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Some product varieties of groups

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We consider varieties $\underline{\underline{V}} = \underline{\underline{A}}_p \underline{\underline{A}}_m \underline{\underline{A}}_p$ with m prime to p. We show that the subvariety lattice of $\underline{\underline{V}}$ is distributive and has descending chain condition and that $\underline{\underline{A}}_p \underline{\underline{A}}_p$ is its only just non-Cross subvariety. When m is prime we determine the join-irreducible subvarieties of $\underline{\underline{V}}$. The method involves fairly detailed description of the structure of non-nilpotent critical groups in $\underline{\underline{V}}$.

1. Introduction

The principal motivation behind many investigations in the theory of varieties of groups since its inception seems to have been a desire to decide whether or not every variety requires only finitely many laws to define it; and a large number of varieties do have this property (which is usually known as 'the finite basis property'). The papers [*1*], [*3*], [*6*], [*11*], [*17*], [*21*], for example, all contain finite basis theorems. There have been conjectures that every variety has the finite basis property and, more cautiously, that every soluble-of-finite-exponent variety does. Recently, however, two (unpublished) counter-examples to this have been produced: the first, by A.Yu. Ol'shanskii, is soluble of length 5 and exponent 120 and the second, by M.R. Vaughan-Lee, is soluble of length 4 and exponent 16. One of the results proved here (Theorem 5.1) goes a small way towards closing the gap between these examples and known finitely based varieties of smaller soluble length.

Even before these examples of Ol'shanskii and Vaughan-Lee were known the range of questions considered, for locally finite varieties at any

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rate, had widened considerably as people found that, from methods developed to prove finite basis theorems, much more information could be obtained; see, for example, [1], [2], [4, 5], [7], [15], [16]. The test questions, on which one can determine the efficacy of one's methods for dealing with a given variety $\underline{\underline{V}}$ then, include those following.

a) *Does* $\underline{\underline{V}}$ *, and all its subvarieties, have the finite basis property?*

b) *Is the lattice* $\Lambda(\underline{\underline{V}})$ *of subvarieties of* $\underline{\underline{V}}$ *distributive?*

If the answer to (a) is 'yes' for $\underline{\underline{V}}$ then every subvariety of $\underline{\underline{V}}$ can be written as a finite join of (finitely) join-irreducible subvarieties.

c) *What are the join-irreducible subvarieties of* $\underline{\underline{V}}$ *?*

If $\underline{\underline{V}}$ is not a Cross variety it has subvarieties which are just non-Cross (Kovács and Newman [16]).

d) *What are the just non-Cross subvarieties of* $\underline{\underline{V}}$ *?*

This list is far from exhaustive, of course - we have not, for example, mentioned Graham Higman's interesting question about the orders of the free groups of $\underline{\underline{V}}$, (§2 in [12]) - but it is with these questions in mind that the present paper has been written. The varieties $\underline{\underline{V}}$ with which we will be concerned are $\underline{\underline{A}}_p \underline{\underline{A}}_{\alpha} \underline{\underline{A}}_{m} \underline{\underline{A}}_p$ where p is a prime not dividing m ; we answer (a), (b) affirmatively and provide answers to (c), (d). The reader is referred to Hanna Neumann [18] for definitions and terminology about varieties of groups and to Curtis and Reiner [8] for representation theory.

The technique employed involves fairly detailed description of the structure of non-nilpotent critical groups in $\underline{\underline{V}}$, and may be regarded as a natural development of the methods of Chapter 3 in [5] (see also [4]): in particular the concepts of bigroup and variety of bigroups used there will be needed here. The structure theorems are proved in §4, while other preliminary results which will be needed in §5 are introduced in §§2, 3; §2 deals with representations of groups in $\underline{\underline{A}}_m \underline{\underline{A}}_p$ over fields of characteristic p and §3 with enough representation theory over the ring of integers modulo p^{α} for our present purposes.

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A convention used needs comment. If A is an abelian normal subgroup of a group G we shall often regard A as a G-module and may, without comment, write A additively. The action of elements of G on A will be written a^g $(a \in A, g \in G)$, but note that other linear transformations of A may be written as right multiplication; thus if e is an endormorphism of the module A we write ae for the image of a under e.

2. Representations of $\underline{\underline{A}}_m\underline{\underline{A}}_p$ groups

For convenience we start by stating a well-known theorem in a form appropriate for our purposes (see Higman [*10*, Lemma]).

LEMMA 2.1. *Let A be an abelian p-group, K a finite group of automorphisms of A and K_1 a normal p'-subgroup of K. If A_0 is the subgroup of A whose elements are fixed by every element of K_1 then A_0 has a complement in A which admits K.*

This section is devoted to proving the following theorem.

THEOREM 2.2. *Let p be a prime and m a natural number prime to p, and let K in $\underline{\underline{A}}_m\underline{\underline{A}}_p$ be a finite group which has a faithful irreducible representation over a field E of characteristic p. All the faithful irreducible representation modules for K over E are principal indecomposables and the representations they afford form a single linear isomorphism class.*

Proof. Start by assuming that E is algebraically closed and let M be a faithful co-monolithic module for K over E, with unique maximal submodule M_0, say. Write S for the normal Hall p'-subgroup of K: notice that S is not 1. By Maschke's Theorem there exists an irreducible submodule N of M_S outside M_0. Now N is one dimensional and, since S is normal in K, Nk is a submodule of M_S whenever $k \in K$. Hence if T is a transversal of K to S

$$M = \sum_{t \in T} Nt . \tag{2.3}$$

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Suppose $k \in K$ is such that $N \cong Nk$. That is, there is a one-to-one linear transformation $\theta : N \to Nk$ such that

(2.4) $$(ns)\theta = (n\theta)s \ , \quad s \in S \ , \quad n \in N \ .$$

However s acts simply as a scalar multiplication, say $ns = n\alpha(s)$ where $\alpha(s) \in E$; so that if $n\theta = n'k$, then $n \mapsto n'$ is a linear transformation and

(2.5) $$(ns)' = \{n\alpha(s)\}' = n'\alpha(s) = n's \ , \quad n \in N \ , \quad s \in S \ .$$

We conclude from (2.4) and (2.5) that

$$(n'k)s = (ns)'k = (n's)k \ , \quad n' \in N \ , \quad s \in S \ ,$$

whence it follows that $[S, k] \leq \ker N$. Now S centralizes K' and therefore $[S, k]$ is normal in K. Lemma 2.1, and the fact that M is faithful and indecomposable, ensures that $[S, k] = 1$; but $C_K(S) = S$ or else K would have a non-trivial normal p-subgroup and could not have a faithful irreducible representation over E. Hence $k \in S$ and it follows from (2.3) that

(2.6) $$M = \bigoplus_{t \in T} Nt \ .$$

We have shown, therefore, that the dimension of M is $|K : S|$. A simple application of Lemma 2.1 and Maschke's Theorem shows that M/M_0 is faithful and, being irreducible, is co-monolithic, so that

$$\dim M = \dim M/M_0 \ ;$$

in other words $M_0 = 0$. The first statement of the theorem is now proved (for closed fields) by observing that a principal indecomposable for K over E is co-monolithic and that it is faithful if the co-monolith is faithful.

The restriction on the field can now be removed. If V is a faithful irreducible module for K over E, and E^* is the closure of E then, by (70.15) of Curtis and Reiner [8], $E^* \otimes_E V$ is completely reducible and each irreducible component is faithful, hence projective, by what has already been proved; and therefore $E^* \otimes_E V$ is projective. This in turn will imply that V is projective. For, if

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$$\begin{array}{ccccc} & & V & & \\ & & \downarrow{\scriptstyle\alpha} & & \\ V_1 & \xrightarrow{\beta} & V_2 & \longrightarrow & 0 \end{array}$$

is a diagram with exact row then there exists γ^* such that the diagram

$$\begin{tikzcd} & E^* \otimes_E V \arrow[dl, "\gamma^*"'] \arrow[d, "1 \otimes \alpha"] & \\ E^* \otimes_E V \arrow[r, "1 \otimes \beta"] & E^* \otimes_E V \arrow[r] & 0 \end{tikzcd}$$

commutes. If $v \in V$ and $(1 \otimes v)\gamma^* = \sum e_i \otimes v_i$ (where $\{1 = e_0, e_1, \ldots\}$ is a basis for E^* over E) it is easily checked that $\gamma : v \mapsto v_1$ is an EK-homomorphism such that $\gamma\beta = \alpha$, as required. Finally then, V is a direct sum of principal indecomposables and is therefore itself a principal indecomposable.

To proceed further, more structure on K is required. The lemma that follows comes either directly from, or by routine modification of, results of Kochendörffer [*13*] and Taunt [*20*].

