



# Self 2-distance Graphs

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*Abstract.* All finite simple self 2-distance graphs with no square, diamond, or triangles with a common vertex as subgraph are determined. Utilizing these results, it is shown that there is no cubic self 2-distance graph.

## 1 Introduction

Let  $(X, \rho)$  be a metric space and let  $D$  be a set of positive real numbers. The *distance graph*  $G(X, D)$  of  $X$  with respect to a distance set  $D$  is the graph whose vertex set is  $X$  and two distinct vertices  $x$  and  $y$  are adjacent if  $\rho(x, y) \in D$ .

The well-known unit distance graph  $G(\mathbb{R}^2, \{1\})$  is the first instance of a distance graph, arising from a question of E. Nelson about its chromatic number in 1950 (see [10, Chapter 3]). It is shown by Moser and Moser [7] and Hadwiger, Debrunner, and Klee [4] that the chromatic number of this graph is between 4 and 7. Unit distance graphs are also investigated on any of the sets  $\mathbb{R}^n$ ,  $\mathbb{Q}^n$ , and  $\mathbb{Z}^n$  as well (see [10] for a detailed history). The other well-studied sort of distance graphs are the distance graphs  $G(\mathbb{Z}, D)$  introduced by Eggleton, Erdős, and Skiltons in [3], where  $D$  is a set of positive integers. Clearly, every graph  $\Gamma$  with associated distance function  $d$  defines a metric space  $(\Gamma, d)$ . Hence, we can define the distance graphs of the graph  $\Gamma$  with respect to a set of positive integer distances. For example, the  $n$ -th power of a graph  $\Gamma$  is defined simply as the distance graph  $G(\Gamma, \{1, \dots, n\})$ . We refer the interested reader to the survey articles [2, 5, 6] for further details concerning these three kinds of distance graphs, respectively.

The  $n$ -th *distance graph* (or  $n$ -*distance graph*) of a graph  $\Gamma$  is defined simply as  $D_n(\Gamma) := G(V(\Gamma), \{n\})$ . The study of  $n$ -th distance graphs was initiated by Simić [9] while solving the graph equation  $D_n(\Gamma) \cong L(\Gamma)$ , where  $L(\Gamma)$  denotes the line graph of  $\Gamma$ . Regarding the same problem, we have classified all graphs whose 2-distance graphs are path or cycle in [1].

A graph is said to be a *self  $n$ -distance graph* if it is isomorphic to its  $n$ -distance graph. The aim of this paper is to investigate self 2-distance graphs under some conditions. More precisely, we will show that self 2-distance graphs with no squares or disjoint triangles are precisely odd cycles of order  $\geq 5$  along with three small graphs (see Theorems 3.7 and 4.7). Also, we show that a self 2-distance graph with no diamond is either an odd cycle of order  $\geq 5$ , or one of the given five small graphs (see Theorem 5.1). One note that our knowledge about  $n$ -distance graphs can be used to answer/pose some problems in groups through their Cayley graphs. Indeed, we

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can observe that the  $n$ -th distance graph of a Cayley graph  $\text{Cay}(G, S)$  of  $G$  equals  $\text{Cay}(G, S^n \setminus S^{n-1})$ , and hence it is itself a Cayley graph. Here,  $S$  is an inverse closed subset of the group  $G$  without identity,  $S' := S \cup \{1\}$  and  $S^m = \{1\} \cup S \cup S^2 \cup \dots \cup S^m$  for all  $m \geq 0$ . Any isomorphism between  $\text{Cay}(G, S)$  and  $\text{Cay}(G, S^n \setminus S^{n-1})$  gives the constraint  $|S^n| < n|S|^{n-1}$  on  $S$ , a problem which is the subject of recent research especially when  $n$  is small, say  $n = 2$ . On the other hand, such an isomorphism brings us the question whether  $S^n \setminus S^{n-1}$  and  $S$  are conjugates via an automorphism of  $G$ , which is a central problem in the theory of Cayley graphs. In case  $S^n \setminus S^{n-1} = S^\theta$  for some automorphism  $\theta \in \text{Aut}(G)$ , we obviously have  $\text{Cay}(G, S^n \setminus S^{n-1}) \cong \text{Cay}(G, S)$ ; that is,  $\text{Cay}(G, S)$  is a self  $n$ -distance graph.

Throughout this paper, we use the following notation. The maximum degree of vertices of a graph  $\Gamma$  is denoted by  $\Delta(\Gamma)$ , and  $N_\Gamma(v)$  is the set of all neighborhoods of the vertex  $v$  in  $\Gamma$ . Also,  $\nabla(\Gamma)$  denotes the number of triangles in a graph  $\Gamma$ . The complement of a graph  $\Gamma$  is denoted by  $\Gamma^c$ . All graphs in this papers are connected finite simple graphs with no multiple edges. Recall that a *diamond* is the edge product  $\mathcal{D} = C_3|C_3$ , where the *edge product*,  $\Gamma_1 | \Gamma_2$ , of two edge-transitive graphs  $\Gamma_1$  and  $\Gamma_2$  is obtained by identification of an edge from  $\Gamma_1$  and  $\Gamma_2$ .

## 2 Preliminary Results

We begin with a simple query about the existence of self 2-distance graphs. Clearly, every odd cycle of length at least 5 is a self 2-distance graph. As we shall see later, odd cycles are exceptional examples in the class of all self 2-distance graphs. We note that the class of self 2-distance graphs is broad as Propositions 2.2 and 2.3 provide ample of them. The following simple key lemma plays an important role in our study.

**Lemma 2.1** *Let  $\Gamma$  be a graph. Then  $\text{diam}(\Gamma) \leq 2$  if and only if  $D_2(\Gamma) = \Gamma^c$ .*

**Proposition 2.2** *Every self-complementary graph with diameter two is a self 2-distance graph.*

**Proposition 2.3** *Every graph is an induced subgraph of a self 2-distance graph.*

**Proof** Let  $\Gamma$  be an arbitrary graph. Consider two disjoint copies  $\Gamma_1$  and  $\Gamma_2$  of  $\Gamma$  and two disjoint copies  $\Gamma_3$  and  $\Gamma_4$  of  $\Gamma^c$ , and let  $v$  be a new vertex. Then the graph  $\Gamma'$  with vertex set

$$V(\Gamma_1) \cup V(\Gamma_2) \cup V(\Gamma_3) \cup V(\Gamma_4) \cup \{v\}$$

and edge set

$$E(\Gamma_1) \cup E(\Gamma_2) \cup E(\Gamma_3) \cup E(\Gamma_4) \cup E,$$

where

$$E = \{ \{v, v_1\}, \{v, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4\} : v_i \in V(\Gamma_i), i = 1, 2, 3, 4 \}$$

is a self 2-distance graph containing  $\Gamma$  as an induced subgraph (see Figure 2.1). ■

**Lemma 2.4** *If  $\Gamma$  is a self 2-distance graph that is not an odd cycle, then  $\Gamma$  has a triangle.*

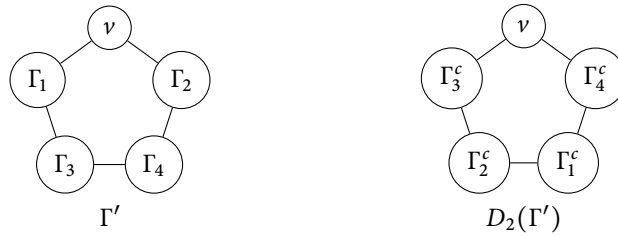


Figure 2.1

**Proof** Since  $\Delta(\Gamma) > 2$ , we can choose a vertex  $v$  of degree  $\geq 3$ . If  $N_\Gamma(v)$  contains an edge, then  $\Gamma$  has a triangle. Thus, we can assume that  $N_\Gamma(v)$  has no edges. But then  $N_\Gamma(v)^c$  is a subgraph of  $D_2(\Gamma) \cong \Gamma$ , which implies that  $\Gamma$  has a triangle, as required. ■

The following lemma will be used in the next section.

