the complex projective plane with reversed orientation and ι is the involution induced by the involution of $\mathbb{T}^2_{\mathbb{C}}$, which gives rise to 16 singularities where we attach the $\overline{\mathbb{C}P}^2$ terms.

Identify $\mathbb{T}^2_{\mathbb{C}} = \mathbb{T}^4$, the usual real torus. For any given hyperbolic matrix $A \in SL(2, \mathbb{Z})$, we are able to take a perturbation A' of A, such that $A' \oplus A' : \mathbb{T}^2_{\mathbb{C}} \to \mathbb{T}^2_{\mathbb{C}}$ descends to a diffeomorphism f of the quotient space X, and there exists a principal $\mathbb{T}^2$ bundle with simply connected total space that admits a fibrewise affine Anosov diffeomorphism with matrix A^2 as specified in the definition, which covers $f : X \to X$. In particular, the above fibrewise affine Anosov diffeomorphism is partially hyperbolic by this construction.

We would also like to remark that from [5], we are able to construct more examples from the above Examples 3.1, 3.2 and 3.4 by connect-summing along invariant tori.

More precisely, in these examples, the base dynamics (up to a finite order) already have or can be perturbed to have a hyperbolic fixed point, near which the map has a local form $(x, y) \mapsto (Ax, f_x(y))$ in a tubular neighbourhood $\mathbb{D} \times \mathbb{T}^d$. Then, by replacing $\mathbb{D} \times \mathbb{T}^d$ of the invariant fibre by $\tilde{\mathbb{D}} \times \mathbb{T}^d$ where $\tilde{\mathbb{D}} := \{(x, l(x)) : x \in \mathbb{D},\ x \in l(x),\ l(x)$ is the line passing through $x\}$, we obtain what we call a blow-up of the original bundle, with boundary $\mathbb{S}^k \times \mathbb{T}^d$ for some k.

Now if we have examples of different types, with the same fibre and base dimensions, whose base dynamics have hyperbolic fixed points where the local forms of the maps as described above are the same, then we can take the blow-ups and glue along the boundaries to get new examples of fibrewise affine Anosov diffeomorphisms. We get partially hyperbolic diffeomorphisms from gluing partially hyperbolic diffeomorphisms by [5].

4. *The homotopy*

In this section, we let $F : E \to E$ be a C^1 fibrewise Anosov diffeomorphism on the principal $\mathbb{T}^d$ bundle $p : E \to B$ that covers $f : B \to B$. We prove the following proposition.

PROPOSITION 4.1. *There is a fibrewise affine Anosov diffeomorphism $G : E \to E$ such that F is homotopic to G.*

We will prove the above proposition by the end of this section.

LEMMA 4.2. *There is a fibre-preserving map $\hat{F} : E \to E$ that is C^1 close to F^N for some $N \in \mathbb{N}_{>0}$ and covers a map $\hat{f} : B \to B$ such that $\hat{f}$ has a fixed point.*

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Proof. We construct such a map $\hat{F}$. Let $b \in B$ denote a recurrent point of f. Consider the horizontal distribution of our choice $H \subset TE$. Take a small neighbourhood of $U \subset B$ that contains b such that for every $e \in p^{-1}b$, $b \in U$, there is a small disc $D_e \subset H_e$ so we have $\exp : D_e \to P(e)$ is a diffeomorphism where $P(e)$ is a disc of the dimension of the base but in the total space E. There exists an $N \in \mathbb{N}_{>0}$ such that $f^N(b) \in U$. Then take a smaller neighbourhood $V \subset U$ containing b such that $f^N(V) \subset U$. Now, by Theorem 2.6, there is a map f' that is C^1 close to f on V, $f' = f$ outside V and has b as a periodic point. Without loss of generality, assume $(f')^N b = b$ and $(f')^N V \subset U$. Denote $\hat{f} = (f')^N$.

Recall that we have defined $P(e)$ for $e \in p^{-1}b$. Now let $P(e') = P(e)$ if $P(e) \cap p^{-1}(pe') = e'$ (there is a unique $P(e)$ because the neighbourhood is foliated by these discs). Now $P(F^N(e))$ is also well defined for all $e \in p^{-1}V$. Then we define

$$\hat{F}(e) = P(F^N(e)) \cap p^{-1}(\hat{f}(b'))$$

for each $e \in p^{-1}b'$ and $b' \in V$. We can check that $\hat{F}$ covers $\hat{f}$ and is C^1 close to F^N because V is small enough. □

Now, by a classical cone field argument, the $\hat{F}$ is fibrewise Anosov. As b is a fixed point of $\hat{f}$, the restriction $\hat{F}_b : p^{-1}b \to p^{-1}b$ is an Anosov diffeomorphism. Recall that $p^{-1}b$ is canonically identified with $\mathbb{T}^d$ up to a translation. Thus, $\hat{F}$ induces a hyperbolic automorphism $(\hat{F}_b)_*$ on $H_1(\mathbb{T}^d; \mathbb{R}) = \mathbb{R}^d$ [9]. However, $(F_b^N)_* = (\hat{F}_b)_*$ because they are C^1 close.

By continuity of F, we know that if b, b' are close enough points in the base, F_b and $F_{b'}$ are homotopic, and thus induce the same automorphism on $H_1(\mathbb{T}^d; \mathbb{R})$. By compactness, we are able to extend this to the entire B, so for all $b \in B$, $(F_b)_* = A$ for a fixed A, which is also hyperbolic. We state this fact as the following lemma.

LEMMA 4.3. *For any $b \in B$, the induced homomorphism $(F_b)_* : H_1(\mathbb{T}^d; \mathbb{R}) \to H_1(\mathbb{T}^d; \mathbb{R})$ is the same hyperbolic automorphism.*

Now suppose $F, G : E \to E$ are two arbitrary continuous maps that cover f. We can define $r : E \to \mathbb{T}^d$ to be such that $F(e) = G(e) + r(e)$. Note that morally r is defined as the amount we need to translate from G to get F. Because the action is free, r is well defined and unique. We show that it is continuous.

LEMMA 4.4. *For two continuous maps $F, G : E \to E$ on a principal bundle with fibre H that cover the same map, we have a continuous map $\bar{r} : E \to H$ such that $F(e).\bar{r}(e) = G(e)$.*

Proof. We use charts. Suppose for a fixed b, $f(b) \in U_j$. Define $r_j = (\psi_j F)^{-1}.\psi_j G : E \to H$. We show r_j is independent of charts.

This is because if, at the same time, $f(b) \in U_k$ for some $k \neq j$, then $\psi_k = h_{jk}.\psi_j$. We have $r_j = r_k$. These r_j terms are continuous as compositions of continuous functions. Thus, we get a globally defined continuous function $\bar{r} : E \to H$ that agrees with the r_j terms everywhere. □

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Now suppose we have a fibre-preserving map $F : E \to E$ that induces the same matrix A on the homology of every fibre. From [6], if E admits an A-map (we will show it does), then we have the following commutative diagram. Here, $A(E)$ means E with cocycles $\{A \circ g_{ji}\}$ as in the notation of Property 2.3. For the existence and well definedness of such bundles and A-maps, we refer to Proposition 4.6 and Theorem 6.2 in the same paper.

$$\begin{array}{ccccccccc} E & \xrightarrow{F} & E & \xrightarrow{\mathrm{id}_{\mathbb{T}^d}\text{-map}} & f^*E & \xrightarrow{\mathrm{id}_{\mathbb{T}^d}\text{-map}} & A(E) & \xrightarrow{A^{-1}\text{-map}} & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{f} & B & \xrightarrow{f^{-1}} & B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & B \end{array} \tag{4.1}$$

Denote the composition in the upper row $\bar{F}$. Then the restriction of $\bar{F}$ to each fibre clearly induces the identity on $H_1(\mathbb{T}^d; \mathbb{R})$.