LEMMA 2.7. *S is a direct product of indecomposable normal homocylic subgroups S_i $(1 \le i \le r)$ of K. Moreover each σS_i is a minimal normal subgroup of K and σK is their direct product, this being the unique decomposition of σK as a direct product of minimal normal subgroups of K.*

In order to prove that the faithful irreducibles of K over E form a single linear isomorphism class it suffices, by Theorem 2.5 in [*1*], to assume that E is algebraically closed. If, then, M is a faithful irreducible module for K over E (2.6), and (44.1) in [*8*], ensures that

$$M \cong N^K \tag{2.8}$$

where N is a one-dimensional submodule of M_S. The proof consists in choosing a basis for M and a set of generators for K, depending on M, and evaluating the matrices representing these generators; it will be obvious that the linear group they generate is independent of M.

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In (2.6) T may be chosen as a complement for S and is to be fixed throughout. Also the subgroups S_i $(1 \le i \le r)$ in Lemma 2.7 are to be fixed. Regard σS_i and $S_i/\Phi(S_i)$ as T-modules over a field of prime order (NOTE: S_i has prime-power order); they are easily seen to be isomorphic and hence, by 12.2.2 of [9], $\ker\sigma S_i = C_T(S_i)$. With at most one exception (by Lemma 2.7 at most one σS_i is central in K) $|T : C_T(S_i)|$ is therefore equal to p; choose a fixed $t_i \in T \setminus C_T(S_i)$. Next, given the faithful irreducible module M choose N so that (2.8) holds; let $L = \ker N$. Note that S/L is cyclic, hence $S_i/S_i \cap L$ are all cyclic and that, because of Lemmas 2.7 and 2.1,

$$(2.9) \qquad |S_i/S_i \cap L| = \exp S_i \ , \quad |S/L| = \exp S \ , \quad 1 \le i \le r \ .$$

Choose an element $s \in S \setminus L$ whose order is $\exp S$.

The following lemma is vital.

LEMMA 2.10. *Let V be a free module of rank ρ over the ring of integers modulo q^α (q a prime), and let V_0 ($\not\subseteq qV$) be a free submodule of V of rank $\rho - 1$. $H = \langle h \rangle$ is a q'-cycle acting faithfully and indecomposably on V. There is a basis $\{v_1, \ldots, v_\rho\}$ of V such that $v_i \in V_0$ $(1 \le i \le \rho-1)$ and*

$$(2.11) \qquad v_i h = v_{i+1} \ , \quad 1 \le i \le \rho-1 \ .$$

Moreover the matrices representing h with respect to all bases with the property (2.11) *are the same.*

Proof. No proper submodule of V, not in qV, admits h. It follows easily that if

$$U_i = V_0 \cap V_0 h \cap \ldots \cap V_0 h^{i-1} \ , \quad 1 \le i \le \rho \ ,$$

then the rank of U_i is $\rho - i$. Choose $0 \ne v_{\rho-1} \in U_{\rho-1}$. Since $U_{\rho-1} = U_{\rho-2} \cap U_{\rho-2}h$ there exists $v_{\rho-2} \in U_{\rho-2}$ such that $v_{\rho-1} = v_{\rho-2}h$ and $v_{\rho-2} \notin U_{\rho-1}$. In a similar fashion choose $v_{\rho-3}, \ldots, v_1$ $(\in V_0)$

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and define $v_\rho = v_{\rho-1}h$. Now $\{v_1, \ldots, v_{\rho-1}\}$ is a basis for V_0 by construction and hence $v_\rho \notin V_0$ so that $\{v_1, \ldots, v_\rho\}$ is a basis for V, and has the desired properties. Notice that the coefficients in the expression for $v_\rho h$ are coefficients of the minimum polynomial for h. This completes the proof of Lemma 2.10.

Choose a basis $\{v_{i1}, \ldots, v_{i\rho(i)}\}$ for S_i respecting $S_i \cap L$ as in the lemma, with t_i playing the rôle of h : for convenience set $s_i = v_{i\rho(i)}$. Then there exists an integer m_i, independent of L, such that

$$s^{m_i} \equiv s_i^{u_i} \bmod L$$

where $(u_i, \exp S_i) = 1$. However $\left\{v_{i1}^{u_i}, \ldots, v_{i\rho(i)}^{u_i}\right\}$ is still a basis for S_i with the properties of Lemma 2.10, so, without loss of generality,

$$s^{m_i} \equiv s_i \bmod L . \tag{2.12}$$

Now s_i is a generator of S_i as T-module and the action of T on s_i depends only on that of t_i; specifically, to each $t \in T$ there exists an integer $\tau(i, t)$, independent of L by Lemma 2.10, such that

$$s_i^t \equiv s_i^{\tau(i,t)} \bmod L .$$

Finally let $N = \mathrm{sp}\{n\}$, $ns = \alpha n$ $(\alpha \in E)$, and $\underline{\underline{B}} = \{n \otimes t : t \in T\}$ be our basis for M (from (2.8) this is possible). Note that

$$(n \otimes t)s_i = n \otimes ts_i = n \otimes s_i^{t^{-1}} t = ns_i^{\tau(i,t^{-1})} \otimes t = \alpha^{m_i\tau(i,t^{-1})}(n \otimes t) .$$

If U is the representation afforded by M then, with respect to $\underline{\underline{B}}$, s_iU has diagonal matrix, and the diagonal entries are powers of α. Our choice of the quantities m_i, $\tau(i, t)$ was independent of M. Hence if U' is a faithful irreducible representation of K and α' is the analogue of α, then α' is a power of α and so, with respect to a

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suitable basis, $s_i'U'$ is a power of s_iU. Since the matrix of tU with respect to $\underline{\underline{B}}$ is a permutation matrix for all $t \in T$ we see that the linear group KU is independent of the faithful irreducible U.

3. Further preliminaries

Let p be a prime and R_α the ring of integers modulo p^α. We need some facts about representations over R_α, analogues of well known facts about representations over fields.

If K is a finite group the group ring $R_\alpha K$ has minimum condition on right ideals. Let $R_\alpha K = \bigoplus_{i=1}^{\rho} A_i$ be a decomposition of $R_\alpha K$ as a direct sum of indecomposable right ideals. Since $R_\alpha K/pR_\alpha K$ and $p^{\alpha-1}R_\alpha K$ are isomorphic vector spaces over R_1 it follows in a familiar fashion that each A_i is a free R_α-module. By (54.11) in [8] each A_i/pA_i is a principal indecomposable of R_1K.

Next suppose that N_1 is an irreducible module for K over R_1, and write $E_1 = \mathrm{End}_K N_1$, a finite field isomorphic to $\mathrm{GF}(p^d)$, say. Let C_1 be the multiplicative group of E_1; then N_1 is a C_1K-module over R_1 and we shall show that if N_1 is projective as R_1K-module it is projective as R_1C_1K-module. Choose a fixed isomorphism $\varphi : \mathrm{GF}(p^d) \to E_1$ thus turning N_1 into a $\mathrm{GF}(p^d)K$-module - call it $\hat{N}_1$. Now $\hat{N}_1$ is projective since it occurs as a direct summand of $\mathrm{GF}(p^d) \otimes_{R_1} N_1$, and N_1 is projective (using somewhat more than 70.15 of [8]; L.G. Kovács (unpublished) has proved our assertion). If $c \in C_1$ there exists $e \in \mathrm{GF}(p^d)$ for which, in $\hat{N}_1$,

$$nc = ne \ , \quad n \in N_1 \ ;$$

and using this and the projectivity of $\hat{N}_1$ one easily checks the commutivity of the diagrams which ensure that N_1, as R_1C_1K-module, is projective.

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The last two paragraphs are now brought together. Let N_α $(\alpha \in I^+)$ be homocyclic groups of exponent p^α, say with $N_\beta = N_\alpha/p^\beta N_\alpha$ $(\alpha \geq \beta)$; E_α is the endomorphism ring, and A_α the automorphism group, of N_α. Define $\mu_{\alpha\beta} : E_\alpha \to E_\beta$ by

$$\left(x+p^\beta N_\alpha\right)\left(e\mu_{\alpha\beta}\right) = xe + p^\beta N_\alpha \ , \quad x \in N_\alpha \ , \quad e \in E_\alpha \ .$$

It is easy to see that $\mu_{\alpha\beta}$ is an onto ring homomorphism such that $\mu_{\alpha\beta}\mu_{\beta\gamma} = \mu_{\alpha\gamma}$ $(\alpha \geq \beta \geq \gamma)$, and that the restriction $\nu_{\alpha\beta}$ of $\mu_{\alpha\beta}$ to A_α as multiplicative homomorphism is onto A_β. Suppose now that K_1 is a subgroup of A_1 such that N_1 , as R_1K_1-module, is irreducible and principal indecomposable. If, as above, $E_1 = \operatorname{End}_{K_1} N_1$ and C_1 is its multiplicative group then N_1 as $R_1C_1K_1$-module is principal indecomposable and hence there are subgroups C_α, K_α of A_α , which centralize each other, such that $\nu_{\alpha 1}$ takes C_α isomorphically onto C_1 and K_α isomorphically onto K_1. We may assume also that $K_\alpha\nu_{\alpha\beta} = K_\beta$, $C_\alpha\nu_{\alpha\beta} = C_\beta$. In this set up we have

LEMMA 3.1. *As $R_\alpha K_\alpha$-module N_α has the double centralizer property. Also if $E_\alpha = \operatorname{End}_{K_\alpha} N_\alpha$ then every element e of E_α can be written uniquely as*

$$e = \sum_{i=0}^{\alpha-1} p^i c_i \ , \quad c_i \in C_\alpha \cup \{0\} \ .$$

Proof. The double centralizer property is proved first, by induction on α ; for $\alpha = 1$ it is true by (26.4) in [8]. We show in fact that if $\xi \in E_\alpha$ and ξ centralizes C_α then $\xi = \sum r_ik_i$ for some $r_i \in R_\alpha$ and $k_i \in K_\alpha$.