**Lemma 2.5** *Let  $\Gamma$  be a graph with no squares as subgraph. Then*

$$|E(L(\Gamma))| = |E(D_2(\Gamma))| + 3\nabla(\Gamma).$$

**Proof** We count the number of edges in the line graph of  $\Gamma$ . First observe that there is a one-to-one correspondence between the edges of  $L(\Gamma)$  and the paths of length two in  $\Gamma$ , among which  $3\nabla(\Gamma)$  edges of  $L(\Gamma)$  corresponds to paths of length two arising from triangles of  $\Gamma$ . All other edges of  $L(\Gamma)$  correspond to induced paths of length two in  $\Gamma$ , each of which determine a unique edge of  $D_2(\Gamma)$ , as required. ■

### 3 Graphs with no Square as Subgraph

Throughout this section, we assume that  $\Gamma \cong D_2(\Gamma)$  is a graph with no square as subgraph. Further, we assume that  $\Gamma$  is not an odd cycle. A simple observation shows that every triangle in  $D_2(\Gamma)$  comes from an induced claw, an induced hexagon, or an induced edge product  $C_5|C_3$ . Moreover, every hexagon in  $\Gamma$  is induced or it induces a graph isomorphic to  $C_5|C_3$ . To achieve the structure of  $\Gamma$ , first we show that  $\Gamma$  is subcubic. Then, by using induction, we show that  $\Gamma$  is essentially a tree; that is, it does not contain cycles longer than 3. Next, by a counting argument, we show that  $\Gamma$  has exactly three pendants, thus reducing the problem to four cases that we analyze. Recall that a pendant is a vertex of degree one. The following lemma will be used in the sequel.

**Lemma 3.1**  $\Delta(\Gamma) = 3$ .

**Proof** Since neither  $\Gamma$  nor  $D_2(\Gamma)$  has a square and  $N_\Gamma(v)^c$  is a subgraph of  $D_2(\Gamma)$  for all  $v \in V(\Gamma)$ , it follows that  $\Delta(\Gamma) \leq 3$ . Now the fact that  $\Gamma$  is neither a cycle nor a path implies that  $\Delta(\Gamma) \geq 3$ , so that  $\Delta(\Gamma) = 3$ . ■

**Lemma 3.2** *If  $\Gamma$  has a  $C_5|C_3$  subgraph, then  $\Gamma$  is isomorphic to  $C_5|C_3$ .*

**Proof** Suppose on the contrary that  $\Gamma \not\cong C_5|C_3$  and  $S \subset V(\Gamma)$  induces a subgraph of  $\Gamma$  isomorphic to  $C_5|C_3$  (see Figure 3.1). Then there exists a vertex  $v \in V(\Gamma)$  adjacent to some vertex of  $S$ . Clearly,  $v$  is not adjacent to the temples for  $\Delta(\Gamma) = 3$ .

First suppose that  $v$  is adjacent to the forehead. If  $v$  is adjacent to any of the jaws, then we get a square, which is a contradiction. Thus,  $N_S(v) = \{a\}$  or  $\{a, d\}$ , which imply that  $\{v, b, d, f\}$  is a square in  $D_2(\Gamma)$ , which is again a contradiction. Therefore,  $v$  is not adjacent to the forehead. Next assume that  $v$  is adjacent to the chin. Clearly,  $v$  is not adjacent to both  $c$  and  $d$ , say  $c$ , for otherwise we have a square  $\{c, d, e, v\}$ . But then  $\{a, f, e, v\} \subseteq N_{D_2(\Gamma)}(c)$ , that is,  $\Delta(D_2(\Gamma)) > 3$ , which is a contradiction. Finally, assume that  $v$  is adjacent to any of the jaws. Then  $v$  is adjacent to exactly one of the jaws, say  $c$ , for otherwise  $\{v, c, d, e\}$  is a square. Since  $D_2(S \cup \{v\}) \not\cong S \cup \{v\}$ , there exists yet another vertex  $u \in V(\Gamma) \setminus S \cup \{v\}$  adjacent to some vertex of  $S \cup \{v\}$ . If  $u$  is adjacent to  $v$ , then  $N_{D_2(\Gamma)}(c)$  contains  $\{a, e, f, u\}$  as  $u$  cannot be adjacent to  $c$  by a degree argument, which is a contradiction. Thus,  $u$  is not adjacent to  $v$  and by the same arguments as before,  $u$  is adjacent to one of the jaws. Since  $u$  and  $c$  are not adjacent,  $u$  and  $e$  must be adjacent, which implies that  $\{b, f, u, v\} \subseteq N_{D_2(\Gamma)}(d)$ , a contradiction. The proof is complete. ■

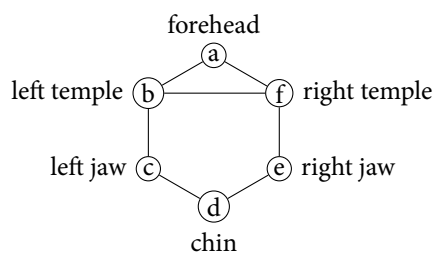


Figure 3.1

**Lemma 3.3** *If  $\Gamma$  has a pentagon, then  $\Gamma$  is isomorphic to  $C_5|C_3$ .*

**Proof** Since  $\Gamma \not\cong C_5$ , there must exist a vertex  $v \in V(\Gamma) \setminus S$  adjacent to some vertex  $u$  of  $S$ , where  $S$  is a pentagon in  $\Gamma$ . Clearly,  $S$  is an induced subgraph of  $\Gamma$ . Let  $a, b$  be the two vertices adjacent to  $u$  in  $S$  and  $c, d$  be the two other vertices. Since  $\Gamma$  has no square, it follows that  $v$  is not adjacent to  $c, d$ . Now it is easy to see that the subgraph induced by  $S \cup \{v\}$  in  $\Gamma$  or  $D_2(\Gamma)$  has a subgraph isomorphic to  $C_5|C_3$  according as  $v$  is adjacent to  $a, b$  or none. Hence, by Lemma 3.2,  $\Gamma \cong C_5|C_3$ , as required. ■

**Lemma 3.4** *If  $\Gamma$  has a hexagon, then  $\Gamma$  is isomorphic to  $C_5|C_3$ .*

**Proof** If  $\Gamma$  has a  $C_5|C_3$  subgraph, then we are done. Thus, we can assume that  $\Gamma$  has no subgraphs isomorphic to  $C_5|C_3$ . Let  $S \subset V(\Gamma)$  denote a hexagon  $\{a, b, c, d, e, f\}$  in  $\Gamma$ . Clearly,  $S$  is an induced subgraph of  $\Gamma$ . Since  $D_2(S) \not\cong S$ , we have a vertex  $u \in V(\Gamma) \setminus S$  adjacent to some vertex  $a$  of  $S$ . Clearly,  $u$  is adjacent to exactly one of  $b$

and  $f$ , say  $b$ , for otherwise either  $\{b, d, f, u\}$  is a square in  $D_2(\Gamma)$ , or  $\{b, a, f, u\}$  is a square in  $\Gamma$ , both of which are impossible. Again, the fact that  $\Gamma$  has no square implies that  $u$  is not adjacent to  $c, d, e, f$ . Moreover,  $u$  is the unique vertex adjacent to both  $a, b$ . Now we have three cases:

*Case 1.* If  $\Gamma$  has a subgraph  $S'$  as drawn in Figure 3.2(c), then  $S'$  is an induced subgraph, and a simple verification shows that  $S'$  is a connected component of  $\Gamma$ , which implies that  $\Gamma = S'$ . Indeed, if a new vertex  $v$  is adjacent to a vertex of  $S'$ , say  $u'$ , then  $\{v, v', d', f'\} \subseteq N_{D_2(\Gamma)}(b')$ , which contradicts Lemma 3.1. But then  $D_2(\Gamma) \not\cong \Gamma$ , which is a contradiction.

*Case 2.* If  $\Gamma$  has a subgraph  $S'$  as drawn in Figure 3.2(b), then since  $S'$  is induced and  $D_2(S') \not\cong S'$ ,  $\Gamma$  has a vertex  $w'$  adjacent to some vertex of  $S'$ . If  $w'$  is adjacent to any of the vertices  $a', b', c', d', u', v'$ , then we get a vertex of degree  $\geq 4$  in  $\Gamma$  or  $D_2(\Gamma)$ , which is impossible. Thus,  $w'$  is adjacent to  $e'$  or  $f'$  and by the previous argument it follows that  $w'$  is adjacent to both  $e'$  and  $f'$ , which is impossible by Case 1.

