EXTENDED CESÀRO OPERATOR BETWEEN SOME HOLOMORPHIC FUNCTION SPACES

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We characterize the boundedness and compactness of the extended Cesàro operator \( T_g \) from \( H^\infty \) to the mixed norm space and Bloch-type space (or little Bloch-type space), where \( g \) is a given holomorphic function in the unit ball of \( \mathbb{C}^n \) and \( T_g \) is defined by

\[
T_g f(z) = \int_0^1 f(tz) g(tz) (dt/t).
\]

1. INTRODUCTION

Let \( B = \{ z \in \mathbb{C}^n; |z| < 1 \} \) be the unit ball of \( \mathbb{C}^n \), and let \( H(B) \) be the family of all holomorphic functions on \( B \). We denote by \( H^\infty \) the space of all bounded functions in \( H(B) \). \( H^\infty \) is a Banach space under the norm

\[
\| f \|_\infty = \sup \{ |f(z)|; z \in B \}.
\]

A positive continuous function \( \varphi \) on \( [0, 1) \) is called normal if there are three constants \( 0 \leq \delta < 1 \) and \( 0 < a < b \) such that

\[
(P_1) \quad \frac{\varphi(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1^-} \frac{\varphi(r)}{(1-r)^a} = 0;
\]

\[
(P_2) \quad \frac{\varphi(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1^-} \frac{\varphi(r)}{(1-r)^b} = \infty.
\]

We extend it to \( B \) by \( \varphi(z) = \varphi(|z|) \). For \( f \in H(B) \) we set

\[
\| f \|_{p,q,\varphi} = \left\{ \int_0^1 M_q^p(f,r) \frac{\varphi(r)}{1-r} dr \right\}^{1/p}, \quad 0 < p < \infty,
\]

and

\[
\| f \|_{\infty,q,\varphi} = \sup_{0<r<1} M_q(f,r) \varphi(r).
\]
Here

\[ M_q(f, r) = \left\{ \int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}, \quad 0 < q < \infty; \]

\[ M_\infty(f, r) = \sup_{\zeta \in \partial B} |f(r\zeta)|. \]

The mixed norm space \( H_{p,q}(\varphi), 0 < p, q \leq \infty, \) is the space of all functions \( f \in H(B) \) for which \( \|f\|_{p,q,\varphi} < \infty. \) When \( 0 < p = q < \infty, \) \( H_{p,q}(\varphi) \) is just the weighted Bergman space

\[ A_p^p(\varphi) = \left\{ f \in H(B) : \|f\|_{A_p^p} = \left\{ \int_B |f(z)|^p \varphi^p(z) \frac{d\nu(z)}{1 - |z|^2} \right\}^{1/p} < \infty \right\}. \]

A function \( f \in H(B) \) is said to belong to the Bloch-type space \( B_\varphi \) if

\[ \|f\|_{B_\varphi} = \sup_{z \in B} \varphi(z) |\nabla f(z)| < \infty; \]

and it is said to belong to the little Bloch-type space \( B_{\varphi,0} \) if

\[ \lim_{|z| \to 1} \varphi(z) |\nabla f(z)| = 0. \]

Here

\[ \nabla f(z) = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right) \]

is the complex gradient of \( f. \) It is easy to check that both \( B_\varphi \) and \( B_{\varphi,0} \) are Banach spaces under the norm \( \|f\|_\varphi = |f(0)| + \|f\|_{B_\varphi}, \) and \( B_{\varphi,0} \) is a closed subspace of \( B_\varphi. \) When \( \varphi(r) = 1 - r^2 \) and \( \varphi(r) = (1 - r^2)^{1-\alpha} \) with \( \alpha \in (0, 1), \) two typical normal weights, the induced spaces \( B_\varphi \) are the Bloch space and Lipschitz type space, respectively.

Let \( D \) denote the open unit disc in the complex plane \( \mathbb{C}. \) For a holomorphic function \( f(z) \) on \( D \) with Taylor expansion \( f(z) = \sum_{j=0}^{\infty} a_j z^j, \) the Cesàro operator acting on \( f \) is

\[ C[f](z) = \sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j. \]

The behaviour of the operator \( C[\cdot] \) have been studied extensively on various spaces of holomorphic functions (see [5, 6, 7, 8, 11, 12]). A little calculation shows

\[ C[f](z) = \frac{1}{z} \int_{0}^{z} f(t) \left( \log(1/1 - t) \right)' dt. \]

Hence, on most holomorphic function spaces, \( C[\cdot] \) is bounded if and only if the integral operator

\[ f \mapsto \int_{0}^{z} f(t) \left( \log(1/1 - t) \right)' dt \]
is bounded. From this point of view it is natural to consider the extended Cesàro operator $T_g$ on $H(D)$ with holomorphic symbol $g$,

$$T_g f(z) = \int_0^z f(t) g'(t) dt.$$  

The boundedness and compactness of this operator on Hardy spaces, Bergman spaces, Bloch-type spaces and Lipschitz spaces have been studied in [1, 2, 10].