LEMMA 4.5. *Let $F : E \to E$ be a fibre-preserving map that induces the same automorphism A, which is not necessarily hyperbolic, on the homology of each fibre. Let $r : E \to \mathbb{T}^d$ be such that $\bar{F} = \mathrm{id}_E + r$, where $\bar{F}$ is as the above. Then r is homotopic to $v \circ p$, where $v : B \to \mathbb{T}^d$ is a translation in the fibre and $p : E \to B$ is the projection, that is, the following diagram commutes up to homotopy.*

$$\begin{array}{ccc} E & \xrightarrow{r} & \mathbb{T}^d \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle v} & \\ B & & \end{array} \tag{4.2}$$

Proof. We have the following diagram.

$$\begin{array}{ccccccc} \pi_1(\mathbb{T}^d) & \overset{i_\#}{\hookrightarrow} & \pi_1(E) & \xrightarrow{p_\#} & \pi_1(B) & \longrightarrow & 1 \\ & \searrow{\scriptstyle \text{trivial map}} & \downarrow{\scriptstyle r_\#} & \swarrow{\scriptstyle v_\#} & & & \\ & & \pi_1(\mathbb{T}^d) & & & & \end{array} \tag{4.3}$$

The upper row comes from the long exact sequence. The trivial map on the left comes from the fact that the restriction of r to each fibre induces the zero map on $H_1(\mathbb{T}^d; \mathbb{R})$.

Our goal is to define $v_\#$. Once it is defined, from the $K(\pi, 1)$-spaces fact (see, for example, Hatcher [7, 1B.9]), we get a unique v, up to homotopy, from any homomorphism $\pi_1(B) \to \pi_1(\mathbb{T}^d)$.

For $\alpha \in \pi_1(B)$, because $p_\#$ is surjective, there is a $\beta \in \pi_1(E)$ such that $p_\#\beta = \alpha$. We define $v_\#\alpha = r_\#\beta$.

To show this is well defined, suppose there is another β' such that $p_\#\beta' = p_\#\beta = \alpha$, we need to show that $r_\#\beta' = r_\#\beta$.

We know $p_\#(\beta'\beta^{-1}) = 1$ so $\beta'\beta^{-1} \in \ker p_\# = \mathrm{im}\, i_\#$. There is a $\gamma \in \pi_1(\mathbb{T}^d)$ such that $i_\#\gamma = \beta'\beta^{-1}$. However, $r_\# i_\#\gamma = r_\#\beta'\beta^{-1} = 1$. □

LEMMA 4.6. *Let $G : E \to E$ be a map that induces the identity id_{Y*} on $H_1(\mathbb{T}^d)$. Then E admits an $\mathrm{id}_{\mathbb{T}^d}$-map.*

Proof. We prove the statement by induction on k for principal torus bundles over $\mathbb{S}^k$. After showing for spheres, a general statement is true because manifolds have CW approximations and we can just do inductions on the dimension of skeletons.

For $k = 1$, the bundle must be trivial, so there is trivially an $\mathrm{id}_{\mathbb{T}^d}$-map.

Now suppose the statement for the bundle over $\mathbb{S}^k$ is true. Then for a bundle over $\mathbb{S}^{k+1}$, we consider the clutching construction. The bundle splits into two trivial bundles over discs $\mathbb{D}^+$ and $\mathbb{D}^-$ of dimension $k + 1$. The restrictions of the torus bundle to the boundary spheres $\mathbb{S}^k$ of $\mathbb{D}^{+/-}$ and each admits an $\mathrm{id}_{\mathbb{T}^d}$-map.

Let us denote the $\mathrm{id}_{\mathbb{T}^d}$-map $g : \mathbb{S}^k \times \mathbb{T}^d \to \mathbb{S}^k \times \mathbb{T}^d$. Parametrize the disc $\mathbb{D}^{k+1}$ by (ρ, θ), where ρ gives the radius and θ is a k-dimensional vector of angles. In this coordinate, $\rho = 1$ is the boundary sphere, and $g(\theta, y) = (\theta, \mathrm{id} + v(\theta))$ on the boundary. The problem actually reduces to the extension of $v(\theta)$ to the entire disc.

However, because the pair $(\mathbb{D}^{k+1} \times \mathbb{T}^d, \mathbb{S}^k \times \mathbb{T}^d)$ is a good pair, which has the homotopy extension property, we are able to find a homotopy $h : \mathbb{D} \times \mathbb{T}^d \times I \to \mathbb{D} \times \mathbb{T}^d$ such that $h(1, \theta, y, 0) = g(\theta, y)$. Thus, $V(\rho, \theta) = h(\rho, \theta, 0, 0)$ is a map that restricts to $v(\theta)$ on the boundary, where $y = 0$ is just the zero section of the trivial bundle. Now $\mathrm{id}_{\mathbb{D}\times\mathbb{T}^d} + V$ is an $\mathrm{id}_{\mathbb{T}^d}$-map on half of the bundle. We can glue via the clutching map. □

As a consequence, we have the following proposition.

PROPOSITION 4.7. *If $F : E \to E$ is a fibre-preserving map that induces the same automorphism A on the homology of each fibre, then F is homotopic to an A-map.*

Proof. From the previous lemmas, we have found a homotopy $\bar{H} : E \times I \to \mathbb{T}^d$ such that $\bar{H}(e, 0) = r(e)$, where $\bar{F}(e) = \mathrm{id}_E(e) + r(e)$, and $\bar{H}(e, 1) = v \circ p(e)$. Then, if we define $\tilde{H} : E \times I \to E$ as $\tilde{H}(e, t) = \mathrm{id}_E(e) + \bar{H}(e, t)$, it gives a homotopy such that $\tilde{H}(e, 0) = \bar{F}(e)$ and $\tilde{H}(e, 1) = \mathrm{id}_E(e) + v \circ p(e)$.

If we name the maps in Diagram (4.1), from the left to right, the first $\mathrm{id}_{\mathbb{T}^d}$-map g_1, the second $\mathrm{id}_{\mathbb{T}^d}$-map g_2 and the A^{-1}-map g_3, then $H := g_1^{-1} g_2^{-1} g_3^{-1} \tilde{H} : E \times I \to E$ gives a homotopy such that $H(e, 0) = F(e)$ and $H(e, 1) = G(e)$ for some A-map G. □

Then Proposition 4.1 follows.

Proof of Proposition 4.1. By the remarks we made right after the proof of Lemma 4.2, if $F : E \to E$ is fibrewise Anosov, then F induces the same hyperbolic automorphism A on the homology of the fibre $\mathbb{T}^d$. Therefore, F is homotopic to G for some fibrewise affine Anosov diffeomorphism G from Proposition 4.7. □

5. *Proof of Theorem 1.1*

From the last section, we get a homotopy H between $F, G : E \to E$, where G is a fibrewise affine Anosov diffeomorphism. First, we construct the h as stated in Theorem 1.1. This is very similar to [4, Proposition 2.1].

For any two maps $F, G : E \to E$ on a principal torus bundle that covers the same map $f : B \to B$, again we let $r : E \to \mathbb{T}^d$ be the map such that $F(e) + r(e) = G(e)$. If $F \simeq G$ and we let $H : E \times I \to E$ denote the homotopy such that $H(e, 0) = F(e)$

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and $H(e, 1) = G(e)$, then we can let $R : E \times I \to \mathbb{T}^d$ be the map such that $H(e, t) + R(e, t) = G(e)$. This is well defined, and continuous in e also by Lemma 4.4 and in t because H is continuous. We have $R(e, 0) = r(e)$ and $R(e, 1) = c$ where $c : E \to \mathbb{T}^d$ denotes a constant map.

Thus, in the case where F is homotopic to G, r is nullhomotopic and it lifts to a map $\tilde{r} : E \to \mathbb{R}^d$. Now if G is fibrewise affine Anosov with the matrix A, we can write $\tilde{r} = \tilde{r}^s + \tilde{r}^u$, where $\tilde{r}^s$ and $\tilde{r}^u$ take values in the stable and unstable subspaces of A, respectively.