*Case 3.*  $\Gamma$  has no subgraphs isomorphic to that of Figure 3.2(b). Then  $u$  is the only vertex of  $\Gamma$  adjacent to  $S$  (see Figure 3.2(a)). Since  $D_2(S \cup \{u\}) \not\cong S \cup \{u\}$ , there exists a vertex  $v \in V(\Gamma) \setminus S \cup \{u\}$  adjacent to  $u$ . But then  $D_2(S \cup \{u, v\})$  is an induced subgraph of  $D_2(\Gamma) \cong \Gamma$  isomorphic to the graph in Figure 3.2(d), from which it follows that  $\deg_{D_2(D_2(\Gamma))}(u) \geq 4$ , a contradiction. ■

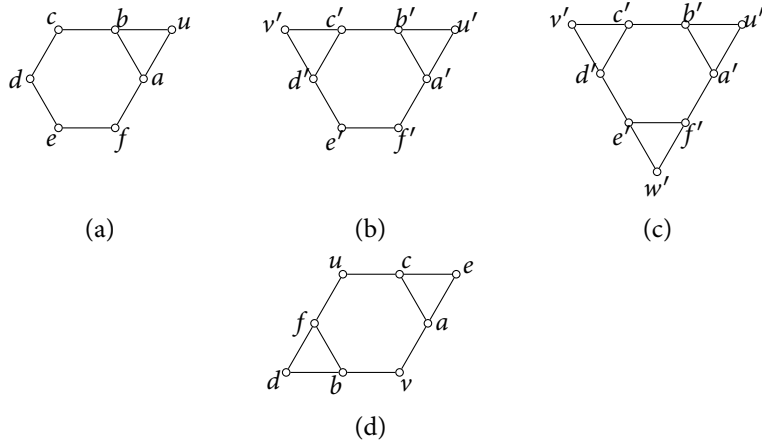


Figure 3.2

**Lemma 3.5** *If  $\Gamma$  is not isomorphic to  $C_5|C_3$ , then  $\Gamma$  has no cycles of length exceeding 3.*

**Proof** By Lemmas 3.3 and 3.4 and the hypothesis,  $\Gamma$  has no cycles of lengths 4, 5, and 6. We proceed by induction to show that  $\Gamma$  has no cycles of length 4 or more. Suppose  $\Gamma$  has no cycles of lengths 4, 5,  $\dots$ ,  $n$  for some  $n \geq 6$ . If  $\Gamma$  has an  $(n + 1)$ -cycle  $C$ , then

$C$  is an induced subgraph of  $\Gamma$ . If  $n + 1$  is even, then clearly  $D_2(\Gamma)$  has two  $(n + 1)/2$ -cycles, which is a contradiction. Thus,  $n + 1$  is odd. Since  $\Gamma$  is not an odd cycle, there exists a vertex  $v \in V(\Gamma)$  adjacent to some vertex  $a \in V(C)$ . Let  $N_C(a) = \{b, c\}$ . If  $v$  is adjacent to some vertex in  $C \setminus \{a, b, c\}$ , then we obtain a cycle of length  $l$  ( $4 \leq l \leq n$ ), which is a contradiction. If  $v$  is not adjacent to  $b, c$ , then  $\Gamma \cong D_2(\Gamma)$  has a subgraph isomorphic to  $D_2(C \cup \{v\})$  that is an  $|C|$ -cycle with two adjacent vertices having a common neighbor. Hence, we can assume that  $v$  is adjacent to either  $b$  or  $c$ , say  $b$ . Since  $\Gamma$  has no square,  $v$  is not adjacent to  $c$ . Let  $N_C(b) = \{a, d\}$ . Then  $c, v, d$  is a path of length two in  $D_2(\Gamma)$ . On the other hand, since  $D_2(C)$  is a subgraph of  $D_2(\Gamma)$ , there is a path of length at most  $n/2$  from  $c$  to  $d$  disjoint from  $\{c, v, d\}$ . Hence,  $D_2(\Gamma)$  has a cycle of length  $l$  such that  $4 \leq l \leq n/2 + 2 \leq n$ , which is a contradiction. The proof is complete. ■

**Lemma 3.6** *Triangles in  $\Gamma$  have disjoint vertices.*

**Proof** If two triangles of  $\Gamma$  have a vertex in common, then they must have an edge in common by Lemma 3.1. But then  $\Gamma$  has a square, which is a contradiction. ■

Now we are ready to prove the main result of this section. To this end, we use the notion of distance between two subgraphs of a graph as the length of the shortest path connecting a vertex of the first subgraph to a vertex of the second subgraph. Also, we call a vertex *pseudo-pendant* if either it is a pendant or it is adjacent to a pendant.

**Theorem 3.7** *Let  $\Gamma$  be a self 2-distance graph with no square. Then either  $\Gamma$  is an odd cycle, or it is isomorphic to one of the following graphs.*

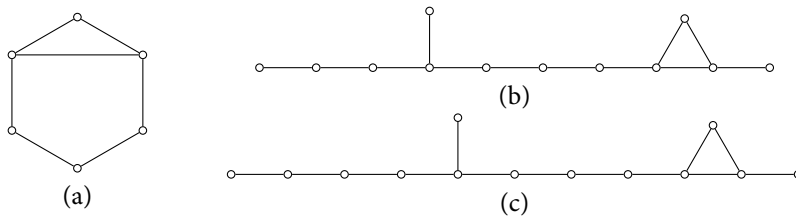


Figure 3.3

**Proof** Suppose  $\Gamma$  is neither an odd cycle nor the edge product  $C_5|C_3$ . Let  $\Gamma'$  be the graph obtained from  $\Gamma$  by contracting all triangles into single vertices. By Lemmas 3.5 and 3.6,  $\Gamma'$  is a tree. Let  $n_i$  be the number of vertices of degree  $i$  in  $\Gamma$  for  $i = 1, 2, 3$ . Clearly,  $|V(\Gamma')| = |V(\Gamma)| - 2\triangleright(\Gamma)$  and  $|E(\Gamma')| = |E(\Gamma)| - 3\triangleright(\Gamma)$ . Since  $\Gamma'$  is a tree, we have  $|E(\Gamma')| = |V(\Gamma')| - 1$ , which implies that  $\triangleright(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ . On the other hand, by Lemma 2.5,  $3\triangleright(\Gamma) = |E(L(\Gamma))| - |E(D_2(\Gamma))| = |E(L(\Gamma))| - |E(\Gamma)|$ .

Now we have

$$\begin{aligned}
 |V(\Gamma)| &= n_1 + n_2 + n_3, \\
 |E(\Gamma)| &= \frac{1}{2} \sum_{v \in V(\Gamma)} \deg_{\Gamma}(v) = \frac{n_1 + 2n_2 + 3n_3}{2}, \\
 |E(L(\Gamma))| &= \sum_{v \in V(\Gamma)} \binom{\deg_{\Gamma}(v)}{2} = n_2 + 3n_3,
 \end{aligned}$$

from which it follows that  $n_1 = 3$ .

If  $\Gamma$  has no triangles, then  $\Gamma$  is a tree with a bipartition  $V(\Gamma) = U \cup V$ , so that  $D_2(\Gamma)$  is disconnected with connected components  $U$  and  $V$ , a contradiction. Hence  $\Gamma$  has some triangles. A triangle in  $\Gamma$  is said to be  $i$ -tailed if it contains  $i$  cubic vertices. Clearly,  $\Gamma$  has no 3-tailed triangle, for otherwise  $D_2(\Gamma)$  must have a hexagon, contradicting Lemma 3.4. We distinguish two cases:

*Case 1.*  $\Gamma$  has no 1-tailed triangle. Then, as there are just three pendants,  $\Gamma$  has only one induced claw along with only one 2-tailed triangle as drawn in Figure 3.4, where  $a, b, d \geq 1, c \geq 0$ . If  $a, b \leq 2$  then  $D_2(\Gamma)$  has a 1-tailed triangle, which is impossible. Thus, we have either  $a \geq 3$  or  $b \geq 3$ . Clearly,  $c \neq 1$  for otherwise  $\deg_{D_2(\Gamma)}(u) = 4$ , which is impossible. A simple verification shows that  $d_{\Gamma}(\text{triangle, claw}) = c$  and

$$d_{D_2(\Gamma)}(\text{triangle, claw}) = \begin{cases} \frac{c+4}{2}, & c \text{ is even,} \\ \frac{c-3}{2}, & c \text{ is odd.} \end{cases}$$

Since  $\Gamma \cong D_2(\Gamma)$ , this implies that  $c = 4$ . If either  $a, b \geq 2$  or  $d \geq 3$ , then we have two claws in  $D_2(\Gamma)$  centered at  $u, o$  or  $u, v$ , respectively, which is impossible. Hence  $d = 1$  or 2, and we can assume that  $a = 1$ . If  $d = 2$  then the claw of  $D_2(\Gamma)$  has no pendants, a contradiction. Thus  $d = 1$  and from  $D_2(\Gamma) \cong \Gamma$  it follows that  $b = 3$  or 4. Therefore,  $\Gamma$  is isomorphic to one of the (self 2-distance) graphs in Figures 3.3(b) and (c).