Notice that $\ker\mu_{\alpha\alpha-1} = p^{\alpha-1}E_\alpha$ and that $\xi\mu_{\alpha\alpha-1}$ centralizes $C_{\alpha-1}$ and so, by induction, there exists $r_i' \in R_\alpha$, $k_i' \in K_\alpha$ so that

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$$\xi = \sum r_i' k_i' + p^{\alpha-1}e \tag{3.2}$$

for some $e \in E_\alpha$. Since $\lambda : x + pN_\alpha \to p^{\alpha-1}x$ is a group isomorphism of N_1 onto $p^{\alpha-1}N_\alpha$ and, from (3.2), $p^{\alpha-1}e$ centralizes C_α, for all $c \in C_\alpha$ and all $x \in N_\alpha$ we have

$$\begin{aligned}(x+pN_\alpha)\big((ec)\mu_{\alpha 1}\big)\lambda &= (xec+pN_\alpha)\lambda = p^{\alpha-1}(xec)\\ &= x\big(p^{\alpha-1}e\big)c = xc\big(p^{\alpha-1}e\big) = (x+pN_\alpha)\big((ce)\mu_{\alpha 1}\big)\lambda .\end{aligned}$$

Hence $(ec)\mu_{\alpha 1} = (ce)\mu_{\alpha 1}$. Therefore $e\mu_{\alpha 1}$ centralizes C_1. As $\ker\mu_{\alpha 1} = pE_\alpha$ there exists $e' \in E_\alpha$, $r_i'' \in R_\alpha$ and $k_i'' \in K_\alpha$ such that

$$e = \sum r_i'' k_i'' + pe' .$$

Combining this with (3.2) yields the desired result. The remainder of Lemma 3.1 can be proved by entirely similar methods.

It is well-known that there exists a linear transformation β_1 of N_1 such that $\beta_1^{-1}e\beta_1 = e^p$ $(e \in E_1)$: as E_1-space N_1 is completely reducible and, on each irreducible component of N_1, e and e^p have the same minimal polynomial and the matrix of each is similar to the companion matrix of this minimum polynomial. More generally we have

LEMMA 3.3. *There exists* $\beta_\alpha \in E_\alpha$ *such that*

$$\beta_\alpha^{-1}c\beta_\alpha = c^p , \quad c \in C_\alpha .$$

The proof of this will follow from the next lemma and the fact that $\ker\nu_{\alpha 1}$ is a p-group (12.2.2 in [9]).

LEMMA 3.4. *Two* p'*-elements of a finite group are conjugate if and only if they are conjugate modulo a normal* p*-subgroup.*

Proof. Suppose that X is a finite group, Y is a normal p-subgroup of X and x_1, x_2 are conjugate modulo Y. It suffices to assume Y is abelian. For each $t \in Yh$ there exists $y_t \in Y$ such that

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$$x_1^t = x_2 y_t \ .$$

We show that for some t, $y_t \in C$, the centralizer of x_2 in Y. For, suppose t, u are such that $y_t y_u^{-1} \in C$; then $x_1^t x_1^{-u} \in C$ and so $[u^{-1}t, x_1^{-u}] \in C$. Now C has a complement Y_1 in Y by Lemma 2.1, which admits x_2. Write $u^{-1}t = cy_1$ $(c \in C, y_1 \in Y_1)$ and then

$$\left[cy_1, x_1^{-u}\right] = \left[cy_1, y_u^{-1}x_2^{-1}\right] = \left[y_1, x_2^{-1}\right] \in Y_1 \cap C = 1 \ ,$$

whence $y_1 = 1$. Therefore $u^{-1}t \in C$. It follows that, if T is a transversal of Y to C, $\{y_t : t \in hT\}$ is also a transversal of Y to C, and hence for some t, y_t centralizes x_2 as we asserted above. For this t, $\left(x_1^t\right)^n = x_2^n y_t^n$ which, if n is the l.c.m. of the $(p'\text{-})$ orders of x_1, x_2, gives $y_t^n = 1$ whence $y_t = 1$, completing the proof.

LEMMA 3.5. *Two faithful principal indecomposable* $R_\alpha K$*-modules* N_α *and* P_α *afford linearly isomorphic representations if and only if* N_α/pN_α *and* P_α/pP_α *afford (faithful) linearly isomorphic representations of* R_1K.

Proof. The 'only if' direction is easy, so suppose that N_α and P_α afford representations T, U respectively and that the representations T', U' thereby induced on N_α/pN_α and P_α/pP_α are linearly isomorphic. That is, with respect to suitably chosen bases, the matrix groups KT' and KU' are equal. Hence there is an automorphism λ of K such that

$$k\lambda T' = kU' \ , \quad k \in K \ .$$

The module N_α/pN_α affording the representation $\lambda T'$ is therefore isomorphic to P_α/pP_α. Hence by (54.14) in [8], and the remarks at the beginning of this section, N_α as module affording the representation λT is isomorphic to P_α. In particular, with respect to suitably chosen

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bases, the matrix groups $K\lambda T$ and KU are equal. Hence T, U are linearly isomorphic.

Suppose that V_1, V_2 are irreducible modules affording faithful representations T_1, T_2 of a group K. One can form the groups X_1, X_2 by split-extending V_1, V_2 by K with actions T_1, T_2 respectively. It is easy to check that T_1 is linearly isomorphic to T_2 if and only if X_1 and X_2 are isomorphic groups. In view of this we have the following corollary to Lemma 3.5.

LEMMA 3.6. *If K is an irreducible linear group acting on a space N_1 over R_1, the split-extension is a uniquely determined critical group. If, moreover, N_1 as R_1K-module is principal indecomposable then to each $\alpha \in I^+$ there exists a unique split-extension $N_\alpha K$ such that $N_\alpha K/pN_\alpha \cong N_1K$; $N_\alpha K$ is critical. Furthermore if N is abelian of exponent p^α and an extension NK exists such that $NK/pN \cong N_1K$ then N is homocyclic and, indeed, principal indecomposable as $R_\alpha K$-module, and $NK \cong N_\alpha K$.*

Proof. The existence of $N_\alpha K$ has already been shown, and the criticality follows from (1.65) in [*14*] of Kovács and Newman. That N is homocyclic follows from the fact that $x + pN \mapsto p^{\alpha-1}x$ is a K-homomorphism. The remainder will be proved by Lemma 3.5 when we show that N is principal indecomposable. This follows easily from the projectivity of N_α.

4. Structure of certain critical groups

Let G be a critical group the last non-trivial term of whose lower nilpotent series, A say, is abelian. Then, by Theorem 3 in [*10*] of Higman, A has a complement B in G. Since G is monolithic, A is self-centralizing in G and, for some prime p, A is a p-group. We will be interested in cases when $B = H \times K$ with H the maximal normal p-subgroup of B and K a group whose faithful irreducible representations over $GF(p)$ are projective: whenever G is

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abelian-by-nilpotent of $G \in \underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_m\underline{\underline{A}}_p$ $(p \nmid m)$, and G is not nilpotent, this is easily seen to be the case (using Theorem 2.2 in the latter case). In this section we describe the structure of A as B-module: in fact we show that it suffices to obtain a description of A_H and A_K . Our aim is to construct another group (denoted later by $G^{\#}$) which generates the same variety as G but which is easier to work with then G itself.

As B-module, then, A is faithful and monolithic and, by 51.37 in [18], co-monolithic also; let A_0 be the unique maximal submodule of A . Write $A^* = pA + [A, H]$ so that $A^* \leq A_0$. Choose $h_{ij} \in Z_i(H) - Z_{i-1}(H)$ $\big(1 \leq i \leq c$, $1 \leq j \leq r(i)$ - H has class c , say,$\big)$ such that

$$A(i) = \big[A, h_{11}, \ldots, h_{1r(1)}, \ldots, h_{i1}, \ldots, h_{ir(i)}\big]$$

is non-trivial for each $i \in \{1, \ldots, c\}$ but that

$$\big[A(i), Z_i(H)\big] = 1 \ , \ 1 \leq i \leq c \ .$$

Let p^δ be the exponent of $A(c)$. Then it is easy to check that the mapping

$$\xi : a + A^* \mapsto \big[a, h_{11}, \ldots, h_{1r(1)}, \ldots, h_{c1}, \ldots, h_{cr(c)}\big]^{p^{\delta-1}}$$

is a non-zero B-homomorphism of A/A^* . Now σA is in $(A/A^*)\xi$ and is therefore centralized by H ; hence K acts irreducibly on σA and, by Lemma 2.1, faithfully also. But our assumptions on K mean that $(\sigma A)_K$ is injective so that

$$(A/A^*)\xi = \sigma A \oplus U$$

where U admits K and therefore B since H acts trivially on $(A/A^*)\xi$; thus $U = 0$ and $(A/A^*)\xi = \sigma A$. It follows that $\ker\xi = A_0/A^*$ and that, as B-modules,

$$A/A_0 \cong \sigma A \ . \tag{4.1}$$

We aim now to delineate the module structure of A commencing with

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the next lemma.