For $f \in H(B)$, the radial derivative of $f$ is

$$\Re f(z) = \sum_{j=1}^n \frac{\partial f(z)}{\partial z_j}.$$  

Given $g \in H(B)$, the operator $T_g$ on $H(B)$ is defined by

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(B), \quad z \in B.$$  

It is trivial that (1.2) is just (1.1) when $n = 1$. In the unit ball, Hu [3] got the characterisation on $g$ for which the induced extended Cesàro operator is bounded or compact on the Bergman space $L^p_{\omega_{\alpha}}$, Zhang [13] studied the same problems between $B(1 - r^2)^p$ and $B(1 - r)^p$, for $0 < p, q < \infty$. And also, Hu discussed the boundedness and compactness of $T_g$ on the mixed norm space $H^{p,q}_{\varphi}(\varphi)$, where $0 < p, q \leq \infty$ (see [4]). The purpose of this work is to obtain the sufficient and necessary conditions on $g \in H(B)$, such that the operator $T_g : H^\infty \to H^{p,q}_{\varphi}(\varphi)$ (respectively, $H^\infty \to B_{\varphi}$, $H^\infty \to B_{\varphi,0}$) is bounded or compact.

In what follows, $C$ will stand for positive constants whose value may change from line to line but not depend on the functions in $H(B)$. The expression $A \simeq B$ means $C^{-1}A \leq B \leq CA$.

2. SOME PRELIMINARY RESULTS

**Lemma 2.1.** ([4]) Let $0 < p, q \leq \infty$ and $\varphi$ be normal. Then for any $f \in H(B)$,

$$\|f\|_{p,q,\varphi} \simeq |f(0)| + \left\{ \int_0^1 M^p_q(\Re f, \varphi) (1 - r^2)^p \frac{\varphi'(r)}{1 - r} dr \right\}^{1/p}.$$  

**Lemma 2.2.** ([9]) Let $\varphi$ be normal and $f \in H(B)$. Then

(A) $f \in B_{\varphi}$ if and only if $\sup_{z \in B} \varphi(z) |\Re f(z)| < \infty$. Moreover,

$$\|f\|_{\varphi} \simeq |f(0)| + \sup_{z \in B} \varphi(z) |\Re f(z)|.$$  

(B) $f \in B_{\varphi,0}$ if and only if $\lim_{|z| \to 1} \varphi(z) |\Re f(z)| = 0$.
**Lemma 2.3.** Let \( \varphi \) be normal, \( 0 < p, q \leq \infty \) and \( g \in H(B) \). Then \( T_g : H^\infty \to H_{p,q}(\varphi) \) (or \( H^\infty \to B_\varphi \)) is compact if and only if for any bounded sequence \( \{f_j\} \subseteq H^\infty \) which converges to 0 uniformly on any compact subset of \( B \), we have \( \lim_{j \to \infty} \|T_g f_j\|_{p,q,\varphi} = 0 \) (or \( \lim_{j \to \infty} \|T_g f_j\|_\varphi = 0 \)).

**Proof.** It can be proved by Montel’s Theorem and the definition of compact operator. The details are omitted here. \( \square \)

3. **Main results**

**Theorem 3.1.** Let \( \varphi \) be normal, \( 0 < p < \infty, 0 < q \leq \infty \) and \( g \in H(B) \). Then the following statements are equivalent:

(A) \( T_g : H^\infty \to H_{p,q}(\varphi) \) is bounded;
(B) \( T_g : H^\infty \to H_{p,q}(\varphi) \) is compact;
(C) \( g \in H_{p,q}(\varphi) \).

In this case, \( \|T_g\| \simeq \|g - g(0)\|_{p,q,\varphi} \).