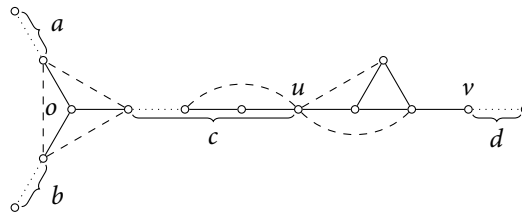


Figure 3.4

*Case 2.*  $\Gamma$  has a 1-tailed triangle. Such a triangle arises from an induced claw two of its non-central vertices are pseudo-pendants in  $\Gamma$ . Since  $\Gamma$  has exactly three pendants, it can be drawn in the plane (see Figure 3.5) with one further triangle  $\Delta$  having an edge in the dotted areas, where  $|A|, |B| \geq 0$ , and  $|C| \geq 1$  denote the number of vertices in the corresponding dotted areas. We note that every triangle in  $D_2(\Gamma)$  arises from an induced claw in  $\Gamma$  and that the mentioned pseudo-pendants are actually pendants, since they correspond to vertices adjacent to  $v'$  in  $D_2(\Gamma)$ . We have three subcases:

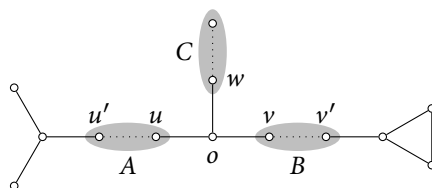


Figure 3.5

(i)  $|A| = 0$ . Clearly,  $|C| = 1$  for otherwise  $\deg_{D_2(\Gamma)}(o) \geq 4$ , which is impossible. Then the graph  $\Gamma$  can be drawn as in Figure 3.6. Note that  $|X| \geq 3$ , for otherwise  $X$  has a vertex of degree  $\geq 4$  in  $D_2(\Gamma)$ , which is impossible. This implies that two triangles in  $D_2(\Gamma)$  are at distance at least five and so we must have  $|Y| \geq 4$ . But then we obtain three induced claws in  $D_2(\Gamma)$  as drawn in Figure 3.6 with dashes, which is a contradiction.

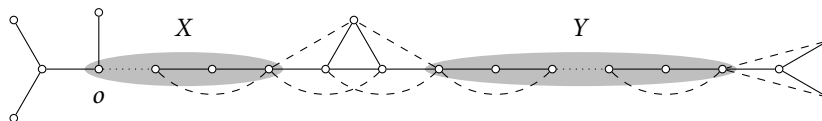


Figure 3.6

(ii)  $|B| = 0$ . As in case (i), we have  $|C| = 1$ . Then  $|X| \geq 2$ , for otherwise  $|X| = 1$ , and consequently two induced claws are connected with two triangles with distance zero while it is not true in  $D_2(\Gamma)$ . Also, if  $|X| = 2$  (resp.  $|Y| = 1$  or  $2$ ), then  $X$  (resp.  $Y$ ) has a vertex of degree 4 in  $D_2(\Gamma)$ , which is impossible. Thus,  $|X|, |Y| \geq 3$ . But then we obtain three induced claws in  $D_2(\Gamma)$  as drawn in Figure 3.7 with dashes, which is a contradiction.

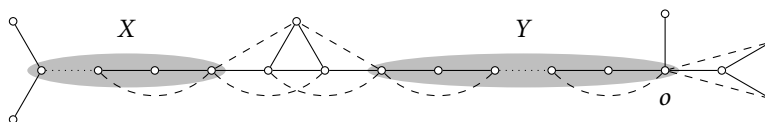


Figure 3.7

(iii)  $|A|, |B| \geq 1$ . Indeed,  $|A|, |B| \geq 2$ , for otherwise either  $u$  or  $v$  (in Figure 3.5) has degree  $\geq 4$  in  $D_2(\Gamma)$ . Clearly,  $\Delta$  is not at distance one from  $\{u, u', v, v'\}$  for otherwise  $u, u', v$  or  $v'$  has degree 4 in  $D_2(\Gamma)$ . If  $|C| \geq 3$ , then  $\{u, v, w\}$  induces a 3-tailed triangle in  $D_2(\Gamma) \cong \Gamma$ , which is impossible. Since  $\deg_{D_2(\Gamma)}(o) \leq 3$ , it follows that  $\Delta$  has no edges with both endpoints in area  $C$ , that is,  $\Delta$  is located in area  $A$  or  $B$ . If  $\Delta$  is disjoint from  $\{u, u', v, v'\}$ , then  $D_2(\Gamma)$  has three induced claws centered at  $v'$  and the two vertices adjacent to  $\Delta$ , which is a contradiction. Thus  $T$  contains one of the vertices  $u, u', v$  or  $v'$ . If  $z$  denotes the vertex adjacent to  $\Delta$  closer to  $o$ , then either



$\deg_{D_2(\Gamma)}(o) = 4$  (when  $|C| = 2$  and  $\Delta$  contains  $u$  or  $v$ ), or  $D_2(\Gamma)$  has a claw with no pendants centered at  $z$ , which is a contradiction. The proof is complete. ■

## 4 Graphs Without Intersecting Triangles

Throughout this section, we assume that  $\Gamma \cong D_2(\Gamma)$  is a graph without intersecting triangles. We further assume that  $\Gamma$  is not an odd cycle. As in Section 3, we proceed by analyzing the existence of special subgraphs in  $\Gamma$ . Indeed, first we show that  $\Gamma$  is subcubic and that  $\Gamma$  has no small cycles other than triangles, which reduces the problem to that of graphs with no squares. Then apply Theorem 3.7. The following lemma is crucial in the proof of our results.

**Lemma 4.1** *We have  $\Delta(\Gamma) = 3$ .*

**Proof** Let  $v$  be a vertex of  $\Gamma$ . Clearly,  $N_\Gamma(v)$  is a union of isolated vertices and at most one edge. Now since  $N_\Gamma(v)^c$  is a subgraph of  $D_2(\Gamma)$ , we must have  $|N_\Gamma(v)| \leq 3$ , as required. ■

**Lemma 4.2** *If  $\Gamma$  has a  $C_5|C_3$  subgraph, then  $\Gamma$  is isomorphic to  $C_5|C_3$ .*

**Proof** Suppose on the contrary that  $\Gamma$  is not isomorphic to  $C_5|C_3$  and consider a subgraph  $S$  of  $\Gamma$  isomorphic to  $C_5|C_3$  as drawn in Figure 3.1. We proceed in two steps.

*Case 1.* The jaws are non-adjacent. Hence there is a vertex in  $\Gamma \setminus S$  adjacent to some vertex of  $S$ . First suppose that the chin  $d$  is adjacent to some new vertex  $g$ . If  $g$  is not adjacent to jaws  $c$  and  $e$ , then we have two triangles  $\{a, c, e\}$  and  $\{c, e, g\}$  with a common edge in  $D_2(\Gamma)$  contradicting the assumption. Hence  $g$  is adjacent to exactly one of the jaws, say  $c$ . But then we have two triangles  $\{a, c, e\}$  and  $\{b, e, g\}$  in  $D_2(\Gamma)$  with a common vertex, which is another contradiction. Therefore  $N_\Gamma(d) = \{c, e\}$ . Next assume that a jaw, say  $c$ , is adjacent to a new vertex  $g$ . By Lemma 4.1,  $b, d, f \notin N_\Gamma(g)$ . If  $g$  is adjacent to  $a$  or  $e$ , then we have two triangles  $\{b, d, g\}$  and  $\{d, f, g\}$  with a common edge in  $D_2(\Gamma)$ , a contradiction. Hence  $N_S(g) = \{c\}$  and the subgraph induced by  $\{a, b, c, d, e, f\}$  in  $D_2(\Gamma)$  is isomorphic to  $C_5|C_3$  with  $g$  adjacent to its chin, which is impossible by the previous discussion. Clearly, the temples are not adjacent to any vertex of  $\Gamma \setminus S$  by the degree argument. Hence the forehead  $a$  must be adjacent to a new vertex  $g$  so that the subgraph induced by  $\{a, b, c, d, e, f\}$  in  $D_2(\Gamma)$  is isomorphic to  $C_5|C_3$  with  $g$  adjacent to its jaws, which contradicts the above arguments.