LEMMA 4.2. *There exists a principal indecomposable submodule* N *of* A_K *such that* $N/pN \cong \sigma A$ *as* R_1K*-modules and* N *generates* A *as* H*-module.*

Proof. Since $(\sigma A)_K$ is projective so is $(A/A_0)_K$ by (4.1). Hence there exists a submodule C_1 of A_K so that

$$C_1 + A_0 = A \ , \quad C_1 \cap A_0 = pA \ .$$

But $C_1/pA \cong A/A_0$ so, for the same reason, there is a submodule C_2 of C_1 such that $C_2 + pA = C_1$, $C_2 \cap pA = pC_1$ and $C_2/pC_1 \cong A/A_0$. In this way construct a descending sequence of submodules C_i such that $C_i/pC_{i-1} \cong A/A_0$. For some j we must have $pC_j = pC_{j-1}$ which means that $C_j/pC_j \cong A/A_0$ is irreducible and the last assertion of Lemma 3.5 then gives that C_j is a free R_α-module. Note that $C_i \nsubseteq A_0$ for any i. Put $N = C_j$ and then, since N admits K, N generates A as H-module, whence N has exponent p^α. This completes the proof of Lemma 4.2.

Let the N of Lemma 4.2 have R_α-basis $\{n_1, \ldots, n_s\}$. Write M_i for the H-submodule of A generated by n_i $(1 \le i \le s)$. Using von Dyck's theorem one checks that group isomorphisms $\delta_i : M_1H \to M_iH$ may be defined by

$$n_1\delta_i = n_i \ , \quad h\delta_i = h \ , \quad h \in H \ .$$

The mapping

$$(m, n) \mapsto \sum_{i=1}^{s} r_i\left(m\delta_i\right) \ , \quad m \in M_1 \ , \quad n \in N$$

where $n = \sum r_i n_i$, is easily seen to be balanced. Hence there exists a homomorphism τ of $M_1 \otimes_{R_\alpha} N$ onto A such that

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$$(m \otimes n)\tau = \sum_{i=1}^{s} r_i(m\delta_i) \ , \quad m \in M_1 \ , \quad n \in N \ .$$

Moreover, if $H \times K$ acts on $M_1 \otimes_{R_\alpha} N$ in the usual outer tensor product fashion:

$$(m \otimes n)^{hk} = m^h \otimes n^k \ , \quad m \in M_1 \ , \quad n \in N \ , \quad hk \in H \times K \ ,$$

then τ is an $H \times K$-epimorphism. Define $G^\#$ to be the group obtained by split-extending $M_1 \otimes_{R_\alpha} N$ $(= A^\#)$ by $H \times K$; then τ extends to a group epimorphsim $\tau : G^\# \to G$.

Next write $E = \mathrm{End}_K N$. We can extend the action of E to the whole of $A^\#$ by identifying E with $1 \otimes E$:

$$(m \otimes n)e = m \otimes ne \ , \quad m \in M_1 \ , \quad n \in N \ , \quad e \in E \ .$$

Then, by Lemma 3.1, $\ker\tau$ admits E . Consequently A may be regarded as an E-space. In this set up we have:

THEOREM 4.3. *If M is the EH-subspace of A generated by n_1 then, as $E(H \times K)$-modules,*

$$A \cong M \otimes_E N \ .$$

Proof. Since each space $p^iN/p^{i+1}N$ $(0 \le i \le \alpha-1)$ is a vector space over the field $E/pE = E_1$, N has an E-basis $\{l_1, \ldots, l_t\}$ with $l_1 = n_1$, say. Then, copying the construction of $G^\#$ and τ , we find an $E(H \times K)$-homomorphism

$$\nu : M \otimes_E N \to A$$

onto A . We will show that ν is one-to-one.

The construction of ν gives first

(4.4) $\qquad n \in N\text{-}pN \ , \quad m \in M$ and $m \otimes n \in \ker\nu \Rightarrow m = 0$;

and second

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$$\sigma\left(M \otimes_E N\right) = \sigma M \otimes_E N \cong \sigma M \otimes_{E_1} N/pN \ . \tag{4.5}$$

Since $\sigma M \otimes_E N$ is centralized by H it follows from (4.4) and (4.5) that

$$0 \neq \left[\sigma\left(M \otimes_E N\right)\right]\nu \leq Z(AH) = \sigma A \ ;$$

hence $\left[\sigma\left(M \otimes_E N\right)\right]\nu = \sigma A$. Write $D = \ker\nu \cap \sigma\left(M \otimes_E N\right)$. Then, regarding $\sigma\left(M \otimes_E N\right)$ as an E_1-space (by (4.5)), and σA also, we find that the co-dimension of D in $\sigma\left(M \otimes_E N\right)$ is precisely $\dim\sigma A = \dim_E N = t$. Suppose that σM contains elements m_1, m_2 which are E-independent and therefore E_1-independent. Using (4.5)

$$\dim(m_1 \otimes N/pN + m_2 \otimes N/pN) = 2\dim N$$

whence, if $D \neq 0$,

$$D_1 = D \cap (m_1 \otimes N/pN + m_2 \otimes N/pN) \neq 0 \ .$$

If $d \in D_1$ then, for suitable $x_1, x_2 \in N/pN$ we have

$$d = m_1 \otimes x_1 + m_2 \otimes x_2 \ ,$$

and (4.4) shows that the relation $\mu : x_1 \mapsto x_2$ is a mapping from $X_1 = \{x_1 \in N/pN : d \in D_1\}$ to $X_2 = \{x_2 \in N/pN : d \in D_1\}$. Indeed since $D_1 \neq 0$ and N/pN is an irreducible E_1K-module, $X_1 = X_2 = N/pN$ and $0 \neq \mu \in E_1$. Hence for some non-zero $x_1 \in N/pN$,

$$(m_1+m_2\mu) \otimes x_1 \in D_1$$

and (4.4) implies $m_1 + m_2\mu = 0$ contrary to the independence of m_1 and m_2 . Therefore $D = 0$ and hence $\ker\nu = 0$. If the E-dimension of N is one then clearly $\ker\nu = 0$; in any case ν is one-to-one and Theorem 4.3 is proved.

The proof just completed shows that σM is a one dimensional E_1-space. Since $\sigma M_1 \leq \sigma M$ we have

COROLLARY 4.6. *The dimension of σM_1 as R_1-space is at most the dimension d of E_1 over R_1 .*

Write $\hat{A} = \left(M_1 \otimes_{R_\alpha} E\right) \otimes_E N$ so that there is an E-isomorphism $\lambda : \hat{A} \mapsto A^{\#}$ which has

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$$\lambda : (m_1 \otimes 1) \otimes n \mapsto m_1 \otimes n \ , \quad m_1 \in M_1 \ , \quad n \in N \ .$$

Moreover if we define the action of $H \times K$ on $\hat{A}$ in the natural way:

$$\big((m_1 \otimes 1) \otimes n\big)^{hk} = \big(m_1^h \otimes 1\big) \otimes n^k \ , \quad m_1 \in M_1 \ , \quad n \in N \ ,$$

then λ is an $H \times K$ homomorphism; and if $\hat{G}$ is obtained by extending $\hat{A}$ by $H \times K$ in this action, λ extends to a group isomorphism $\lambda : \hat{G} \rightarrowtail G^{\#}$. In this set up one easily checks, using Theorem 4.3

LEMMA 4.7. *There exists an EH-submodule* L *of* $M_1 \otimes_{R_\alpha} E$ *such that* $\ker\lambda\tau = L \otimes_E N$.

Next let $\beta = \beta_\alpha$ of Lemma 3.3 and let $\gamma : E \to E$ be defined by

$$\gamma : e \mapsto e^{\beta} \ , \quad e \in E \ .$$

Then it is simple to check that

$$\lambda\big(1 \otimes \beta^i\big)\lambda^{-1} = \big(1 \otimes \gamma^i\big) \otimes \beta^i \ , \quad 0 \le i \le d-1 \ . \tag{4.8}$$

LEMMA 4.9. $\bigcap_{i=0}^{d-1} (\ker\tau)\beta^i = 0$.