PROPOSITION 5.1. *Suppose $G : E \to E$ is a fibrewise affine Anosov diffeomorphism with the matrix A as specified in the definition. For any $F : E \to E$ that is homotopic to G and covers the same map on the base with G, there is a continuous surjective map $h : E \to E$ homotopic to id_E which fibres over id_B such that $h \circ F = G \circ h$.*

Proof. We define $\tilde{w}^s, \tilde{w}^u : E \to E$,

$$\tilde{w}^s(e) = -\sum_{n=0}^{\infty} A^n \tilde{r}^s(F^{-(n+1)}(e)) \quad \text{and} \quad \tilde{w}^u(e) = \sum_{n=0}^{\infty} A^{-(n+1)} \tilde{r}^u(F^n(e)).$$

Then we have

$$A\tilde{w}^s(e) - \tilde{w}^s(F(e)) = -\sum_{n=0}^{\infty} A^{n+1} \tilde{r}^s(F^{-(n+1)}(e)) + \sum_{n=0}^{\infty} A^n \tilde{r}^s(F^{-n}(e)) = \tilde{r}^s(e).$$

Similarly, we can check that $A\tilde{w}^u(e) - \tilde{w}^u(F(e)) = \tilde{r}^u(e)$. Let $\tilde{w} = \tilde{w}^s \oplus \tilde{w}^u$. Then we have $A\tilde{w}(e) - \tilde{w}(F(e)) = \tilde{r}(e)$.

We show that $\tilde{w} : E \to E$ is continuous. Let $\mathbb{R}^d$ be endowed with the Riemannian metric lifted from $\mathbb{T}^d$. Let d_E denote the metric on E and $d_{\mathbb{R}^d}$ the metric on $\mathbb{R}^d$. Since E is assumed to be compact, there is a uniform bound M of the distance from $\tilde{r}(e)$ to $0 \in \mathbb{R}^d$ for all $e \in E$. For a given $\varepsilon > 0$, take $N \in \mathbb{N}$ so we have $\sum_{n>N} \lambda^n M < \varepsilon/4$. Then, since both $-\sum_{n=0}^{N} A^n \tilde{r}^s(F^{-(n+1)}(e))$ and $\sum_{n=0}^{N} A^{-(n+1)} \tilde{r}^u(F^n(e))$ are uniformly continuous, there is a $\delta > 0$ such that if $d_E(e_1, e_2) < \delta$ for any $e_1, e_2 \in E$, we have both

$$d_{\mathbb{R}^d}\bigg(\sum_{n=0}^{N} A^n \tilde{r}^s(F^{-(n+1)}(e_1)), \sum_{n=0}^{N} A^n \tilde{r}^s(F^{-(n+1)}(e_2))\bigg) < \varepsilon/4$$

and

$$d_{\mathbb{R}^d}\bigg(\sum_{n=0}^{N} A^{-(n+1)} \tilde{r}^u(F^n(e_1)), \sum_{n=0}^{N} A^{-(n+1)} \tilde{r}^u(F^n(e_2))\bigg) < \varepsilon/4.$$

Then $d_{\mathbb{R}^d}(\tilde{w}(e_1), \tilde{w}(e_2)) < \varepsilon$.

We can then project $\tilde{w} : E \to \mathbb{R}^d$ and get $p_1\tilde{w} = w : E \to \mathbb{T}^d$, where $p_1 : \mathbb{R}^d \to \mathbb{T}^d$ denotes the covering projection. Note $p_1 : \mathbb{R}^n \to \mathbb{T}^n$ is linear. Then,

$$Aw(e) - w(F(e)) = Ap_1\tilde{w}(e) - p_1\tilde{w}(F(e)) = p_1A\tilde{w}(e) - p_1\tilde{w}(F(e)) = p_1\tilde{r}(e) = r(e).$$

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Let $h : E \to E$ be such that $h(e) = e + w(e)$, which obviously covers the identity. We have

$$G \circ h(e) = G(e + w(e)) = G(e) + Aw(e) = G(e) + r(e) - r(e) + Aw(e)$$
$$= F(e) + w(F(e)) = h \circ F(e).$$

Since $\tilde{w}$ is bounded, it is homotopic to a constant map $E \to \mathbb{R}^d$, and thus h is homotopic to the identity via a homotopy that preserves each fibre. Then h induces a map of non-zero degree on the top homology, and thus it is surjective. □

We denote $\tilde{F}_b : \mathbb{R}^d_b \to \mathbb{R}^d_{f(b)}$ the lift of the restriction $F_b : \mathbb{T}^d_b \to \mathbb{T}^d_{f(b)}$. We use $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e})$ and $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$ to denote the stable and unstable leaves of $\tilde{F}_b$ passing through a point $\tilde{e}$ in $\mathbb{R}^d_b$. We will also use $\widetilde{W}^s_{\tilde{F}_b}(\tilde{F}^n_b(\tilde{e}))$ and $\widetilde{W}^u_{\tilde{F}_b}(\tilde{F}^n_b(\tilde{e}))$ to denote the leaves along an orbit of b lifted fibrewise. If the leaves of which map we are talking about are clear, we would also write $\widetilde{W}^s_b(\tilde{e})$ and $\widetilde{W}^u_b(\tilde{e})$.

Let $d(s; \cdot, \cdot)$ and $d(u; \cdot, \cdot)$ denote the distance between two points along stable and unstable leaves, respectively.

If any pair of stable and unstable leaves of $\tilde{F}_b$ has a unique intersection, we say that we have the global product structure (GPS) of the leaves of $\tilde{F}_b$ in the fibre over b.

We want to show the global product structure at all $b \in B$ to prove the injectivity of h (see Lemma 5.10). Before that, we check the following properties of either h or the fibrewise lift $\tilde{h}_b : \mathbb{R}^d_b \to \mathbb{R}^d_b$, which we will use repeatedly.

Property 5.2. The restriction h_b of h to a fibre maps a stable (or an unstable) leaf of F_b to a stable (or an unstable) leaf of G_b.

Proof. It is sufficient to show that h_b takes a local stable disk to a local stable disk, that is for any $e \in \mathbb{T}^d_b$, if $e' \in W^s_{F_b,\varepsilon}(e)$, we will have $h_b(e') \in W^s_{G_b,\varepsilon}(h_b(e))$ for some $\varepsilon > 0$.

Suppose the statement is not true, say $h_b(e') \notin W^s_{G_b,\varepsilon}(h_b(e))$. Then,

$$d_{\mathbb{T}^d_{f^n(b)}}(G^n_b(h_b(e)), G^n_b(h_b(e'))) = d_{\mathbb{T}^d_{f^n(b)}}(h_{f^n(b)}F^n_b(e), h_{f^n(b)}F^n_b(e')) \to 0,$$

as $n \to \infty$. However, the G_b is the fibrewise affine Anosov diffeomorphism which has straight leaves in the eigen-direction of A, so the left-hand side of the equation goes to infinity. We have reached a contradiction. □

Remark 5.3. From the the above, we also have that the lift $\tilde{h}_b$ takes a stable (or an unstable) leaf of $\tilde{F}_b$ to a stable (or an unstable) leaf of $\tilde{G}_b$.

Property 5.4. [4] Since $\tilde{h}_b$ is homotopic to the identity along the fibres, it is proper.

We claim that the existence of an intersection for any pair of stable and unstable leaves in the fibrewise covering spaces follows exactly from Lemma 1.6 of Franks [3], which we include for completeness. Since the proof there is rather long, we only include a sketch. Note that we are only going to use the local product structure and the semiconjugacy. This works for any point $b \in B$.

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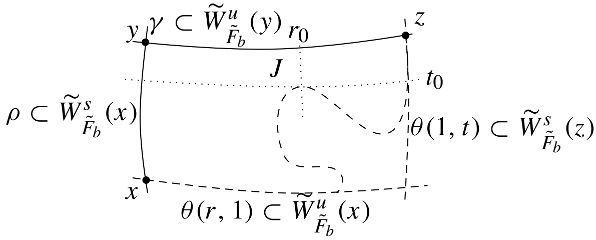

FIGURE 1. Definition of θ.

LEMMA 5.5. *For any points $\tilde{e}, \tilde{e}' \in \mathbb{R}^d_b$, we have $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e}) \cap \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}') \neq \emptyset$ and $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) \cap \widetilde{W}^s_{\tilde{F}_b}(\tilde{e}') \neq \emptyset$.*

Sketch of proof. Fix a stable leaf S of $\tilde{F}_b$, and we want to show that the set

$$Q := \{\tilde{e} \mid \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) \cap S \neq \emptyset\}$$

is $\mathbb{R}^d_b$. Because of the local product structure, there is an open set O that contains S such that $Q = \{\tilde{e} \mid \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) \cap O \neq \emptyset\}$. If $\tilde{e} \in Q$, there is a foliated neighbourhood U containing $\tilde{e}$ which we could extend to a path of foliated neighbourhoods whose union contains $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$, so $U \subset Q$. Thus, Q is open. We want to show that Q is also closed.