*Case 2.* The jaws are adjacent. Since the subgraph induced by  $S$  is not a self 2-distance graph, one of the foreheads or chin, say  $a$ , must be adjacent to a new vertex  $g$ . Then we have two triangles  $\{c, d, e\}$  and  $\{c, e, g\}$  with a common edge in  $D_2(D_2(\Gamma))$ , which is a contradiction. ■

**Lemma 4.3** *The graph  $\Gamma$  does not have any hexagons.*

**Proof** Suppose on the contrary that  $\Gamma$  has a hexagon  $S$  as in Figure 4.1 with vertices  $a, b, c, d, e, f$ . Since there is no subgraph isomorphic to  $C_5|C_3$  in  $\Gamma$ , the only possible

chords of  $S$  are  $\{a, d\}$ ,  $\{b, e\}$ , or  $\{c, f\}$ . Since  $S$  is not a self 2-distance graph, we can assume that  $a$  is adjacent to a new vertex  $g$ . Clearly,  $g$  is adjacent to exactly one of  $b$  or  $f$ , say  $b$ , for otherwise either  $\Gamma$  or  $D_2(\Gamma)$  has two triangles with a common edge. Now, by using Lemmas 4.1 and 4.2,  $V(S) \cup \{g\}$  induces a subgraph in  $D_2(\Gamma)$  as drawn with dashes in Figure 4.1. Hence the degree of  $g$  in  $D_2(D_2(\Gamma))$  is at least 4, which is a contradiction. ■

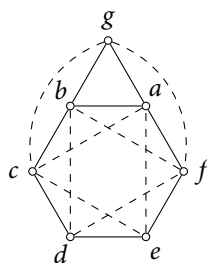


Figure 4.1

**Lemma 4.4** *The graph  $\Gamma$  does not have any pentagons.*

**Proof** Suppose on the contrary that  $\Gamma$  has a pentagon  $S$  with vertices  $a, b, c, d, e$ . We consider two cases:

*Case 1.*  $S$  does not have any chord. Since  $\Gamma$  is not an odd cycle, we can assume that  $a$  is adjacent to a new vertex  $f$ . By Lemma 4.2,  $f$  is not adjacent to  $b$  and  $e$ , from which it follows that  $D_2(\Gamma)$  has a subgraph isomorphic to  $C_5|C_3$ , a contradiction.

*Case 2.*  $S$  has a chord. Clearly,  $S$  has a unique chord, say  $\{b, e\}$ . Since  $S$  is not a self 2-distance graph, it has a vertex adjacent to a new vertex  $f$ . First suppose that  $a$  and  $f$  are adjacent. Since  $D_2(\Gamma)$  does not have a subgraph isomorphic to  $C_5|C_3$ , either  $c, d \in N_\Gamma(f)$  or  $c, d \notin N_\Gamma(f)$ . In both cases  $V(S) \cup \{f\}$  induces a hexagon in  $D_2(\Gamma)$ , contradicting Lemma 4.3 (see Figure 4.2(a)). Therefore,  $f$  is adjacent to  $c$  or  $d$ , say  $c$ . By Lemmas 4.1 and 4.2,  $N_S(f) = \{c\}$ . If there is a vertex  $g$  adjacent to  $d$ , then again  $N_S(g) = \{d\}$ . Now, by using Lemma 4.2, it follows that  $V(S) \cup \{f, g\}$  induces a subgraph in  $D_2(\Gamma)$  as drawn with dashes in Figure 4.2(b). Hence,  $a$  is adjacent to  $b, e, f, g$  in  $D_2(D_2(\Gamma))$  contradicting Lemma 4.1. Therefore,  $d$  is not adjacent to vertices other than  $c$  and  $e$ . This implies that the vertex  $f$  is adjacent to another vertex  $g$  as in Figure 4.2(c). Again, by using Lemma 4.2,  $V(S) \cup \{f, g\}$  induces a subgraph in  $D_2(\Gamma)$  as drawn with dashes in Figure 4.2(c). Hence,  $a$  is adjacent to  $b, e, f, g$  in  $D_2(D_2(\Gamma))$ , which contradicts Lemma 3.1. The proof is complete. ■

**Lemma 4.5** *The graph  $\Gamma$  does not have any heptagons.*

**Proof** Suppose on the contrary that  $\Gamma$  has a heptagon  $S$  with vertices  $a, b, c, d, e, f, g$ . By Lemmas 4.3 and 4.4,  $S$  is an induced subgraph. Since  $\Gamma$  is not an odd cycle, there

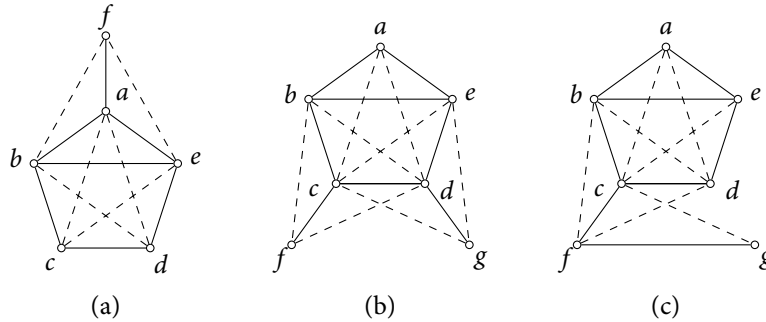


Figure 4.2

exists a new vertex  $h$  adjacent to some vertex of  $S$ . A simple verification shows that  $h$  is adjacent to two consecutive vertices of  $S$  in  $\Gamma$  or  $D_2(\Gamma)$ . Hence, we can assume that  $h$  is adjacent to vertices  $d$  and  $e$  of  $S$  in  $\Gamma$ . By Lemmas 4.3 and 4.4, one gets  $N_S(h) = \{d, e\}$ . For the same reasons, the subgraph of  $D_2(\Gamma)$  induced by  $V(S) \cup \{h\}$  is as drawn in Figure 4.3 with dashed lines. But then  $D_2(D_2(\Gamma))$  has two triangles  $\{a, e, h\}$  and  $\{a, d, h\}$  with a common edge, which is a contradiction. ■

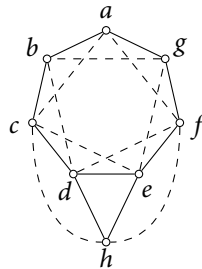


Figure 4.3

**Lemma 4.6** *The graph  $\Gamma$  does not have any octagons.*

**Proof** Suppose on the contrary that  $\Gamma$  has an octagon  $S$  with  $a, b, c, d, e, f, g, h$  as its vertices. By Lemmas 4.3–4.5,  $S$  is an induced subgraph of  $\Gamma$ . Since  $\Gamma$  is not an even cycle, there exists a new vertex  $i$  adjacent to some vertex of  $S$ . Clearly,  $i$  is adjacent to two consecutive vertices of  $S$  for otherwise we have a pentagon in  $D_2(\Gamma)$ , contradicting Lemma 4.4. Hence, we can assume that  $i$  is adjacent to vertices  $d$  and  $e$  of  $S$ . Now, by using Lemmas 4.1 and 4.4, it follows that the subgraph of  $D_2(\Gamma)$  induced by  $V(S) \cup \{i\}$  is as drawn in Figure 4.4 with dashed lines. But then  $i$  is adjacent to vertices  $a, d, e, h$  in  $D_2(D_2(\Gamma))$  contradicting Lemma 4.1. The proof is complete. ■