Proof. It suffices to show that $\bigcap_{i=0}^{d-1} \big(L \otimes_E N\big)\lambda\beta^i\lambda^{-1} = 0$ which, by (4.8), will follow if $\bigcap_{i=0}^{d-1} L\big(1 \otimes \gamma^i\big) = 0$. Write $U = L \cap \sigma(M_1 \otimes E)$ so that we need only prove

$$\bigcap_{i=0}^{d-1} U\big(1 \otimes \gamma^i\big) = 0 \ . \tag{4.10}$$

Analogously to (4.4) and (4.5) we have:

$$m_1 \in M_1 \ , \quad e \in E\text{-}pE \ \text{ and } \ m_1 \otimes e \in L \ \Rightarrow \ m_1 = 0 \ ; \tag{4.11}$$

$$\sigma\big(M_1 \otimes_{R_\alpha} E\big) \cong \sigma M_1 \otimes_{R_1} E_1 \ . \tag{4.12}$$

The force of (4.12) is that to prove (4.10) we may assume that we are working in a vector space over a field E_1 , where γ is now the Galois

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automorphism $e_1 \mapsto e_1^p$ of E_1. The following rather technical lemma will prove (4.10).

LEMMA 4.13. *Let* V *be a proper non-zero submodule of* $\sigma M_1 \otimes_{R_1} E_1$ *and* l *be maximal with respect to the property that, whenever* S *is a linearly independent subset of* σM_1 *containing at most* $l - 1$ *elements,*

$$(4.14) \qquad 0 = V \cap \bigoplus_{t \in S} t \otimes E_1 .$$

Then, if S' *is a linearly independent subset of* σM_1 *containing* l *elements*

$$0 = V \cap V1 \otimes \gamma \cap \bigoplus_{t \in S'} t \otimes E_1 .$$

Proof. Observe first that the co-dimension of V in $\sigma M_1 \otimes E_1$ is at least $l - 1$ and that, if S' is a linearly independent subset of σM_1 containing l elements,

$$0 \neq V(S') = V \cap \bigoplus_{t \in S'} t \otimes E_1$$

We emulate the argument following (4.5). For all $v \in V(S')$ then,

$$v = \sum_{t \in S'} t \otimes x_t$$

for some $x_t \in E_1$. By virtue of (4.14) for each $t \in S'$ the correspondence $x_t \mapsto x_{t'}$ is one-to-one for each $t' \in S'$, and is indeed, an E_1-endomorphism of E_1. Consequently there exist elements $S'(t)$ of E_1, linearly independent over R_1 by (4.14), such that

$$(4.15) \qquad V(S') = \left\{ \sum_{t \in S'} t \otimes xS'(t) \,:\, x \in E_1 \right\} .$$

Next define, for a linearly independent subset S of σM_1 containing exactly $l - 1$ elements, and a basis B of σM containing S,

$$S_b = S \cup \{b\} , \quad b \in \overline{S} = B - S .$$

Then, using (4.15), the $V(S_b)$ clearly generate their direct sum and each $V(S_b)$ has E_1-dimension 1. Hence $\bigoplus_{b \in \overline{S}} V(S_b)$ has co-dimension $l - 1$

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and therefore

$$V = \bigoplus_{b \in \overline{S}} V(S_b) \; ; \tag{4.16}$$

in particular V has co-dimension exactly $l - 1$.

We now show that for all S and B as above

$$V \cap V(S_b) 1 \otimes \gamma = 0 \;, \quad b \in \overline{S} \;. \tag{4.17}$$

Suppose that $w \in V(S_b)$ and $w1 \otimes \gamma \in V$. From (4.15) and (4.16)

$$w1 \otimes \gamma = \left(\sum_{t \in S_b} t \otimes xS_b(t) \right) 1 \otimes \gamma = \sum_{j \in S} \sum_{t \in S_j} t \otimes x_j S_j(t)$$

for some $x, x_j \in E_1$. This implies at once that for $j \neq b$, $x_j = 0$ since $j \otimes x_j S_j(j)$ occurs once only on the right and not at all on the left. Hence

$$\sum_{t \in S_b} t \otimes (x\gamma) S_b(t)^p = \sum_{t \in S_b} t \otimes x_b S_b(t)$$

so that

$$(x\gamma) S_b(t)^p = x_b S_b(t) \;, \quad t \in S_b \;.$$

If t, t' are distinct elements of S_b (under the hypotheses $|S_b| \geq 2$) then $x_b \left\{ S_b(t) S_b(t)^{-p} - S_b(t') S_b(t')^{-p} \right\} = 0$ from which $x_b \neq 0$ implies $y \left\{ S_b(t) S_b(t)^{-p} - S_b(t') S_b(t')^{-p} \right\} = 0$ for all $y \in E_1$. In other words

$$x_b \neq 0 \;\Rightarrow\; S_b(t) S_b(t)^{-p} - S_b(t') S_b(t')^{-p} = 0 \;,$$

which implies $\left(S_b(t')^{-1} S_b(t) \right)^p = S_b(t')^{-1} S_b(t)$, and this means that $S_b(t')^{-1} S_b(t) \in R_1$ contrary to the independence of $S_b(t'), S_b(t)$ over R_1 . We conclude that $x_b = 0$ and therefore that $w = 0$, proving (4.17). Since an arbitrary set S' of l independent elements of σM_1 can always be constructed as $S' = S_b$ for suitable S and B , (4.17) yields

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(with γ replaced by γ^{-1})

$$V \cap V1 \otimes \gamma \cap \bigoplus_{t \in S'} t \otimes E_1 = V1 \otimes \gamma \cap V(S') = \left(V \cap V(S')1 \otimes \gamma^{-1}\right)1 \otimes \gamma = 0$$

and the proof of Lemma 4.13 is complete.

Return to the proof of Lemma 4.9. By (4.11) U satisfies the hypotheses of Lemma 4.13 with $l = 2$. Now $U \cap U1 \otimes \gamma$ has co-dimension at most 2 since U, $U1 \otimes \gamma$ each have co-dimension 1; but (4.15) shows that $U \cap U1 \otimes \gamma$ has co-dimension at least 2, and hence exactly 2. An easy induction using Lemma 4.13 shows that $U \cap U1 \otimes \gamma \cap \ldots \cap U1 \otimes \gamma^{i}$ has co-dimension $i + 1$ in $\sigma M_1 \otimes E_1$ and therefore, by Corollary 4.6, is zero for $i = d$. This proves (4.10) and with it Lemma 4.9.

Lemma 4.9 provides a subdirect decomposition of $G^{\#}$ which we now describe. First note that K^{β^i} centralizes E, and therefore, by Lemma 3.1, $\ker\tau$ admits K^{β^i} $(0 \le i \le d-1)$; whence $(\ker\tau)^{\beta^i}$ admits K $(0 \le i \le d-1)$. Consequently in $\hat{G}$,

$$(L \otimes N)\lambda\beta^{i}\lambda^{-1} = \left(L1 \otimes \gamma^{i}\right) \otimes N, \quad 0 \le i \le d-1$$

admits HK. Write $M(i) = (M_1 \otimes E)/L1 \otimes \gamma^{i}$, $(0 \le i \le d-1)$, so that $M(i)$ is an EH-module "Galois conjugate" to M. Then $G_i = \hat{G}/\left(L1 \otimes \gamma^{i}\right) \otimes N$ is the group obtained by extending $M(i) \otimes_E N$ by $H \times K$ in the outer-tensor product fashion, and

LEMMA 4.18. *$G^{\#}$ is isomorphic to a subdirect product of $G_0, \ldots, G_{d-1}$.*

Note that $G_0 \cong G$. Indeed under suitable restrictions each of $G_0, \ldots, G_{d-1}$ is isomorphic to G.

LEMMA 4.19. *If the faithful irreducible representations of K over $\mathrm{GF}\left(p^d\right)$ form a single linear isomorphism class then each of $G_0, \ldots, G_{d-1}$*

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is isomorphic to G .

Proof. If E_α is the ring in Lemma 3.1, taking N_1 to be the additive group of $\mathrm{GF}\left(p^d\right)$ and K_1 its multiplicative group, then Lemma 3.5 shows that the faithful indecomposable representations of K over E_α are linearly isomorphic. In particular there exists ζ centralizing E such that $K = K^{\beta\zeta}$. Since $\beta\zeta$ has the same action on E as β does we may, without loss of generality, assume $K = K^\beta$.

Now there exist H-isomorphisms $\eta_i : M \to M(i)$ with the property:

$$(me)\eta_i = (m\eta_i)e\gamma^i \ , \quad m \in M \ , \quad e \in E \ , \quad i \in \{0, \ldots, d-1\} \ ,$$

and it is easy to check that there exist isomorphisms $\theta_i : M \otimes_E N \to M(i) \otimes_E N$ with the property that

$$(m \otimes n)\theta_i = m\eta_i \otimes n\beta^i \ , \quad m \in M \ , \quad n \in N \ , \quad i \in \{0, \ldots, d-1\} \ .$$

Moreover if $a \in M \otimes_E N$, $\left(a^{hk}\right)\theta_i = \left(a\theta_i\right)^{hk^{\beta^i}}$ $(h \in H \ , \ k \in K)$, and hence the mapping $\theta_i : G \to G_i$ defined by

$$(hka)\theta_i = hk^{\beta^i}a\theta_i \ , \quad h \in H \ , \quad k \in K \ , \quad a \in A$$

is an isomorphsim.

COROLLARY 4.20. *Under the conditions of* (4.19), $\mathrm{var}G = \mathrm{var}G^\#$.