**Proof:** The implication (B) \( \Rightarrow \) (A) is trivial.

(A) \( \Rightarrow \) (C). Suppose \( T_g : H^\infty \to H_{p,q}(\varphi) \) is bounded, by the fact that \( g(z) = g(0) + T_g(1)(z) \) we know \( g \in H_{p,q}(\varphi) \). Moreover,

\[
(T_g f)(z) = f(z)g(z) = f(z)g(0) + T_g f(z).
\]

Let \( g \in H_{p,q}(\varphi) \), \( \{f_j\} \subseteq H^\infty \) satisfying \( \|f_j\|_{\infty} \leq 1 \). By Montel’s Theorem, there exists some subsequence of \( \{f_j\} \) converging to \( f \) uniformly on any compact subset of \( B \). Without loss of generality, we suppose the subsequence is \( \{f_j\} \) itself. Then \( f \in H(B) \) and \( \|f\|_{\infty} \leq 1 \). Hence

\[
M_g^p((f_j - f)g, r) \leq 2^p M_g^p(g, r).
\]

By \( g \in H_{p,q}(\varphi) \), Lemma 2.1 and the dominated convergence theorem we obtain

\[
\int_0^1 M_g^p((f_j - f)g, r)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} \, dr \to 0 \quad (j \to \infty).
\]

Lemma 2.1 implies, as \( j \to \infty \),

\[
\|T_g f_j - T_g f\|_{p,q,\varphi} \leq C \int_0^1 M_g^p((f_j - f)g, r)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} \, dr
\]

\[
= C \int_0^1 M_g^p(f_j - f)(g, r)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} \, dr \to 0.
\]
Therefore, $T_g : H^\infty \to H_{p,q}(\varphi)$ is compact.

Furthermore, for any $f \in H^\infty$, Lemma 2.1 yields

$$
\|T_g f\|_{p,q,\varphi} \leq C \int_0^1 M_q^p(\Re g, r) \sup \{ |f(z)|^p ; |z| = r \} (1 - r^2)^{p \varphi^p(r)} (1 - r) \, dr
\leq C \| g - g(0) \|_{p,q,\varphi} \| f \|_\infty.
$$

This, together with (3.1), means $\|T_g\| \approx \| g - g(0) \|_{p,q,\varphi}$. The proof is completed. \( \diamond \)

Remark. When $p = \infty$, the implication (C) $\Rightarrow$ (B) does not hold for any $0 < q \leq \infty$ in general. For example, we let $n = 1$, and choose some $g$ and $\varphi$ satisfying (C) but $T_g : H^\infty \to H_{p,q}(\varphi)$ is not compact. In fact, set $g(z) = (z/(1 - z)^{1+1/p})$, where $z \in D$ and $0 < q \leq \infty$, $\varphi(r) = 1 - r$. Then for $0 < q < \infty$,

$$
M_q(g, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{r^q d\theta}{|1 - re^{i\theta}|^{q+1}} \right\}^{1/q} \sim \frac{1}{1 - r},
$$

as $r \to 1^-$. For $q = \infty$,

$$
\sup_{0 \leq \theta < 2\pi} \frac{r}{|1 - re^{i\theta}|} \sim \frac{1}{1 - r} \text{ as } r \to 1^-.
$$

Hence, for $0 < q \leq \infty$ and $1/2 \leq r < 1$,

$$
M_q(g, r)\varphi(r) \sim \frac{1}{1 - r} \varphi(r) = 1.
$$

Write $f_j(z) = z^j$, $z \in D$. Then $\|f_j\|_\infty \leq 1$ and $\{f_j\}$ converges to 0 uniformly on any compact subset of $D$. However, for each $j$, Lemma 2.1 and $g(0) = 0$ yield

$$
\|T_g f_j\|_{\infty, q, \varphi} \approx \sup_{0 < r < 1} M_q(\Re T_g(f_j), r)(1 - r^2)\varphi(r)
\geq C \sup_{(1/2) < r < 1} r^j M_q(g, r)(1 - r^2)\varphi(r)
\geq C \lim_{r \to 1^-} r^j = C,
$$

where the constant $C$ is independent of $j$.

**Theorem 3.2.** Let $\varphi$ be normal and $g \in H(B)$. Then the following statements are equivalent:

- (A) $T_g(H^\infty) \subseteq B_{\varphi,0}$;
- (B) $T_g : H^\infty \to B_{\varphi,0}$ is bounded;
- (C) $T_g : H^\infty \to B_{\varphi}$ is compact;
- (D) $T_g : H^\infty \to B_{\varphi,0}$ is compact;
In this case, \( \|T_g\| \approx \sup_{z \in B} |Rg(z)|. \)

**Proof:** The implications (B) \( \Rightarrow \) (A) and (D) \( \Rightarrow \) (C) are obvious.

(B) \( \Rightarrow \) (E). Suppose \( T_g : H^\infty \to B_{\varphi,0} \) is bounded, then \( g = g(0) + T_g(1) \in B_{\varphi,0} \).