Suppose x is a point in the closure of Q but not in Q. We show that $\widetilde{W}^u_{\tilde{F}_b}(x) \cap S \neq \emptyset$. Because of the local product structure, we are able to find a sequence of points in Q approaching x which is also contained in the same stable leaf $\widetilde{W}^s_{\tilde{F}_b}(x)$. We pick one point from the sequence and denote it by y. Pick another point z in $\widetilde{W}^u_{\tilde{F}_b}(y) \cap S$. We connect y and z by a path $\gamma : [0, 1] \to \widetilde{W}^u_{\tilde{F}_b}(y)$ and connect x and y by a path $\rho : [0, 1] \to \widetilde{W}^s_{\tilde{F}_b}(x) = \widetilde{W}^s_{\tilde{F}_b}(y)$.

We aim to define a function $\theta : [0, 1] \times [0, 1] \to \mathbb{R}^d_b$ such that $\theta(r, 0) = \gamma(r)$ and $\theta(0, t) = \rho(t)$, and $\theta(r, t) = \widetilde{W}^s_{\tilde{F}_b}(\gamma(r)) \cap \widetilde{W}^u_{\tilde{F}_b}(\rho(t))$. If we are able to define θ at $(1, 1)$, then $\widetilde{W}^s_{\tilde{F}_b}(z) \cap \widetilde{W}^u_{\tilde{F}_b}(x) = \widetilde{W}^u_{\tilde{F}_b}(x) \cap S \neq \emptyset$ (see Figure 1).

Because of the local product structure, the set where θ can be defined is open. Now let

$$t_0 = \sup\{\hat{t} \mid \theta(r, t) \text{ is defined for } 0 \leq r \leq 1 \text{ and } 0 \leq t \leq \hat{t}\}, \quad \text{and}$$

$$r_0 = \sup\{\hat{r} \mid \theta(r, t_0) \text{ is defined for } 0 \leq r \leq \hat{r}\}.$$

If we let $J := \{(r, t) \mid t < t_0\}$, $\tilde{h}_b(\theta(J))$ is bounded because the projections of it to the linear leaves $\widetilde{W}^u_{\tilde{G}_b}(\tilde{h}_b(y))$ and $\widetilde{W}^s_{\tilde{G}_b}(\tilde{h}_b(y))$ are bounded. Then, $\theta(J)$ is bounded because $\tilde{h}_b$ is proper. Next pick a sequence $\{(r_n, t_n) \mid r_n \leq r_{n+1}\}$ in J converging to (r_0, t_0) such that $\theta(r_n, t_n)$ converges to a point $w \in \mathbb{R}^d_b$. We define $\theta(r_0, t_0) := w$ by continuity. What is left is to check that θ is well defined.

Let N be a product neighbourhood of w. We can assume $\theta(r_n, t_n) \in N$ for all $n > 0$ (see Figure 2).

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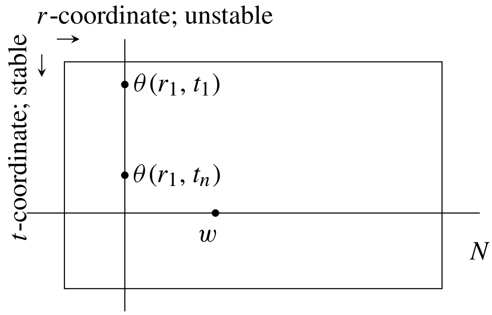

FIGURE 2. Extending θ to (r_0, t_0).

Since θ is defined at (r_n, t_n) for $n > 0$ and is, by definition, $\widetilde{W}^s_{\tilde{F}_b}(\gamma(r_n)) \cap \widetilde{W}^u_{\tilde{F}_b}(\rho(t_n))$. Thus, $\theta(r_1, t_n)$ lies on $\widetilde{W}^s_{\tilde{F}_b}(\gamma(r_1))$ for all $n > 0$. Then, $\theta(r_1, t_n)$ converges to $\widetilde{W}^s_{\tilde{F}_b}(\theta(r_1, t_1)) \cap \widetilde{W}^u_{\tilde{F}_b}(w)$. (Suppose not. If $\theta(r_1, t_n)$ converges to $\bar{w} := \widetilde{W}^s_{\tilde{F}_b}(\theta(r_1, t_1)) \cap \widetilde{W}^u(w')$ for a point $w' \neq w$, then $\theta(r_n, t_n)$ would converge to a point inside $\widetilde{W}^u_{\tilde{F}_b}(\bar{w}) \neq \widetilde{W}^u_{\tilde{F}_b}(w)$.) Similarly, $\theta(r_n, t_1)$ converges to $\widetilde{W}^s(w)_{\tilde{F}_b} \cap \widetilde{W}^u_{\tilde{F}_b}(\theta(r_1, t_1))$. Thus, it is either defined already so we have, or we could define that,

$$\theta(t_1, r_0) = \widetilde{W}^s_{\tilde{F}_b}(\theta(r_1, t_1)) \cap \widetilde{W}^u_{\tilde{F}_b}(w), \quad \theta(t_0, r_1) = \widetilde{W}^s(w)_{\tilde{F}_b} \cap \widetilde{W}^u_{\tilde{F}_b}(\theta(r_1, t_1)).$$

Now suppose somehow we are able to define for another sequence $\{(r'_n, t'_n)\}$ that converges to (r_0, t_0) such that $\theta(r_0, t_0) = \lim_{n\to\infty} \theta(r'_n, t'_n)$. Then, we have

$$\begin{aligned}\lim_{n\to\infty} \theta(r'_n, t'_n) &= \lim_{n\to\infty} \widetilde{W}^u_{\tilde{F}_b}(\theta(r_1, t'_n)) \cap \widetilde{W}^s_{\tilde{F}_b}(\theta(r'_n, t_1)) \\ &= \widetilde{W}^u_{\tilde{F}_b}(\theta(r_1, t_0)) \cap \widetilde{W}^s_{\tilde{F}_b}(\theta(r_0, t_1)) = w.\end{aligned}$$

This is saying that we could always extend θ to a limit point, so Q is closed.

Next we are going to show the uniqueness of the intersection for each pair of stable and unstable leaves.

Remark 5.6. Since we have already shown the existence of the intersections, if we take any pair of stable and unstable leaves, they must intersect, say at a point $\tilde{e} \in p^{-1}b$. It is then sufficient to show that this pair of stable and unstable leaves has no other intersections besides $\tilde{e}$.

However, for any point $\tilde{e} \in p^{-1}b$, there is a pair of stable and unstable leaves that intersects at $\tilde{e}$, namely, $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e})$ and $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$, which are exactly the pair of leaves that we considered in the above paragraph.

Therefore, to show the unique intersection for an arbitrary pair of leaves, we only need to show that for any $\tilde{e}$, $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e}) \cap \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) = \{\tilde{e}\}$.

The steps of the proof can be summarized as follows.

(1) If b is a recurrent point of f in the base, then in $p^{-1}b$, any pair of stable and unstable leaves has a unique intersection. This is Lemma 5.7.

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(2) We show that the set

$$\{b \in B \mid \text{the foliations induced by } F_b \text{ has the global product structure}\}$$

is open (with a uniform constant δ such that at every point where we already have GPS, any point in a ball with radius $< \delta$ has GPS). This is Corollary 5.14.

(3) Corollary 5.14 also implies that the above set is closed because the δ is uniform. This is Corollary 5.15.

Now we proceed with the proof.

LEMMA 5.7. *Let $b \in B$ be a recurrent point of f. Then $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) \cap \widetilde{W}^s_{\tilde{F}_b}(\tilde{e}) = \{\tilde{e}\}$.*

Proof. We use the fact that the leaves are 'very close' to the leaves of an Anosov diffeomorphism.