Proof. Use Lemmas 4.19 and 4.18.

L.G. Kovács has constructed for us a group K which has a faithful irreducible representation whose Galois conjugates, regarded as $\mathrm{GF}\left(p^d\right)$ representations, are not linearly isomorphic; indeed for this K one can easily construct a critical group G such that $G_0, \ldots, G_{d-1}$ are not pair-wise isomorphic. Whether or not G and $G^\#$ nevertheless generate the same variety in general we have been unable to determine.

Summarizing this section then, we have

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THEOREM 4.21. *Let G be a non-nilpotent critical group the last non-trivial term of whose lower nilpotent series, A say, is abelian. A is a p-group, with a complement B in G. If $B = H \times K$ where H is the largest normal p-subgroup of B and K is such that its faithful irreducible representations over* $\mathrm{GF}(p)$ *are projective and form a single linear isomorphism class, then*

$$\mathrm{var}G = \mathrm{var}G^{\#} .$$

5. $\underline{\underline{A}}_p \underline{\underline{A}}_{\alpha m} \underline{\underline{A}}_p$

The first four sections can now be used to answer the test questions raised in the introduction. Much of the proof is technical in nature.

THEOREM 5.1. *Every subvariety of $\underline{\underline{V}} = \underline{\underline{A}}_p \underline{\underline{A}}_{\alpha m} \underline{\underline{A}}_p$ (where p is a prime not dividing m) is finitely based.*

THEOREM 5.2. *The lattice of subvarieties $\Lambda(\underline{\underline{V}})$ of $\underline{\underline{V}}$ is distributive.*

THEOREM 5.3. *The only just non-Cross subvariety of $\underline{\underline{V}}$ is $\underline{\underline{A}}_p \underline{\underline{A}}_p$.*

The join irreducible subvarieties of $\underline{\underline{V}}$ can be described with the apparatus we have developed, but in general this is tedious without being especially illuminating. We shall content ourselves with the case when m is a prime q. Even in this case a certain amount of preamble is necessary - it will be needed in the proof of Theorem 5.1 also.

A bigroup is a group G together with an idempotent endomorphism of G; alternatively G is a triple (G, A, B) where A is a normal subgroup of G and B a complement for A in G. Bigroups and varieties of bigroups are discussed in [4, 5] and the reader is referred there for more complete information.

Let $\mathsf{F} = (F, A_1, B_1)$ and $\mathsf{D} = (D, A_2, B_2)$ be bigroups in which A_1, A_2 are abelian groups, that is Z-modules over the ring of integers Z. Let $\mathsf{F} \# \mathsf{D}$ be the group obtained by split-extending $A_1 \otimes_Z A_2$ by $B_1 \times B_2$ with the usual outer-tensor product action:

$$(a_1 \otimes a_2)^{b_1 b_2} = a_1^{b_1} \otimes a_2^{b_2} \ , \quad a_i \in A_i \ , \quad b_i \in B_i \ , \quad i = 1, 2 \ .$$

Notice that if G is a non-nilpotent critical group in $\underline{\underline{V}}$ then, using the notation of §4, $G^\# \cong \mathsf{F} \# \mathsf{D}$ where $\mathsf{F} = (M_1H, M_1, H)$ and $\mathsf{D} = (NK, N, K)$; tensoring over R_α instead of Z makes no difference since $M_1 \otimes_Z N \cong M_1 \otimes_{R_\alpha} N$ as HK-modules. If S is a variety of bigroups denote by S^D the variety of groups generated by

$$\{\mathsf{F} \# \mathsf{D} : \mathsf{F} \in S\} \ .$$

We record without proof the following facts:-

$\mathsf{F}_1 \le \mathsf{F}$ *implies* $\mathsf{F}_1 \# \mathsf{D} \le \mathsf{F} \# \mathsf{D}$;

if ζ *is a homomorphism of* F *then the natural homomorphisms* $B_1 \times B_2 \to B_1\zeta \times B_2$ *and* $A_1 \otimes_Z A_2 \to A_1\zeta \otimes_Z A_2$ *extend to a homomorphism* $\mathsf{F} \# \mathsf{D} \to \mathsf{F}\zeta \# \mathsf{D}$;

$\left(\prod_i \mathsf{F}_i\right) \# \mathsf{D}$ *is a subdirect product of* $\prod_i \mathsf{F}_i \# \mathsf{D}$.

The next lemma follows easily from these three.

LEMMA 5.4. *If* S *is generated by the set* $\{\mathsf{F}_i : i \in I\}$ *then* S^D *is generated by* $\{\mathsf{F}_i \# \mathsf{D} : i \in I\}$.

COROLLARY 5.5. $(S_1 \vee S_2)^\mathsf{D} = S_1^\mathsf{D} \vee S_2^\mathsf{D}$.

The join irreducibles in $\Lambda\left(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_q\underline{\underline{A}}_p\right)$ are either locally nilpotent or not; the former are described in [*16*] and the latter will be described in terms of join irredicuble subvarieties of the variety of bigroups $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$ (for which see (4.3.15) of [5]) and irreducible linear groups in $\underline{\underline{A}}_q\underline{\underline{A}}_p$:

THEOREM 5.6. *The non-locally nilpotent join-irreducible subvarieties of* $\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_q\underline{\underline{A}}_p$ *not in* $\underline{\underline{A}}_{p^{\alpha-1}}\underline{\underline{A}}_q\underline{\underline{A}}_p$ *are precisely those of the form* S^D *where* S *is a join-irreducible subvariety of* $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$ *not in*

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$\underline{\underline{E}} \circ \underline{\underline{A}}_p$ *and* $\mathsf{D} = (D, A_2, B_2)$ *has* A_2 *of exponent* p^α *and* $B_2 \in \underline{\underline{A}}_q\underline{\underline{A}}_p$ *acting faithfully and irreducibly on* A_2/pA_2 .

Proof of Theorem 5.3. This result is covered by a more general (unpublished) result of J.M. Brady: a soluble just non-Cross variety of finite exponent, which is not $\underline{\underline{A}}_p\underline{\underline{A}}_p$ or $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, is contained in $\underline{\underline{A}}_p\left(\underline{\underline{N}}_c \wedge \underline{\underline{B}}_n\right)$ $(p \nmid n)$. We include a proof of Theorem 5.3 since it comes easily; it suffices to show that every subvariety $\underline{\underline{U}}$ of $\underline{\underline{V}}$ not containing $\underline{\underline{A}}_p\underline{\underline{A}}_p$ is Cross. By Theorem 5 of Kovács and Newman [*16*], p-groups in $\underline{\underline{U}}$ have bounded class; and, in particular, nilpotent critical groups have bounded order (51.35 in [*18*]). If G is a non-nilpotent critical group in $\underline{\underline{U}}$ then 51.38 in [*18*] ensures that H has bounded order (using the notation of §4). If T is the Sylow p-subgroup of K then $(\sigma G)_T$ is a direct sum of regular representations of T , by Theorem 2.2 and 65.16 in [*8*], and therefore $\text{var}G$ contains a group, isomorphic to $C_p \text{wr} T$, whose class is at least $|T|$. Hence $|T|$ is bounded, and Lemma 2.7 then shows that $|K|$ is bounded. Consequently Theorem 4.3 ensures that $|G|$ is bounded, thus showing that $\underline{\underline{U}}$ contains but finitely many critical groups and is Cross.

Proof of Theorem 5.2. If $\underline{\underline{W}}_i$ $(i = 1, 2, 3)$ are subvarieties of $\underline{\underline{V}}$ we need to show that

(5.7) $$\underline{\underline{W}}_1 \wedge (\underline{\underline{W}}_2 \vee \underline{\underline{W}}_3) \leq (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2) \vee (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_3) ,$$

since the other inclusion is obvious. Let G be a critical group belonging to the left-hand side of (5.7). By (1.12) and (1.14) of Kovács and Newman [*14*] there exist subsets $\Sigma_i \subseteq \underline{\underline{W}}_i$ $(i = 1, 2, 3)$ of critical groups whose monoliths are similar to σG such that

$$G \in \text{var}\Sigma_1 \wedge (\text{var}\Sigma_2 \vee \text{var}\Sigma_3) ;$$

and if $\Phi_i = \{F(X) : X \in \Sigma_i\}$ $(i = 1, 2, 3)$ (where $F(X)$ is the Fitting subgroup of X) then, using (1.14) in [*14*],

(5.8) $$F(G) \in \text{var}\Phi_1 \wedge (\text{var}\Phi_2 \vee \text{var}\Phi_3) .$$