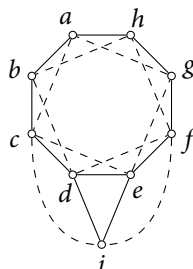


Figure 4.4

**Theorem 4.7** *Let  $\Gamma$  be a self 2-distance graph without intersecting triangles. Then either  $\Gamma$  is an odd cycle or it is isomorphic to one of the graphs in Figure 3.3.*

**Proof** A simple verification shows that squares in  $D_2(\Gamma)$  arise from hexagons or octagons. Hence, by Lemmas 4.3 and 4.6,  $\Gamma$  has no squares, and the result follows by Theorem 3.7. ■

**Corollary 4.8** *There are no cubic self 2-distance graphs.*

**Proof** By Theorem 4.7, there must exist two intersecting triangles. Since the graph is cubic, these triangles must have an edge in common, say  $\{u, v\}$ . Now the fact that  $\Gamma$  is not the complete graph on four vertices implies that  $\Gamma$  has an induced subgraph as in Figure 4.5. Then  $\deg_{D_2(\Gamma)}(u) = 2$ , which is a contradiction. ■

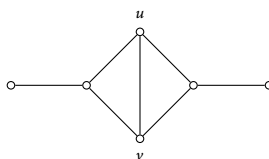


Figure 4.5

## 5 Graphs with no Diamond as Subgraph

In this section, we go further into the study of self 2-distance graphs with a forbidden subgraph, which relies on our earlier results. Indeed, we shall classify all self 2-distance graphs with no diamond as subgraphs. Recall that a diamond is the edge product of two triangles, namely  $C_3|C_3$ . A diamond with vertices  $a, b$  of degree 3 and vertices  $c, d$  of degree two is denoted by  $\mathcal{D}(a, b; c, d)$ . To achieve our goal, first we show that the graphs under investigation are subquartic with a vertex of degree 4, and then analyze the second neighborhood of such a vertex of degree 4.

**Theorem 5.1** *Let  $\Gamma$  be a self 2-distance graph with no diamond as subgraph. Then  $\Gamma$  is either an odd cycle, or it is isomorphic to one of the graphs in Figure 3.3 or Figure 5.1.*

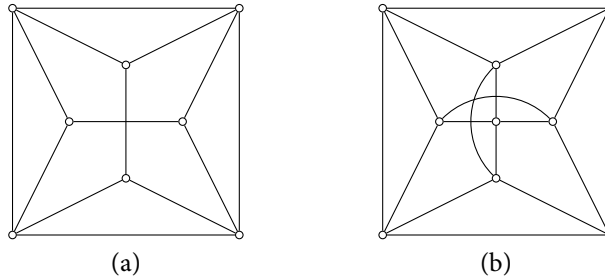


Figure 5.1

**Proof** Suppose on the contrary that  $\Gamma$  is none of the graphs in the theorem. First we observe that  $\Delta(\Gamma) \leq 4$ . Indeed, if  $v \in V(\Gamma)$  is an arbitrary vertex, then by assumption the subgraph induced by  $N_\Gamma(v)$  is a union of disjoint edges and isolated vertices. If  $|N_\Gamma(v)| \geq 5$ , then we can choose five vertices  $a, b, c, d, e \in N_\Gamma(v)$  such that there are no edges among  $a, b, c, d, e$  unless possibly  $\{a, b\}$  or  $\{c, d\}$ . Since  $N_\Gamma(v)^c$  is a subgraph of  $D_2(\Gamma)$ , it follows that  $D_2(\Gamma)$  has the diamond  $\mathcal{D}(a, e; c, d)$ , which is a contradiction. Thus,  $|N_\Gamma(v)| \leq 4$ . If  $\Delta(\Gamma) \leq 3$ , then all triangles in  $\Gamma$  are disjoint so that  $\Gamma$  is either an odd cycle or it is isomorphic to one of the graphs in Figure 3.3 by Theorem 4.7, which contradicts our assumption. Hence we assume that  $\Delta(\Gamma) = 4$ . Clearly, the subgraph induced by the neighbors of every vertex of  $\Gamma$  of degree 4 is a union of two disjoint edges. In what follows, we choose a fixed vertex  $v \in V(\Gamma)$  of degree 4. Then  $N_\Gamma(v)$  is a union of two disjoint edges, say  $\{a, b\}$  and  $\{c, d\}$ , and that  $N_\Gamma(a) \cap N_\Gamma(b) = N_\Gamma(c) \cap N_\Gamma(d) = \{v\}$ . Let  $X = \{a, b\}$ ,  $Y = \{c, d\}$ , and  $M_\Gamma(v)$  be the set of all vertices of  $\Gamma \setminus \{v\}$  adjacent to an element of  $X$  and an element of  $Y$ . Suppose further that  $|M_\Gamma(v)|$  is maximum among all vertices of degree 4. We proceed in some steps:

*Claim 1* If  $e, f \in M_\Gamma(v)$  are distinct, then  $N_{N_\Gamma(v)}(e) \neq N_{N_\Gamma(v)}(f)$ . If it is not the case,  $(N_\Gamma(v) \cup \{e, f\}) \setminus N_{N_\Gamma(v)}(e)$  has a diamond in  $D_2(\Gamma)$ , which is a contradiction.

*Claim 2* If  $e, f \in M_\Gamma(v)$  are distinct, then  $N_\Gamma(e), N_\Gamma(f) \subseteq N_\Gamma(v)$ . If it is not true, we can assume that  $e$  is adjacent to a vertex  $g$  other than  $v, a, b, c, d$ . First assume that  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) = \emptyset$ , say  $N_{N_\Gamma(v)}(e) = \{a, c\}$  and  $N_{N_\Gamma(v)}(f) = \{b, d\}$ .

Suppose  $g \neq f$ . If  $g$  is adjacent to both  $a, c$ , then we have the diamond  $\mathcal{D}(e, g; a, c)$  in  $\Gamma$ , which is a contradiction. Also, in the case where  $g$  is adjacent to none of  $a, c$ , we have the diamond  $\mathcal{D}(a, c; f, g)$  in  $D_2(\Gamma)$ , a contradiction. Hence  $g$  is adjacent to exactly one of  $a$  or  $c$ , say  $a$ . First assume that  $g$  and  $d$  are adjacent. As  $\deg_\Gamma(d) = 4$ , the vertices  $f$  and  $g$  are adjacent too. Clearly,  $e$  and  $f$  are not adjacent; otherwise, we have the diamond  $\mathcal{D}(e, g; a, f)$  in  $\Gamma$ . Then the subgraph  $S$  induced by  $\{v, a, b, c, d, e, f, g\}$  is isomorphic to the graph in Figure 5.1(a), which implies that there exists another vertex  $h$  adjacent to some vertex of  $S$ . Since  $\emptyset \neq N_S(h) \subseteq \{b, c, e, f\}$ , one can

verify that  $N_S(h) = \{b, f\}$ ,  $\{c, e\}$ , or  $\{b, c, e, f\}$ . In all three cases,  $D_2(S')$  is isomorphic to the graph in Figure 5.1(b) in which  $S'$  is the subgraph of  $\Gamma$  induced by  $\{v, a, b, c, d, e, f, g, h\}$ . Since all vertices in  $D_2(S')$  have degree 4, we must have  $V(\Gamma) = V(S')$  so that  $\Gamma \cong D_2(\Gamma) = D_2(S')$ , which contradicts our assumption. Hence,  $g$  is not adjacent to  $d$ , from which it follows that  $g$  and  $f$  are also non-adjacent, for otherwise we get the diamond  $\mathcal{D}(b, g; c, d)$  in  $D_2(\Gamma)$ . Now, replacing  $\Gamma, v, a, b, c, d, e, f, g$  by  $D_2(\Gamma), b, c, g, d, e, a, v, f$ , respectively, we observe that  $g$  and  $f$  are adjacent, which is impossible both in cases where  $d$  and  $g$  are adjacent and cases where they are non-adjacent.