Furthermore,

\[
\sup_{z \in B} \varphi(z)|Rg(z)| \approx \|g - g(0)\|_{\varphi} = \|T_g(1)\|_{\varphi} \leq C\|T_g\|.
\]

(E) \( \Rightarrow \) (B). Let \( g \in B_{\varphi,0} \), then for any \( f \in H^\infty \), \( T_g f \in B_{\varphi,0} \). Moreover,

\[
\|T_g f\|_{\varphi} \approx \sup_{z \in B} \varphi(z)|Rg(z)|\|f(z)\| \leq \|f\|_{\infty}\sup_{z \in B} \varphi(z)|Rg(z)| \leq C\|f\|_{\infty}.
\]

This, together with (3.2), shows \( \|T_g\| \approx \sup_{z \in B} \varphi(z)|Rg(z)|. \)

(A) \( \Rightarrow \) (B). Suppose \( \{f_j\} \subseteq H^\infty \), \( f \in H^\infty \) and \( h \in B_{\varphi,0} \) satisfying \( \lim_{j \to \infty} \|f_j - f\|_{\infty} = 0 \) and \( \lim_{j \to \infty} \|T_g f_j - h\|_{\varphi} = 0 \). Then

\[
f_j(z) \to f(z) \quad (j \to \infty), \quad z \in B.
\]

And

\[
|T_g f_j(0) - h(0)| + \sup_{z \in B} \varphi(z)|Rg(z)f_j(z) - Rh(z)| \to 0 \quad \text{as} \quad j \to \infty.
\]

So \( h(0) = 0 \) and for every \( z \in B \)

\[
f_j(z)Rg(z) \to Rh(z) \quad (j \to \infty).
\]

By (3.3), we have

\[
\lim_{j \to \infty} f_j(z)Rg(z) = f(z)Rg(z), \quad z \in B.
\]

Thus, (3.4) and (3.5) imply \( f(z)Rg(z) = Rh(z) \). Therefore,

\[
h(z) = \int_0^1 Rh(tz)\frac{dt}{t} = \int_0^1 f(tz)Rg(tz)\frac{dt}{t} = (T_g f)(z).
\]

Consequently, \( T_g : H^\infty \to B_{\varphi,0} \) is a closed operator. By the closed graph theorem, \( T_g : H^\infty \to B_{\varphi,0} \) is bounded.

(C) \( \Rightarrow \) (E). Suppose \( g \notin B_{\varphi,0} \). Then there would be some \( \varepsilon_0 > 0 \) and some sequence \( \{z_j\} \subseteq B \) satisfying \( \lim_{j \to \infty} |z_j| = 1 \), but for each \( j \), \( \varphi(z_j)|Rg(z_j)| > \varepsilon_0 \). Set

\[
f_j(z) = \frac{1 - |z|^2}{1 - \langle z, z_j \rangle}, \quad z \in B.
\]
It is easy to check that \( \{f_j\} \) is a bounded sequence in \( H^\infty \) and \( f_j \to 0 \) uniformly on any compact subset of \( B \) as \( j \to \infty \). Since \( T_g : H^\infty \to B_\varphi \) is compact, by Lemma 2.3,

\[
(3.6) \quad \|T_g f_j\|_\varphi \to 0 \quad (j \to \infty).
\]

On the other hand,

\[
\|T_g f_j\|_\varphi \simeq |T_g f_j(0)| + \sup_{z \in B} \varphi(z) |\Re g(z)| |f_j(z)| \\
\geq \varphi(z^j) |\Re g(z^j)| |f_j(z^j)| \\
\geq \varphi(z^j) |\Re g(z^j)| \\
\geq \varepsilon_0.
\]

This is a contradiction to (3.6).

\( (E) \Rightarrow (D) \). Suppose \( g \in B_{\varphi,0} \), then for any \( f \in H^\infty \), \( T_g f \in B_{\varphi,0} \). And also, for every \( \varepsilon > 0 \), there exists some \( r > 0 \) such that

\[
(3.7) \quad \varphi(z) |\Re g(z)| < \varepsilon \quad \text{whenever } |z| > r.
\]

Let \( \{f_j\} \) be any bounded sequence in \( H^\infty \), say \( \|f_j\|_\infty \leq 1 \) and \( f_j \to 0 \) uniformly on any compact subset of \( B \) as \( j \to \infty \). Then for the above \( \varepsilon \), there is a positive integer \( J \) such that for \( |z| \leq r \) and \( j > J \),

\[
(3.8) \quad |f_j(z)| < \frac{\varepsilon}{\|g\|_\varphi + 1}.
\]

Thus, combining (3.7) and (3.8), we have

\[
\|T_g f_j\|_\varphi \simeq \sup_{z \in B} \varphi(z) |\Re g(z)| |f_j(z)| < \varepsilon \quad \text{if } j > J.
\]

By Lemma 2.3, \( T_g : H^\infty \to B_{\varphi,0} \) is compact. The proof is completed. \( \square \)

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