Of course if G is nilpotent then the members of each Σ_i are nilpotent

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and, by Theorem 4 in [16],

$$G \in (\mathrm{var}\Sigma_1 \wedge \mathrm{var}\Sigma_2) \vee (\mathrm{var}\Sigma_1 \vee \mathrm{var}\Sigma_3) \leq (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2) \vee (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_3)$$

as required; thus assume that G is not nilpotent. Put $\mathsf{F} = (M_1H, M_1, H)$ in the notation of §4 and then it is clear that F, $F(G)$ generate the same variety (indeed the bigroups they carry generate the same variety - *cf*. (3.1.6) in [5]); also write F_X for the analogous bigroup corresponding to $X \in \Sigma_i$ $(i = 1, 2, 3)$; and write $\mathsf{D} = (NK, N, K)$. Hence, if

$$\Psi_i = \{\mathsf{F}_X : X \in \Sigma_i\} , \quad i = 1, 2, 3 ,$$

then it follows from (4.3.4) in [5] and (5.8) that

$$\mathsf{F} \in \mathrm{var}\Psi_1 \wedge (\mathrm{var}\Psi_2 \vee \mathrm{var}\Psi_3) .$$

But by (4.3.14) in [5], $\Lambda\left(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p\right)$ is distributive and therefore

$$\mathsf{F} \in (\mathrm{var}\Psi_1 \wedge \mathrm{var}\Psi_2) \vee (\mathrm{var}\Psi_1 \wedge \mathrm{var}\Psi_3) .$$

However Corollary 5.5 shows that

$$\begin{aligned} G^\# \cong \mathsf{F} \# \mathsf{D} &\in (\mathrm{var}\Psi_1 \wedge \mathrm{var}\Psi_2)^{\mathsf{D}} \vee (\mathrm{var}\Psi_1 \wedge \mathrm{var}\Psi_3)^{\mathsf{D}} \\ &\leq \left((\mathrm{var}\Psi_1)^{\mathsf{D}} \wedge (\mathrm{var}\Psi_2)^{\mathsf{D}}\right) \vee \left((\mathrm{var}\Psi_1)^{\mathsf{D}} \wedge (\mathrm{var}\Psi_3)^{\mathsf{D}}\right) ; \end{aligned}$$

and $\left(\mathrm{var}\Psi_i\right)^{\mathsf{D}} \leq \underline{\underline{W}}_i$ by Lemma 5.4. Finally, then,

$$G \in (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2) \vee (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_3) .$$

Proof of Theorem 5.1. Since $\underline{\underline{V}}$ is finitely based (two applications of Theorem 3.1 of Higman [11] show this) it suffices to prove that $\Lambda(\underline{\underline{V}})$ has descending chain condition. The following, easily proved, lemma will be used.

LEMMA 5.9. *A locally finite variety* $\underline{\underline{X}}$ *has descending chain condition on subvarieties if and only if to every set* $\left\{G_i : i \in I^+\right\}$ *of non-isomorphic critical groups in* $\underline{\underline{X}}$ *there exists* $i \in I^+$ *such that*

(5.10) $$G_i \in \mathrm{var}\{G_j : j \geq i+1\} .$$

It will be convenient to restate here some facts from earlier

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sections, in the process establishing some notation.

(5.11). *If* $G \in \underline{\underline{V}}$ *is critical and not nilpotent the Fitting subgroup* F $(= AH)$ *of* G *has a complement* K (see §4).

(5.12). $K = ST$ *where* $S \in \underline{\underline{A}}_m$ *is the centralizer in* K *of* σK *and* $T \in \underline{\underline{A}}_p$ *is a complement for* S *in* K .

(5.13). *By Lemma* 2.7 S *is a direct product of homocyclic subgroups* S_{ij} $(1 \le j \le r_i\ ,\ 1 \le i \le r)$ *which are normal in* K *and, as normal subgroups of* K *, indecomposable.* (*Assume that* $\exp S_{ij} = \exp S_{kl}$ *if and only if* $i = k$.) *The* σS_{ij} *are precisely all the minimal normal subgroups of* K *and* σK *is their direct product. Put* $S_i = \prod\{S_{ij} : 1 \le j \le r_i\}$ $(1 \le i \le r)$.

Now for some $\rho \in \{1, \ldots, r\}$ let W be a normal subgroup of K maximal with respect to containing $S_1S_2 \ldots S_\rho$ and avoiding $S_{\rho+1} \ldots S_r$ (so that

$$W = S_1S_2 \ldots S_\rho \cdot C_T(S_{\rho+1} \ldots S_r)\Big) .$$

For $i \in \{1, \ldots, \rho\}$ and arbitrary $j \in \{1, \ldots, r_i\}$ let X_{ij} be a normal subgroup of K maximal with respect to containing all S_{kl} with $(k, l) \ne (i, j)$ and avoiding S_{ij} (so that

$$X_{ij} = \left(\prod_{(k,l)\ne(i,j)} S_{kl}\right) \cdot C_T(S_{ij})\Bigg) .$$

Put

$$\Gamma = K/W\ ,\quad \Delta_{ij} = K/X_{ij}\ ,\quad 1 \le i \le \rho\ ,\quad 1 \le j \le r_i\ .$$

LEMMA 5.14. *Suppose* $Z(K) = 1$. *Then* $\Delta_{ij} \cong \Delta_{i1}$ $(1 \le j \le r\ )$ *and if* $\Sigma = \Gamma \times \prod_{i=1}^{\rho} \Delta_i^{r_i}$ (*where* $\Delta_i = \Delta_{i1}$ *and* $\Delta_i^{r_i}$ *denotes the* r_i*-fold*

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direct power of Δ_i) *there is an embedding* $\mu : K \to \Sigma$ *such that*

$$(\sigma K)\mu = \sigma\Sigma .$$

Proof. Since $Z(K) = 1$ none of S_{ij} can be central, and each Δ_{ij} is isomorphic to S_{ij} split-extended by an automorphism of order p. Consequently up to isomorphism Δ_{ij} is independent of j. By (5.13) $W \cap \bigcap_{i,j} X_{ij} = 1$, the σS_{ij} being the only minimal normal subgroups of K. This completes the proof.

LEMMA 5.15. *Suppose* $Z(K) = 1$. *Put* $C_0 = C_T(S_{\rho+1} \dots S_r)$ *and* $|T : C_0| = p^\delta$. *Then provided* $r_i \geq p^{\delta+s\rho}$ $(1 \leq i \leq \rho)$, K *contains a subgroup* $\Sigma_0 \cong \Gamma \times \prod_{i=1}^{\rho} \Delta_i^s$ *with* $\sigma\Sigma_0 \leq \sigma K$.

Proof. Note first that the number of inequivalent non-trivial irreducible representations of an elementary abelian group of order p^u is at most $p^u - 1$. Hence, since the minimal normal subgroups σS_{ij} of K afford inequivalent representations of T, by (5.13), a subgroup of index p^u in T centralizes at most $p^u - 1$ of the σS_{ij} and therefore centralizes at most (p^u-1) S_{ij}'s. The proof depends on repeated use of this fact.

Since $r_1 \geq p^{\delta+s\rho} > p^\delta - 1$ there exists $j(1, 1) \in \{1, \dots, r_1\}$ such that

$$C_1 = C_T(S_{1j(1,1)}) \not\geq C_0 .$$

But $|T : C_1| = p$ so that $|T : C_0 \cap C_1| = p^{\delta+1}$. Suppose inductively that for some ξ, η with $1 \leq \xi \leq \rho$ and $0 \leq \eta < s$ we have chosen subgroups C_v (of index p) of T, with $1 \leq v \leq (\xi-1)s + \eta$ such that

i) if $v = (\lambda-1)s + \mu$ $(1 \leq \lambda \leq \xi$, $0 \leq \mu \leq \eta)$ then
$C_v = C_T(S_{\lambda j(\lambda,\mu)})$ for some $j(\lambda, \mu) \in \{1, \dots, r_\lambda\}$;

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ii) if $D_v = \bigcap_{w=0}^{v} C_w$ then $|T : D_v| = p^{\delta+v}$.

For $\xi = 1$, $\eta = 1$ we have done this.

Now $r_\xi \geq p^{\delta+s\rho} = p^{[\delta+(\xi-1)s+\eta]+[(\rho-\xi+1)s-\eta]} > p^{\delta+(\xi-1)s+\eta} - 1$. Hence there exists $j(\xi, \eta+1) \in \{1, \ldots, r_\xi\}$ such that

$$C_{(\xi-1)s+\eta+1} = C_T(S_{\xi j(\xi,\eta+1)}) \not\supseteq D_{(\xi-1)s+\eta} .$$

Also $C_{(\xi-1)s+\eta+1}$ has index p in T and therefore

$$|T : D_{(\xi-1)s+\eta+1}| = p^{\delta+(\xi-1)s+\eta+1}$$

as required. In case $\xi < \rho$ and $\eta = s - 1$ the proof of the inductive step is similar – we choose the next C from among the centralizers of the $S_{\xi+1j}$.

It may happen that $D_{\rho s} \neq 1$. If that is the case continue choosing centralizers C_v , for $v > \rho s$, so that none contains the intersection of all previous C's : this can be done since the intersection of the centralizers of all S_{ij} is 1 (by (5.13)). Indeed if $|C_0| = p^\gamma$ then γ is the first value of v for which $D_v = 1$. Put

$$I_v = \bigcap_{w \neq v} C_w .$$

It is a simple matter to compute that

I_0 is a complement for C_0 in T ;

and

for $v > 0$, each I_v has order p and C_0 is their direct product.