Therefore,  $g = f$ , so we conclude that  $e$  and  $f$  are not adjacent to any vertex other than  $v, a, b, c, d, e, f$ . Since the subgraph induced by  $\{v, a, b, c, d, e, f\}$  is not a self 2-distance graph, one of the vertices  $a, b, c, d$ , say  $a$ , is adjacent to a vertex  $h$  different from  $v, a, b, c, d, e, f$ . But then  $\deg_\Gamma(a) = 4$ , which implies that  $e$  and  $h$  are adjacent, a contradiction. Thus,  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) \neq \emptyset$ . Assume, by Claim 1, that  $a, c \in N_\Gamma(e)$  and  $a, d \in N_\Gamma(f)$ . Then  $\deg_\Gamma(a) = 4$ , which implies that  $e$  and  $f$  are adjacent. If  $g \neq f$ , then  $\deg_\Gamma(e) = 4$  so that  $c$  and  $g$  are adjacent too. Now, replacing  $\Gamma, v, a, b, c, d, e, f, g$  by  $D_2(\Gamma), b, e, d, f, c, v, a, g$ , respectively, we observe that  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) = \emptyset$ , which is impossible, as mentioned before. Hence,  $g = f$ , and we conclude that  $e$  and  $f$  are not adjacent to any vertex other than  $v, a, b, c, d, e, f$ . Accordingly,  $c$  and  $d$  are not adjacent to any vertex other than  $v, a, b, c, d, e, f$ , for otherwise  $c$  or  $d$ , say  $c$ , is adjacent to a new vertex  $h$  different from  $v, a, b, c, d, e, f$ . Then  $\deg_\Gamma(c) = 4$ , which implies that  $e$  and  $h$  are adjacent, a contradiction. Since the subgraph induced by  $\{v, a, b, c, d, e, f\}$  is not a self 2-distance graph,  $b$  is adjacent to a vertex  $h$  different from  $v, a, b, c, d, e, f$ . Now, replacing  $\Gamma, v, a, b, c, d, e, f, h$  by  $D_2(\Gamma), b, e, d, f, c, v, a, h$ , we get  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) = \emptyset$ , which contradicts the above discussions. This completes the proof of Claim 2.

One can easily see that  $|M_\Gamma(v)| \leq 4$ . In the sequel, we discuss the size of  $M_\Gamma(v)$ .

*Case 1.*  $|M_\Gamma(v)| = 4$ . Utilizing Claim 1, we can assume that  $N_\Gamma(a) \cap N_\Gamma(c) = \{v, e\}$ ,  $N_\Gamma(b) \cap N_\Gamma(d) = \{v, f\}$ ,  $N_\Gamma(b) \cap N_\Gamma(c) = \{v, g\}$ , and  $N_\Gamma(a) \cap N_\Gamma(d) = \{v, h\}$  for some distinct vertices  $e, f, g, h$  different from  $v, a, b, c, d$ . As  $\deg_\Gamma(a) = \deg_\Gamma(b) = \deg_\Gamma(c) = \deg_\Gamma(d) = 4$ , the subgraph induced by  $\{e, f, g, h\}$  is the square  $\{e, g, f, h\}$ . Then  $g, h \in N_\Gamma(e) \cap N_\Gamma(f) \setminus N_\Gamma(v)$ , contradicting Claim 2.

*Case 2.*  $|M_\Gamma(v)| = 3$ . Using Claim 1, we can assume that  $N_\Gamma(a) \cap N_\Gamma(c) = \{v, e\}$ ,  $N_\Gamma(b) \cap N_\Gamma(d) = \{v, f\}$ , and  $N_\Gamma(b) \cap N_\Gamma(c) = \{v, g\}$  for some distinct vertices  $e, f, g$  different from  $v, a, b, c, d$ . Since  $\deg_\Gamma(b) = \deg_\Gamma(c) = 4$ ,  $g$  is adjacent to  $e$  and  $f$ . Then  $g \in N_\Gamma(e) \cap N_\Gamma(f) \setminus N_\Gamma(v)$ , which contradicts Claim 2.

*Case 3.*  $|M_\Gamma(v)| = 2$ . Then  $M_\Gamma(v) = \{e, f\}$  for some vertices  $e$  and  $f$ . First assume that  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) = \emptyset$ , say  $N_{N_\Gamma(v)}(e) = \{a, c\}$  and  $N_{N_\Gamma(v)}(f) = \{b, d\}$ . From Claim 2 and the fact that the subgraph of  $\Gamma$  induced by  $\{v, a, b, c, d, e, f\}$  is not self 2-distance, it follows that there exists a new vertex  $g$  adjacent to  $a, b, c$ , or  $d$ , say  $a$ . Then  $\deg_\Gamma(a) = 4$ , which implies that  $g$  and  $e$  are adjacent, contradicting Claim 2. Thus  $N_{N_\Gamma(v)}(e) \cap N_{N_\Gamma(v)}(f) \neq \emptyset$ , say  $N_{N_\Gamma(v)}(e) = \{a, c\}$  and  $N_{N_\Gamma(v)}(f) = \{a, d\}$ . Then  $\deg_\Gamma(a) = 4$  so that  $e$  and  $f$  are adjacent contradicting Claim 2.

*Case 4.*  $|M_\Gamma(v)| = 1$ . Suppose that  $M_\Gamma(v) = \{e\}$  and  $N_{N_\Gamma(v)}(e) = \{a, c\}$ . First we observe that neither  $a$  nor  $c$  is adjacent to a vertex other than  $v, a, b, c, d, e$ , for otherwise we can assume that  $a$  is adjacent to a new vertex  $f$ , from which it follows that  $e$  and  $f$  are adjacent as  $\deg_\Gamma(a) = 4$ . But then  $a, v \in M_{D_2(\Gamma)}(b)$  contradicting the choice of  $v$  as  $D_2(\Gamma) \cong \Gamma$ . Now we discuss on the neighbors of  $b$  and  $d$ .

If two vertices  $f$  and  $g$  other than  $v, a, b, c, d, e$  are adjacent to  $b$  or  $d$ , say  $b$ , then  $\deg_{D_2(\Gamma)}(a) = 4$  and  $b, v \in M_{D_2(\Gamma)}(a)$ , which is again a contradiction. Hence, we can assume that neither  $b$  nor  $d$  is adjacent to two vertices other than  $v, a, b, c, d, e$ . Next assume that  $b$  and  $d$  are adjacent to vertices  $f$  and  $g$  other than  $v, a, b, c, d, e$ , respectively. If  $f$  and  $g$  are adjacent, then  $a, v \in M_{D_2(\Gamma)}(b)$ , contradicting the choice of  $v$ . Hence assume that  $f$  and  $g$  are not adjacent and consequently  $b$  and  $d$  are not adjacent to  $g$  and  $f$  in  $D_2(\Gamma)$ , respectively. Also,  $a$  and  $g$  are not adjacent in  $D_2(\Gamma)$ , for otherwise  $N_{D_2(\Gamma)}(a) = \{c, d, f, g\}$  and consequently  $d$  and  $f$  must be adjacent in  $D_2(\Gamma)$ , which is impossible. Now it is easy to see that  $D_2(D_2(\Gamma))$  has the diamond  $\mathcal{D}(a, b; g, v)$ , which is a contradiction. Hence, we can assume that at most one of  $b$  and  $d$  is adjacent to a new vertex. Suppose  $b$  is such an element adjacent to a vertex  $f$  other than  $v, a, b, c, d, e$ . We show that  $e$  and  $f$  are not adjacent in  $D_2(\Gamma)$ . If it is not the case, there must exist a vertex  $g$  different from  $v, a, b, c, d$  adjacent to  $e$  and  $f$  in  $\Gamma$ . But then  $N_{D_2(\Gamma)}(a) = \{c, d, f, g\}$  so that  $d$  is adjacent to  $f$  in  $D_2(\Gamma)$  as  $c$  and  $g$  are adjacent in  $D_2(\Gamma)$ , giving rise to the diamond  $\mathcal{D}(d, f; a, e)$  in  $D_2(\Gamma)$ , a contradiction. Now either we have the diamond  $\mathcal{D}(c, f; d, e)$  in  $D_2(D_2(\Gamma))$  when  $c$  and  $f$  are not adjacent in  $D_2(\Gamma)$ , or  $e, f \in M_{D_2(D_2(\Gamma))}(v)$  when  $c$  and  $f$  are adjacent in  $D_2(\Gamma)$ , which is a contradiction.