Assume $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$, $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e})$ intersect at two points, say $\tilde{e}, \tilde{e}'$. Then, $\tilde{F}^n_b(\tilde{e}') \in \widetilde{W}^u_{\tilde{F}_b}(\tilde{F}^n_b(\tilde{e})) \cap \widetilde{W}^s_{\tilde{F}_b}(\tilde{F}^n_b(\tilde{e}))$, which means the leaves along the orbit also intersect twice.

We are able to find a subsequence $\{f^{n_k}(b)\}$ that converges to b. By Lemma 4.2, for each large enough n_k, there is an $\hat{F}^{(k)}$ such that $F^{n_k}_b$ is C^1 close to $\hat{F}^{(k)}_b$. Denote $\bar{F}^n_b = \varphi_b \circ \psi \circ F^n_b : p^{-1}b \to p^{-1}b$, the projection of F^n at b to the fibre $p^{-1}b$. Since $F^{n_k}_b$ and $\hat{F}^{(k)}_b$ are C^1 close, the vectors in the distributions of $\bar{F}^{n_k}_b$ at a point in $p^{-1}b$ lie in the γ_k-cones of the distributions of $\hat{F}^{(k)}_b$ for some small γ_k, that is, the angles between the respective stable and unstable distributions are small. To suppress notation, let us still use $\tilde{e}, \tilde{e}'$ for the projection of $\tilde{e}, \tilde{e}'$ to the fibre $p^{-1}b$.

For any $\varepsilon > 0$, we can take k large enough so we have small enough γ_k such that

$$d(s; \bar{F}^{n_k}_b(\tilde{e}), \bar{F}^{n_k}_b(\tilde{e}')) \cdot \gamma_k < \varepsilon.$$

This means $\widetilde{W}^s_{\hat{F}^{(k)}_b}(\hat{F}^{(k)}_b(\tilde{e}))$ lies in the normal neighbourhood $N(\widetilde{W}^s_{\bar{F}_b}(\bar{F}^{n_k}_b(\tilde{e})), \varepsilon)$.

Now the unstable leaf $\widetilde{W}^u_{\hat{F}^{(k)}_b}(\hat{F}^{(k)}_b(\tilde{e}))$ must pass through $N(\widetilde{W}^s_{\bar{F}_b}(\bar{F}^{n_k}_b(\tilde{e})), \varepsilon)$ near $\bar{F}^{n_k}_b(\tilde{e}')$ because of the existence of the unstable cone. Then, $\widetilde{W}^s_{\hat{F}^{(k)}_b}(\hat{F}^k_b(\tilde{e}))$ and $\widetilde{W}^u_{\hat{F}^{(k)}_b}(\hat{F}^{(k)}_b(\tilde{e}))$ should intersect the second time because of the local product structure, which is not the case. Thus, $\widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$ and $\widetilde{W}^s_{\tilde{F}_b}(\tilde{e})$ cannot intersect more than once. □

Next we prove a useful technical lemma, which is implicit in [4] (Proof of Theorem 1).

LEMMA 5.8. *Let $\mathscr{F}_1$, $\mathscr{F}_2$ be a pair of transverse foliations on $\mathbb{T}^d$ with continuous leaves and the local product structure, such that its lift $\tilde{\mathscr{F}}_i$, $i = 1, 2$ also has the global product structure on $\mathbb{R}^d$. Let d_1, d_2 denote the distance along a leaf of $\tilde{\mathscr{F}}_1$, $\tilde{\mathscr{F}}_2$ between two points in the same leaf, respectively. Then for any compact set $K \subset \mathbb{R}^d$, there is a constant $M_K > 0$ such that*

$$\sup\{d_1(\tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}^1(\tilde{e})),\ \widetilde{W}^1 \text{ is a leaf of } \tilde{\mathscr{F}}_1\} < M_K, \quad \text{and}$$
$$\sup\{d_2(\tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}^2(\tilde{e})),\ \widetilde{W}^2 \text{ is a leaf of } \tilde{\mathscr{F}}_2\} < M_K,$$

that is, the distance along the leaves are bounded in a compact set.

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Proof. We show it for $\tilde{\mathscr{F}}_2$. From the global product structure, we get a homeomorphism $\alpha = (\alpha_1, \alpha_2) : \mathbb{R}^d \to \mathbb{R}^l \times \mathbb{R}^{d-l}$, where α_1, α_2 are the projections, $\tilde{\mathscr{F}}_2$ is of dimension $d - l$ and codimension l, and the latter $\mathbb{R}^l \times \mathbb{R}^{d-l}$ has the usual topology.

Suppose there is not such an upperbound M_K for the distance along leaves. Then there exists a sequence of pairs $\{(\tilde{e}_n, \tilde{e}'_n)\}$ such that $\tilde{e}_n, \tilde{e}'_n \in K \cap \widetilde{W}^2(\tilde{e}_n)$ and $d_1(\tilde{e}_n, \tilde{e}'_n) > n$.

By the compactness of K, there is a subsequence $\{(\tilde{e}_{n_k}, \tilde{e}'_{n_k})\}$ that converges to a pair of points $(\tilde{e}_0, \tilde{e}'_0)$. Since $\alpha_1(\tilde{e}_n) = \alpha_1(\tilde{e}'_n)$ (because they are on the same leaf of $\tilde{\mathscr{F}}_2$), we know $\alpha_1(\tilde{e}_0) = \alpha_1(\tilde{e}'_0)$, that is, $\tilde{e}'_0 \in \widetilde{W}^2(\tilde{e}_0)$.

Now we can cover the path between $\tilde{e}_0$ and $\tilde{e}'_0$ that realizes $d_2(\tilde{e}_0, \tilde{e}'_0)$ in $\widetilde{W}^2(\tilde{e}_0)$ with a finite number of cubes with length of edges smaller than the constant of the local product structure, and denote the cubes by $\{P_1, \ldots, P_t\}$. Then, $\alpha(\bigcup_{i=1}^t P_i)$ covers a neighbourhood $B_\varepsilon \times \alpha_2(\widetilde{W}^2(\tilde{e}_0) \cap K)$, where B_ε is a small l-dimensional disk. Thus, $\alpha^{-1}(B_\varepsilon \times \alpha_2(\widetilde{W}^2(\tilde{e}_0) \cap K))$ also covers a piece between $\tilde{e}_{n_k}$ and $\tilde{e}'_{n_k}$ of $\widetilde{W}^2(\tilde{e}_{n_k})$ for all $k > L$ for some large L, because of the local product structure. However, $d_2(\tilde{e}_{n_k}, \tilde{e}'_{n_k}) > k$. They cannot be all covered by a finite number of cubes of a fixed diameter. We have reached a contradiction. □

COROLLARY 5.9. *Let $C \subset B$ be a compact set contained in a trivialized neighbourhood of the bundle we consider. Consider the leaves of F lifted continuously over C. Suppose we have the global product structure at all $b \in C$. Then, for any compact set $K \subset C \times \mathbb{R}^d$, there is an $M_C > 0$ such that*

$$\sup_{b \in C} \sup\{d(u; \tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}^u_b(\tilde{e}))\} < M_C \quad \textit{and}$$

$$\sup_{b \in C} \sup\{d(s; \tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}^s_b(\tilde{e}))\} < M_C.$$

Proof. We are able to run the above argument, starting with a homeomorphism $\alpha = id \times (\alpha_1, \alpha_2) : C \times \mathbb{R}^d \to C \times (\mathbb{R}^l \times \mathbb{R}^{d-l})$ because of the global product structure.

If we want to check for the unstable foliation, then instead of $B_\varepsilon \times \alpha_2(\widetilde{W}^2(\tilde{e}_0) \cap K)$, we consider a neighbourhood $A \times B_\varepsilon \times \alpha_2(\widetilde{W}^u_{b_0}(\tilde{e}_0) \cap K)$ covered by a finite number of cubes given by the local product structure, where $p^{-1}b_0$ contains $\tilde{e}_0, \tilde{e}'_0$, A is a small neighbourhood around b_0 and B_ε is an l-dimensional disk. The rest of the argument is the same. □

LEMMA 5.10. *If we have the global product structure at a point $b \in B$, then the restriction $\tilde{h}_b : \mathbb{R}^d_b \to \mathbb{R}^d_b$ is injective.*

Proof. Note that if we have the global product structure at one point b, then we also have it along the orbit of b, if considering the lifts fibrewise. We are then able to follow a similar argument to Franks [3] to deduce the injectivity of $\tilde{h}_b$.