Hence if

$$\Gamma^* = I_0 S_{\rho+1} \ldots S_r , \quad \Delta^*_{\xi\eta} = I_v S_{\xi j(\xi,\eta)} , \quad 1 \leq v = \xi s + \eta ,$$

then clearly Γ^*, $\Delta^*_{\xi\eta}$ generate their direct product and, since $\Gamma^* \cong \Gamma$,

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$\Delta^*_{\xi\eta} \cong \Delta_\xi$, the proof of Lemma 5.15 is complete.

Suppose now that $S = \left\{G_j : j \in I^+\right\}$ is a set of non-isomorphic critical groups in $\underline{\underline{V}}$. If infinitely many of them are nilpotent then (5.10) is satisfied for S by Theorem 4 in [16]; hence we may suppose (by taking an infinite subset of S instead if necessary) that all groups in S are non-nilpotent. Further we may suppose that, in the notation of (5.13), the quantities r, $\exp S_i$ $(1 \le i \le r)$ are independent of $G \in S$ (again by replacing S by an infinite subset if necessary). Choose $\rho \in \{1, \ldots, r\}$ by

$$\rho+1 \le i \le r \iff \{r_i : G \in S\} \text{ is bounded.}$$

Then the groups Γ corresponding to $G \in S$ have bounded order and hence we may suppose that Γ is (up to isomorphism) independent of G . By (5.13) $Z(K)$ are all cyclic, hence there are but finitely many choices for $Z(K)$ and so we may assume that for $G \in S$, $Z(K)$ are all isomorphic. As a final simplification we may assume that the sequence of ρ-tuples $\left(r_1, \ldots, r_\rho\right)$ is ordered by components in the natural ordering of S .

With S whittled down this far we have

LEMMA 5.16. *For each* $x \in I^+$ *there exists* $j(x) > x$ *and embeddings* $\mu_{xl} : K_x \to K_l$ *for* $l \ge j(x)$ *such that*

$$\left(\sigma K_x\right)\mu_{xl} \le \sigma K_l , \quad l \ge j(x) .$$

Proof. For $G \in S$ we may write

$$K = Z(K) \times \hat{K}$$

where $\hat{K}$ has trivial centre and satisfies the conditions (5.12), (5.13); and $Z(K)$ is cyclic, independent of G , and of course equal to some S_{ij} . By Lemmas 5.14 and 5.15 applied to the $\hat{K}$'s if we choose $j = j(x)$ so that

$$r_i\left(G_j\right) \ge p^{\delta+r_i\left(G_x\right)\rho} , \quad 1 \le i \le \rho ,$$

then there is a monomorphism $\mu_{xl} : \hat{K}_x \to \hat{K}_l$ $\left(l \ge j(x)\right)$, and this does

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what we want.

Finally consider the sequence in I^+ defined by

$$l(1) = 1 \ , \quad l(n) = j\big(l(n-1)\big) \ , \quad n \in I^+ \ .$$

In the notation introduced at the beginning of this section if $G_i \in S$ then $G_i \cong \mathsf{F}_i \# \mathsf{D}_i$. By (4.2.29) in [5], and the analogue of Lemma 5.9 for bivarieties, there exists $n \in I^+$ such that

$$\mathsf{F}_{l(n)} \in \operatorname{svar}\{\mathsf{F}_{l(n+1)}, \mathsf{F}_{l(n+2)}, \ldots\} \ .$$

Consequently there exists i $\big(= l(n)\big)$ such that

(5.17) $$\mathsf{F}_i \in \operatorname{svar}\{\mathsf{F}_{l(i)}, \mathsf{F}_{l(i)+1}, \ldots\} \ .$$

Now if $G \in \{G_{l(i)}, G_{l(i)+1}, \ldots\}$ Lemma 5.16 shows that K has a subgroup $K^* \cong K_i$ with

$$\sigma K^* \leq \sigma K \ .$$

Since N is principal indecomposable, N_{K^*} has a component N^* which is principal indecomposable. By Lemma 2.1 N^* is faithful for K^* and by Theorem 2.2

$$N^*K^* \cong N_iK_i \ .$$

Hence for each $G \in \{G_l : l \geq j(i)\}$, $G^{\#}$ has a subgroup isomorphic to $\mathsf{F} \# \mathsf{D}_i$. It follows from (5.17) and Lemma 5.4 that $G_i \in \operatorname{var}\{G_l : l \geq j(i)\}$. By Lemma 5.9 the proof of Theorem 5.1 is complete.

Proof of Theorem 5.6. Suppose that $\underline{\underline{U}} \leq \underline{\underline{V}}$ is join-irreducible and not locally nilpotent. Since $\underline{\underline{U}}$ is generated by its critical groups it is generated by a set S of non-nilpotent critical groups, and we may further assume that for all $G \in S$, $\exp A$ is constant (in the notation of §4): we may as well assume that $\exp A$ are all equal to p^α. We now show that the number of similarity classes of the σG $(G \in S)$ is finite and therefore, of course, the groups of S may be assumed to belong to a

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single similarity class. The next lemma accomplishes this.

LEMMA 5.18. *Let* S' *be a set of non-nilpotent critical groups in* $\underline{\underline{V}}$ *such that for all* $G \in S'$, $\exp A = p^{\alpha}$ *and, furthermore, the sequence* $\{|K| : G \in S'\}$ *is unbounded. Then* $\mathrm{var}S' = \underline{\underline{V}}$.

Proof. Let G_0 be an arbitrary critical group in $\underline{\underline{V}}$. We show that G_0 is in $\mathrm{var}G$ for some $G \in S'$, this being sufficient to prove the lemma. It follows from Lemmas 5.14 and 5.15 (as in the proof of Lemma 5.16) that for all suitably large K , K_0 is isomorphic to a subgroup $\overline{K}$ of K with

(5.19) $$\sigma\overline{K} \leq \sigma K \text{ .}$$

Moreover since the index of $C_T(\overline{K})$ in T is bounded in terms of K_0 , we may assume that

$$|C_T(\overline{K})| \geq |H_0| \text{ .}$$

Put $\overline{H} = C_T(\overline{K})$. Then $N_{\overline{HK}}$ is a direct sum of principal indecomposables of $\overline{HK}$ over R_α - call one $\overline{A}$ say. Now since $\overline{A}$ is monolithic and co-monolithic and since $\overline{K}$ acts faithfully and irreducibly on $\sigma\overline{A}$ (by (5.19) and Lemma 2.1) we can use §4 to deduce that

$$\overline{A} \cong \overline{M}_1 \otimes_{R_\alpha} \overline{N}$$

where $\overline{M}_1$ is a principal indecomposable of $\overline{H}$ and $\overline{N}$ a principal indecomposable of $\overline{K}$ (this follows since $\overline{M}_1$ is the regular of $\overline{H}$, hence $\sigma\overline{M}_1$ is one dimensional trivial and so $\sigma\left(\overline{M}_1 \otimes \overline{N}\right)_{\overline{K}} \cong \overline{N}$ whence $\ker\tau = 0$) . If $\overline{\mathrm{F}} = (\overline{AH}, \overline{A}, \overline{H})$ and $\overline{\mathrm{D}} = (\overline{NK}, \overline{N}, \overline{K})$ then

$$\overline{AHK} \cong \overline{\mathrm{F}} \;\#\; \overline{\mathrm{D}} \text{ ;}$$

and since F_0 is clearly a homomorphic image of $\overline{\mathrm{F}}$, $G_0^{\#}$ (and therefore G_0) is a homomorphic image of $\overline{\mathrm{F}} \# \overline{\mathrm{D}}$. This shows that G_0 is in $\mathrm{var}G$ for some $G \in S'$ as required.

We may assume, therefore, that for some fixed $\mathrm{D} = (D, A_2, B_2)$ with A_2 a faithful principal indecomposable $R_\alpha B_2$-module and $B_2 \in \underline{\underline{A}}_q\underline{\underline{A}}_p$,

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$$G \in S \Rightarrow G^{\#} \cong F \# D, \exists F \in \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p .$$

Hence by Lemma 5.4 if $S = \text{svar}\{F : G \in S\}$,

$$\underline{\underline{U}} = S^{D} .$$

By Corollary 5.5 and (1.14) of [14], S is join irreducible.

Conversely if S is join irreducible suppose

$$S^{D} = \underline{\underline{U}}_1 \vee \underline{\underline{U}}_2 .$$

By (1.12) of [14] we may assume that each of $\underline{\underline{U}}_1, \underline{\underline{U}}_2$ is generated by critical groups the similarity class of whose monoliths is determined by D . That is we may assume $\underline{\underline{U}}_1, \underline{\underline{U}}_2$ generated by groups of the form $F \# D$ $\left(F \in \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p\right)$, and hence for suitable $S_i \in \Lambda\left(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p\right)$,

$$\underline{\underline{U}}_i = S_i^{D} , \quad i = 1, 2 .$$

Corollary 5.5 and (1.14) of [14] then shows that $S = S_1 \vee S_2$ whence $S_1 \geq S_2$, say, and finally $\underline{\underline{U}}_1 \geq \underline{\underline{U}}_2$. In other words S^{D} is join-irreducible.

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