Therefore neither  $b$  nor  $d$  is adjacent to a vertex other than  $v, a, b, c, d, e$ . Then the second neighborhood of  $v$  is consist of  $e$  only. Since the subgraph induced by  $\{v, a, b, c, d, e\}$  is not a self 2-distance graph, the vertex  $e$  must be adjacent to some vertices other than  $v, a, b, c, d$ . If  $e$  is adjacent to two vertices  $f$  and  $g$  different from  $v, a, b, c, d$ , then  $D_2(\Gamma)$  has the diamond  $\mathcal{D}(a, c; f, g)$ , which is a contradiction. Hence  $N_\Gamma(e) = \{a, c, f\}$  for some vertex  $f$ . As  $\deg_{D_2(\Gamma)}(e) \leq 4$  and the subgraph of  $\Gamma$  induced by  $\{v, a, b, c, d, e, f\}$  is not self 2-distance, there must exist another vertex  $g$  such that  $N_\Gamma(f) = \{e, g\}$ . Then  $N_{D_2(\Gamma)}(e) = \{b, d, v, g\}$  so that  $v$  and  $g$  must be adjacent in  $D_2(\Gamma)$ , which is impossible as  $d_\Gamma(v, g) = 4$ .

*Case 5.*  $M_\Gamma(v) = \emptyset$ . First suppose that three vertices among  $a, b, c, d$  are adjacent to new vertices, say  $a, b, c$ , are adjacent to distinct vertices  $e, f, g$  different from  $v, a, b, c, d$ , respectively. If  $g$  is adjacent to  $e$  or  $f$ , say  $e$ , then  $N_{D_2(\Gamma)}(a) = \{c, d, f, g\}$  and hence  $c$  and  $f$  must be adjacent in  $D_2(\Gamma)$ ; that is,  $c$  and  $f$  are connected in  $\Gamma$  via a path of length 2. Clearly,  $f$  and  $g$  are not adjacent for otherwise we have the diamond  $\mathcal{D}(d, g; a, b)$  in  $D_2(\Gamma)$ , a contradiction. Hence, there exists a vertex  $h$  other than  $v, a, b, c, d, e, f, g$ , which is adjacent to both  $c$  and  $f$ . Then  $N_\Gamma(c) = \{v, d, g, h\}$  so that  $g$  and  $h$  must be adjacent. But then  $f$  and  $g$  are adjacent in  $D_2(\Gamma)$ , which results in the diamond  $\mathcal{D}(a, f; c, g)$  in  $D_2(\Gamma)$ , a contradiction. Thus, we deduce that there is no edges from  $(N_\Gamma(a) \cup N_\Gamma(b)) \setminus \{v, a, b\}$  to  $(N_\Gamma(c) \cup N_\Gamma(d)) \setminus \{v, c, d\}$ , from which we obtain the diamond  $\mathcal{D}(a, b; v, g)$  in  $D_2(D_2(\Gamma))$ , a contradiction. Next assume that exactly two vertices among  $a, b, c, d$  are adjacent to vertices other than  $v, a, b, c, d$ . We have two cases up to symmetry:

(i)  $a$  and  $b$  are adjacent to two distinct other vertices  $e$  and  $f$ , respectively. If  $e$  or  $f$ , say  $e$ , is adjacent to a vertex  $g$  in the third neighborhood of  $v$ , then  $N_{D_2(\Gamma)}(a) = \{c, d, f, g\}$  where  $\{c, d, f\}$  induces an independent set in  $D_2(\Gamma)$ , a contradiction. On the other hand, if  $a$  or  $b$ , say  $a$ , is adjacent to another vertex  $g$ , then  $N_{D_2(\Gamma)}(b) = \{c, d, e, g\}$  with  $\{c, d, e\}$  an independent set in  $D_2(\Gamma)$ , which is again a contradiction.

(ii)  $a$  and  $c$  are adjacent to two distinct other vertices  $e$  and  $f$ , respectively. If  $e$  and  $f$  are adjacent, then  $e, f \in M_{D_2(D_2(\Gamma))}(v)$ , which contradicts the choice of  $v$  as  $D_2(D_2(\Gamma)) \cong \Gamma$ . Hence we can assume that there is no edges from  $N_\Gamma(a) \setminus \{v, b\}$  to  $N_\Gamma(c) \setminus \{v, d\}$ . If  $a$  or  $c$ , say  $a$ , is adjacent to another vertex  $g$ , then  $e$  and  $g$  are also adjacent and hence  $N_{D_2(\Gamma)}(b) = \{c, d, e, g\}$  with  $\{c, d, e\}$  an independent set in  $D_2(\Gamma)$ , which is a contradiction. Thus,  $N_\Gamma(a) = \{v, b, e\}$  and  $N_\Gamma(c) = \{v, d, f\}$ . Since the subgraph induced by  $\{v, a, b, c, d, e, f\}$  is not a self 2-distance graph, we can assume that  $e$  is adjacent to another vertex  $g$ . Since  $c, a, d, f, v$  and  $e, b, d, a, g$  are induced paths in  $D_2(\Gamma)$  we observe that  $N_{D_2(D_2(\Gamma))}(d) \supseteq \{c, e, g, v\}$ . On the other hand,  $\deg_{D_2(D_2(\Gamma))}(d) \leq 4$  and  $g$  is any neighbor of  $e$  other than  $a$ . So, we must have  $N_\Gamma(e) = \{a, g\}$  and  $N_{D_2(D_2(\Gamma))}(d) = \{c, e, g, v\}$ . In particular, the subgraph induced by  $\{c, e, g, v\}$  in  $D_2(D_2(\Gamma))$  is a union of two disjoint edges. As  $c$  and  $e$  are adjacent in  $D_2(D_2(\Gamma))$ ,  $v$  and  $g$  must be adjacent in  $D_2(D_2(\Gamma))$  too, which is possible only if  $f$  and  $g$  are adjacent in  $D_2(\Gamma)$ . This implies  $f$  and  $g$  are non-adjacent in  $\Gamma$ , and we obtain the diamond  $\mathcal{D}(d, g; c, v)$  in  $D_2(D_2(\Gamma))$ , which is a contradiction. Finally, assume that only one of the vertices  $a, b, c, d$  is adjacent to a vertex other than  $v, a, b, c, d$ , say  $a$  is adjacent to a new vertex  $e$ . If  $a$  is adjacent to another vertex  $f$ , then, as before,  $N_{D_2(\Gamma)}(b) = \{c, d, e, f\}$  with  $\{c, d, e\}$  an independent set in  $D_2(\Gamma)$ , which is a contradiction. Hence,  $N_\Gamma(a) = \{v, b, e\}$  so that  $e$  is adjacent to a vertex  $f$  different from  $v, a, b, c, d$ , from which we obtain the diamond  $\mathcal{D}(c, d; e, f)$  in  $D_2(D_2(\Gamma))$ , which is a contradiction. The proof is complete. ■

## 6 Open Problems

We devote the last section of this paper to some open problems arising in our study of self 2-distance graphs. Two triangles with a common vertex but no common edges is called a *butterfly*.

**Conjecture 1** *The number of finite non-cyclic self 2-distance graphs with no subgraphs isomorphic to a butterfly is finite.*

**Conjecture 2** *The number of finite non-cyclic self 2-distance graphs with no induced subgraphs isomorphic to a square, a diamond, a complete graph with four vertices, or a butterfly is finite.*

A graph  $\Gamma$  with  $v$  vertices is *strongly regular* of degree  $k$  if there are integers  $\lambda$  and  $\mu$  such that every two adjacent vertices have  $\lambda$  common neighbors and every two non-adjacent vertices have  $\mu$  common neighbors. The numbers  $v, k, \lambda, \mu$  are the parameters of the corresponding graph.



**Theorem 6.1** *Every strongly regular self 2-distance graph is a self-complementary graph and has parameters  $(4t + 1, 2t, t - 1, t)$ , where the number of vertices is a sum of two squares.*

**Proof** The result follows from [8] and the fact that every strongly regular graph has diameter at most two. ■

We have shown, in Corollary 4.8, that there is no self 2-distance cubic graph. Indeed, we believe that a more general case also holds for regular graphs with odd degrees, while the same result does not hold for regular graphs of even degrees by the above theorem.

**Conjecture 3** *There are no regular self 2-distance graphs of odd degree.*

We note that if the above conjecture is true then for any finite group  $G$  and any inverse closed subset  $S$  of  $G \setminus \{1\}$  of odd size, the sets  $S^2 \setminus (S \cup \{1\})$  and  $S$  belong to different orbits of the poset of subsets of  $G$  under the action of automorphism group of  $G$ .

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