Recall that $\tilde{h}_b$ preserves the stable and unstable leaves (Property 5.2). Suppose after fixing the lifts, we can find $\tilde{e} \neq \tilde{e}'$ such that $\tilde{h}_b(\tilde{e}) = \tilde{h}_b(\tilde{e}')$. Let $\tilde{e}'' := \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}) \cap \widetilde{W}^s_{\tilde{F}_b}(\tilde{e}')$. We know $\widetilde{W}^u_{\tilde{G}_b}(\tilde{h}_b(\tilde{e})) \cap \widetilde{W}^s_{\tilde{G}_b}(\tilde{h}_b(\tilde{e}')) = \tilde{h}_b(\tilde{e}'')$, that is, they also intersect only once at

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$\tilde{h}_b(\tilde{e}'')$. Then, $\tilde{h}_b(\tilde{e}) = \tilde{h}_b(\tilde{e}') = \tilde{h}_b(\tilde{e}'')$. Thus, we have found $\tilde{e}, \tilde{e}'' \in \widetilde{W}^u_{\tilde{F}_b}(\tilde{e})$ such that $\tilde{h}_b(\tilde{e}) = \tilde{h}_b(\tilde{e}'')$.

Since B is assumed to be a compact manifold (which is normal), for each finite open cover, we can find a finite refinement that contains compact sets. Now we use compact trivialized neighbourhoods, and h_b is lifted continuously over each of them, denoted as $\tilde{h}_C$ on a compact set C. Then, $\tilde{h}_C$ is homotopic to the identity along the fibres, and thus is proper (Property 5.4). Denote the set $(\mathrm{id}, \tilde{h}_C)^{-1}(C \times [0, 1]^d)$ as D, which is compact.

Then, by Corollary 5.9, we have an $M_C > 0$ such that

$$\sup_{b\in C} \sup\{d(u; \tilde{e}, \tilde{e}'') : \tilde{e}, \tilde{e}'' \in (\tilde{h}_b^{-1}[0, 1]^d \cap \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}))\} < M_C.$$

Then again by the compactness of B, there is a uniform bound $M > 0$ such that

$$\sup_{b\in B} \sup\{d(u; \tilde{e}, \tilde{e}'') : \tilde{e}, \tilde{e}'' \in (\tilde{h}_b^{-1}[0, 1]^d \cap \widetilde{W}^u_{\tilde{F}_b}(\tilde{e}))\} < M.$$

Denote $\tilde{w}_n := \tilde{h}_{f^n(b)}\tilde{F}^n_b(\tilde{e}) = \tilde{h}_{f^n(b)}\tilde{F}^n_b(\tilde{e}'')$. We can always use deck transformations to translate the $\tilde{w}_n$ terms to $[0, 1]^d$. Thus, we should also have $d(u; \tilde{F}^n_b(\tilde{e}), \tilde{F}^n_b(\tilde{e}'')) < M$ for all n. We have reached a contradiction. □

Note again the above shows that if at b we have the global product structure, $\tilde{h}_b$ is a homeomorphism. It is surjective for free because it has non-zero degree on the top homology. In particular, $\tilde{h}_b$ is a homeomorphism for all recurrent points $b \in B$ now.

Then we show that the property of having the global product structure is open.

Remark 5.11. Our main idea of the next proof is just to approximate the leaves where we do not know if they have the global product structure, with the leaves that already have the global product structure, with the existence of the cones.

However, since the universal cover is non-compact, we could quickly lose control of the error. Thus, we emphasize the use of the semiconjugacy (now already a diffeomorphism at a nearby point where the GPS exists), which is bounded from the identity, so the nonlinear leaves is of a bounded distance from the linear leaves of our fibrewise affine Anosov diffeomorphism, which is essential to make the approximation work.

PROPOSITION 5.12. *Let $\mathscr{F}_1, \mathscr{F}_2$ be a pair of transverse foliations on $\mathbb{T}^d$ with C^1 leaves and the local product structure. Suppose the lifts $\tilde{\mathscr{F}}_i$ of $\mathscr{F}_i$, $i = 1, 2$ have the global product structure on $\mathbb{R}^d$. Let $\mathscr{F}'_1$ and $\mathscr{F}'_2$ be another pair of transverse foliations.*

Then there exists an $\alpha > 0$ such that, if

(1) $\max_x \angle(T\mathscr{F}_i(x), T\mathscr{F}'_i(x)) < \alpha$ *for* $i = 1, 2$;

(2) *there is a homeomorphism* $h : \mathbb{T}^d \to \mathbb{T}^d$ *which is homotopic to the identity and takes* $\mathscr{F}_i, i = 1, 2$ *to another pair of linear foliations* $\mathscr{F}''_i, i = 1, 2$,

then the pair of lifts $\tilde{\mathscr{F}}'_i$ of $\tilde{\mathscr{F}}'_i$, $i = 1, 2$ has the global product structure.

Remark 5.13. We tried to state this in a more general manner. In our setting, the $\mathscr{F}_i$ terms are the foliations of the fibrewise Anosov diffeomorphism at a point, say b, where we do not have the GPS yet, the $\mathscr{F}'_i$ terms are the foliations of F at a point that is close to b where we have the GPS, and the $\mathscr{F}''_i$ terms are the foliations of G with straight leaves.

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Note that this is purely a statement about the foliations which has nothing to do with the dynamics.

Proof of Proposition 5.12. We first fix the notation. We will denote leaves at $\tilde{e}$ of $\tilde{\mathscr{F}}_i$ by $\widetilde{W}_1^i(\tilde{e})$ and leaves of $\tilde{\mathscr{F}}_i'$ by $\widetilde{W}_2^i(\tilde{e})$, $i = 1, 2$. Then, for any $\tilde{e}_0 \in \mathbb{R}^d$, we want to show that $\widetilde{W}_2^1(\tilde{e}_0) \cap \widetilde{W}_2^2(\tilde{e}_0) = \{\tilde{e}_0\}$. We start from selecting the following constants.

- ε: Let $\varepsilon > 0$ be smaller than the constant of the local product structure.
- M_1: Let M_1 be the constant such that $\|h - \mathrm{id}\| < M_1$.
- D: We fix a $D > 10(M_1 + \varepsilon)$.
- M_2: Suppose K is a compact set of diameter $\leq D$. There is an M_2 such that

$$\sup\{d_1(\tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}_1^1(\tilde{e})),\ \widetilde{W}_1^1 \text{ is a leaf of } \tilde{\mathscr{F}}_1\} < M_2, \quad \text{and}$$

$$\sup\{d_2(\tilde{e}, \tilde{e}') : \tilde{e}, \tilde{e}' \in (K \cap \widetilde{W}_1^2(\tilde{e})),\ \widetilde{W}_1^2 \text{ is a leaf of } \tilde{\mathscr{F}}_2\} < M_2,$$

  from Corollary 5.8.
- M: Let $M = 10 \max\{M_2, D\}$.

We could choose $\alpha < \varepsilon/10M$.

We denote $\mathcal{O} := \tilde{h}(\tilde{e}_0)$ and the linear leaves of $\tilde{\mathscr{F}}_i''$, $i = 1, 2$ passing through $\mathcal{O}$ by $\widetilde{W}_0^1$, $\widetilde{W}_0^2$. There is then a natural coordinate of every point if we consider the projection of it onto these straight leaves. For a point $\tilde{e} \in \mathbb{R}^d$, we want to denote the distance between the projection of $\tilde{e}$ to $\widetilde{W}_0^1$ and $\mathcal{O}$ along $\widetilde{W}_0^1$ as $\|\tilde{e} - \mathcal{O}\|_1$ and the distance between the projection of $\tilde{e}$ to $\widetilde{W}_0^2$ and $\mathcal{O}$ along $\widetilde{W}_0^2$ as $\|\tilde{e} - \mathcal{O}\|_2$.

We claim that for any point $\tilde{e}_1 \in \widetilde{W}_2^1(\tilde{e}_0)$ and any point $\tilde{e}_2 \in \widetilde{W}_2^2(\tilde{e}_0)$ of the foliations $\tilde{\mathscr{F}}_i'$,

$$\text{either } \|\tilde{e}_1 - \tilde{e}_2\|_1 > 0 \text{ or } \|\tilde{e}_1 - \tilde{e}_2\|_2 > 0.$$

Case 1. We start with the special case when $\|\tilde{e}_1 - \mathcal{O}\|_1$, $\|\tilde{e}_1 - \mathcal{O}\|_2$, $\|\tilde{e}_2 - \mathcal{O}\|_1$ and $\|\tilde{e}_2 - \mathcal{O}\|_2$ are all $\leq M_1 + \varepsilon$.

There exist points $\tilde{e}_1' \in \widetilde{W}_1^1(\tilde{e}_0)$ and $\tilde{e}_2' \in \widetilde{W}_1^2(\tilde{e}_0)$ such that $\|\tilde{e}_1 - \mathcal{O}\|_1 = \|\tilde{e}_1' - \mathcal{O}\|_1$, $\|\tilde{e}_2 - \mathcal{O}\|_2 = \|\tilde{e}_2' - \mathcal{O}\|_2$. With the α we chose, $\widetilde{W}_2^2(\tilde{e}_1)$ stays in a cone of slope $\leq \varepsilon/10M$ around the leaf $\widetilde{W}_1^2(\tilde{e}_0)$ for at least a distance of M_2, which is the upperbound of the distance along a leaf of $\tilde{\mathscr{F}}_i$ inside a set of diameter $D > 10M_1$. We then have $\|\tilde{e}_2 - \tilde{e}_2'\|_1 \leq 2M_2(\varepsilon/10M) << \varepsilon/2$. Also, $\|\tilde{e}_1 - \tilde{e}_1'\|_1 = 0$.

Note that either $\|\tilde{e}_1' - \tilde{e}_2'\|_1 \geq \varepsilon$ or $\|\tilde{e}_1' - \tilde{e}_2'\|_2 \geq \varepsilon$ because of the local product structure and the fact that $\tilde{W}_1^2(\tilde{e}_0)$ and $\tilde{W}_1^1(\tilde{e}_0)$ (with the GPS) do not meet a second time besides at $\tilde{e}_0$. If both $\|\tilde{e}_1' - \tilde{e}_2'\|_1 < \varepsilon$ and $\|\tilde{e}_1' - \tilde{e}_2'\|_2 < \varepsilon$, this means $W_1^i(\tilde{e}_0)$ have entered a product neighbourhood where they must intersect for another time, contradicting the global product structure.

If $\|\tilde{e}_1' - \tilde{e}_2'\|_1 \geq \varepsilon$, then we have

$$\|\tilde{e}_1 - \tilde{e}_2\|_1 \geq \|\tilde{e}_1' - \tilde{e}_2'\|_1 - \|\tilde{e}_1 - \tilde{e}_1'\|_1 - \|\tilde{e}_2 - \tilde{e}_2'\|_1 \geq \varepsilon - 2M_2 \frac{\varepsilon}{10M} > 0.$$

Similarly, if we have $\|\tilde{e}_1' - \tilde{e}_2'\|_2 \geq \varepsilon$, then $\|\tilde{e}_1 - \tilde{e}_2\|_2 \geq \varepsilon$.

Case 2. When any of $\|\tilde{e}_1 - \mathcal{O}\|_1$, $\|\tilde{e}_1 - \mathcal{O}\|_2$, $\|\tilde{e}_2 - \mathcal{O}\|_1$ or $\|\tilde{e}_2 - \mathcal{O}\|_2$ is greater than $M_1 + \varepsilon$, we can prove this claim by a double induction. See Figure 3.

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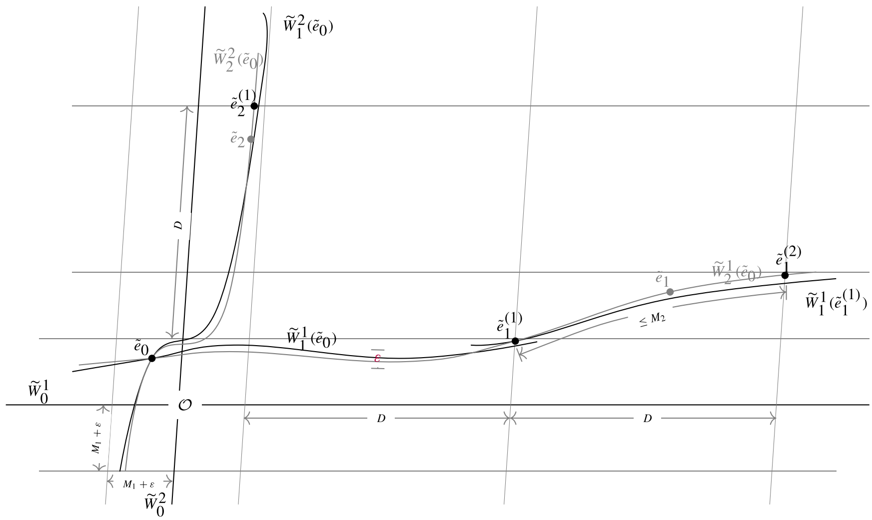

FIGURE 3. Leaves.

A brief idea of this is that we show leaves of $\tilde{\mathscr{F}}_i'$ follow along the leaves of $\tilde{\mathscr{F}}_i$ for distance D with an error $< \varepsilon$. Then we switch to other leaves of $\tilde{\mathscr{F}}_i$ to follow, to keep the error of our estimate small. At the same time, note that the pair of leaves of $\tilde{\mathscr{F}}_i$ is far apart, so they never meet again by GPS, and hence the leaves of $\tilde{\mathscr{F}}_i'$ would not meet neither.

Here, we see the necessity of the existence of the homeomorphism. If it is not injective, it may take a entire leaf to a bounded subset of the linear leaf. Then, we would not have the leaves of $\tilde{\mathscr{F}}_i'$ terms separating from each other.

The detailed induction is as follows.

First, for the base case, we let $\tilde{e}_1,\ \tilde{e}_2$ be points such that

$$M_1 + \varepsilon < \|\tilde{e}_1 - \mathcal{O}\|_1 \leq M_1 + \varepsilon + D \quad \text{and} \quad M_1 + \varepsilon < \|\tilde{e}_2 - \mathcal{O}\|_2 \leq M_1 + \varepsilon + D.$$

Note that we would not need to consider cases where $M_1 + \varepsilon < \|\tilde{e}_1 - \mathcal{O}\|_1 \leq M_1 + \varepsilon + D$ and $\|\tilde{e}_2 - \mathcal{O}\|_2 \leq M_1 + \varepsilon$, or $\|\tilde{e}_1 - \mathcal{O}\|_1 \leq M_1 + \varepsilon$ and $M_1 + \varepsilon < \|\tilde{e}_2 - \mathcal{O}\|_2 \leq M_1 + \varepsilon + D$, since they have been dealt with in Case 1. For any such points $\tilde{e}_1 \in \widetilde{W}_2^1(\tilde{e}_0)$ and $\tilde{e}_2 \in \widetilde{W}_2^2(\tilde{e}_0)$, we again have $\tilde{e}_1' \in \widetilde{W}_1^1(\tilde{e}_0)$ and $\tilde{e}_2' \in \widetilde{W}_1^2(\tilde{e}_0)$, such that $\|\tilde{e}_1 - \mathcal{O}\|_1 = \|\tilde{e}_1' - \mathcal{O}\|_1$ and $\|\tilde{e}_2 - \mathcal{O}\|_2 = \|\tilde{e}_2' - \mathcal{O}\|_2$.

We have the estimate $\|\tilde{e}_1 - \tilde{e}_1'\|_2 < 4M_2\alpha < 4M_2(\varepsilon/10M) < \varepsilon$, and similarly $\|\tilde{e}_2 - \tilde{e}_2'\|_1 < \varepsilon$, while $\|\tilde{e}_1 - \tilde{e}_1'\|_1 = \|\tilde{e}_2 - \tilde{e}_2'\|_2 = 0$. We have $\|\tilde{e}_1' - \mathcal{O}\|_2 \leq M_1$ because again $\tilde{h}$ has a bounded distance M_1 from the identity and, as a result, any point on $\widetilde{W}_1^1(\tilde{e}_0)$ has a bounded distance M_1 from the linear leaf $\widetilde{W}_0^1$. In this case, we always have $\|\tilde{e}_1 - \mathcal{O}\|_2 < M_1 + \varepsilon$ and $\|\tilde{e}_2 - \mathcal{O}\|_1 < M_1 + \varepsilon$ because, for instance,

$$\|\tilde{e}_1 - \mathcal{O}\|_2 \leq \|\tilde{e}_1 - \tilde{e}_1'\|_2 + \|\tilde{e}_1' - \mathcal{O}\|_2 < M_1 + \varepsilon.$$

Then,

$$\|\tilde{e}_1 - \tilde{e}_2\|_1 \geq \|\tilde{e}_1 - \mathcal{O}\|_1 - \|\tilde{e}_2 - \mathcal{O}\|_1 > (M_1 + \varepsilon) - (M_1 + \varepsilon) > 0.$$

Now we denote by $\tilde{e}_1^{(k)}$ the point on $\widetilde{W}_2^1(\tilde{e}_0)$ such that $\|\tilde{e}_1^{(k)} - \mathcal{O}\|_1 = M_1 + kD < (1/10 + k)D$; $\tilde{e}_2^{(k)}$ the point on $\widetilde{W}_2^2(\tilde{e}_0)$ such that $\|\tilde{e}_2^{(k)} - \mathcal{O}\|_2 = M_1 + kD < (1/10 + k)D$, $k \geq 1$. We then take leaves $\widetilde{W}_1^1(\tilde{e}_1^{(k)})$ and $\widetilde{W}_1^2(\tilde{e}_2^{(k)})$ for each $k \geq 1$. These are the 'new leaves' of $\tilde{\mathscr{F}}_i$ we switch to track.

From the induction hypothesis: for $i \geq 1$, $j \geq 1$, and such selected $\tilde{e}_1^{(i)}$, $\tilde{e}_2^{(j)}$, we have

$$\|\tilde{e}_1^{(i)} - \mathcal{O}\|_2 < (2i-1)(M_1+\varepsilon) \quad \text{and} \quad \|\tilde{e}_2^{(j)} - \mathcal{O}\|_1 < (2j-1)(M_1+\varepsilon).$$

Note that this is implicit in the base case.

Now consider the case where $i \geq 1$ and $j \geq 1$, and the points $\tilde{e}_1$, $\tilde{e}_2$ such that

$$M_1 + \varepsilon + iD < \|\tilde{e}_1 - \mathcal{O}\|_1 \leq M_1 + \varepsilon + (i+1)D \quad \text{and}$$

$$M_1 + \varepsilon + jD < \|\tilde{e}_2 - \mathcal{O}\|_2 \leq M_1 + \varepsilon + (j+1)D.$$

For any $\tilde{e}_1 \in \widetilde{W}_2^1(\tilde{e}_0)$ and $\tilde{e}_2 \in \widetilde{W}_2^2(\tilde{e}_0)$, we could also find $\tilde{e}_1' \in \widetilde{W}_1^1(\tilde{e}_1^{(i)})$ and $\tilde{e}_2' \in \widetilde{W}_1^2(\tilde{e}_2^{(j)})$ such that $\|\tilde{e}_1 - \mathcal{O}\|_1 = \|\tilde{e}_1' - \mathcal{O}\|_1$ and $\|\tilde{e}_2 - \mathcal{O}\|_2 = \|\tilde{e}_2' - \mathcal{O}\|_2$.

Again, $\|\tilde{e}_1 - \tilde{e}_1'\|_2 \leq 2M_2\alpha < 2M_2(\varepsilon/10M) < \varepsilon$ and $\|\tilde{e}_2 - \tilde{e}_2'\|_1 < \varepsilon$. Here, we have

$$\begin{aligned}\|\tilde{e}_1 - \mathcal{O}\|_2 &\leq \|\tilde{e}_1 - \tilde{e}_1'\|_2 + \|\tilde{e}_1' - \tilde{e}_1^{(i)}\|_2 + \|\tilde{e}_1^{(i)} - \mathcal{O}\|_2 \\ &\leq \varepsilon + 2M_1 + (2i-1)(M_1+\varepsilon) \\ &< (2i+1)(M_1+\varepsilon),\end{aligned}$$

where $\|\tilde{e}_1' - \tilde{e}_1^{(i)}\|_2 \leq 2M_1$ because every point on the leaf $\widetilde{W}_1^1(\tilde{e}_1^{(i)})$ is of bounded distance $< M_1$ from a linear leaf passing through $h(\tilde{e}_1^{(1)})$ and then $\tilde{e}_1'$ can only be in a cylinder of radius M_1 that contains this linear leaf. Similarly, we have $\|\tilde{e}_2 - \mathcal{O}\|_1 < (2j+1)(M_1+\varepsilon)$.

If $i \geq j$, then

$$\begin{aligned}\|\tilde{e}_1 - \tilde{e}_2\|_1 &\geq \|\tilde{e}_1 - \mathcal{O}\|_1 - \|\tilde{e}_2 - \mathcal{O}\|_1 \\ &> \|\tilde{e}_1^{(i)} - \mathcal{O}\|_1 - \|\tilde{e}_2 - \mathcal{O}\|_1 \\ &\geq iD - (2j+1)(M_1+\varepsilon) \\ &\geq (10i - (2j+1))(M_1+\varepsilon) > 0.\end{aligned}$$

If $i \leq j$, then $\|\tilde{e}_1 - \tilde{e}_2\|_2 > 0$. □

COROLLARY 5.14. *For our fibrewise Anosov diffeomorphism $F : E \to E$, suppose at $b_0 \in B$ we have the global product structure. Then, there exists a δ (uniform for all $b_0 \in B$) such that if $d_B(b_0, b) < \delta$, then we also have the global product structure at b.*

Proof. We apply the above proposition, where $\mathscr{F}_i$, $i = 1, 2$ is the pair of the stable and unstable foliations of F_{b_0} and $\mathscr{F}_i'$, $i = 1, 2$ is the stable and unstable foliations of F_b. We get an α.

Because of the smoothness of F and thus of its induced foliations and the compactness of the bundle, we are able to pick a $\delta > 0$ such that if $d(b, b_0) < \delta$ with b, b_0 in the same trivialized neighbourhood, then at any point $e \in p^{-1}b_0$ and $e' \in p^{-1}b$ such that $\psi(e) = \psi(e')$, we have $\max \angle_e(p_i(e), d\psi(p_i'(e))) < \alpha$, where $\{p_i\}_{i=1}^d$ and $\{p_i'\}_{i=1}^d$ are

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bases of the tangent space of $p^{-1}b_0$ and $p^{-1}b$, respectively. Also, p_i, p_i' are bases elements of the stable bundle if $i = 1, \ldots, l$ and are bases elements of the unstable bundle if $i = l + 1, \ldots, d$. □

COROLLARY 5.15. *Suppose $\{b_n\}$ is a sequence of points in the base B that converges to b_0, where at each b_n, the stable and unstable foliations of F have the global product structure. Then at b_0, we also have the global product structure.*

Proof. This also follows from Corollary 5.14, because for that fixed δ, we could find an N such that $d(b_0, b_N) < \delta$. □

We prove Theorem 1.1 by showing that the h itself is a homeomorphism.

Proof of Theorem 1.1. The set

$$\{b \in B \mid \text{the foliations induced by } F_b \text{ has the global product structure}\}$$

is non-empty by Lemma 5.7, is open by Corollary 5.14 and is closed by Corollary 5.15. Hence, this set is the full set of B. By Lemma 5.10, $\tilde{h}_b$ is injective for all $b \in B$.

We know h is surjective from Proposition 5.1. If we take e, e' such that $h(e) = h(e')$, then e and e' must be in the same fibre. Let $b := p(e) = p(e')$. Consider the lifts to $\mathbb{R}^d_b$. Then there is a deck transformation $j \in \mathbb{Z}^d$, such that $\tilde{h}_b(\tilde{e}) = \tilde{h}_b(\tilde{e}') + j = \tilde{h}_b(\tilde{e}' + j)$, so $\tilde{e} = \tilde{e}' + j$. Thus, $e = e'$. Therefore, h is also injective. □

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