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EFFECTIVE COMPUTATION OF THE GELFAND-KIRILLOV DIMENSION*

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In this note we propose an effective method based on the computation of a Gröbner basis of a left ideal to calculate the Gelfand-Kirillov dimension of modules.

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Introduction

Computation of the Gelfand-Kirillov dimension of modules over a noetherian ring is a hard problem. In this note we propose an effective method based on the computation of a Gröbner (or standard) basis of a left ideal. To simplify the exposition we assume that the noetherian ring R has a filtration $(R_m)_{m\in\mathbb{N}}$ such that the associated graded ring $gr(R)$ is an affine commutative domain and the filtration is standard. A ring of this kind is essentially a quotient ring of an enveloping algebra $U(\mathbf{g})$ of a finite dimensional Lie algebra $\mathbf{g}$. Therefore we use $U(\mathbf{g})$ as frame to develop the theory.

The main result is given in Theorem 2.5. In it we describe an algorithm to give a standard basis of a left ideal J of $U(\mathbf{g})$. The algorithm follows that given by Buchberger [2] in the case of a commutative polynomial ring over a field. As a byproduct of the method developed we give some criteria to compute the Gelfand-Kirillov dimension of $M = U(\mathbf{g})/J$ (i.e. the degree of the Hilbert-Samuel polynomial) and the Bernstein number $a(GK(M)!)$ where a is the leading coefficient of the afore-mentioned polynomial.

1. Almost commutative algebras

In the study of non commutative rings, a fundamental tool has been to find associated commutative rings and translate properties from the commutative case to the non commutative one. Let us focus on the Weyl algebra and the universal enveloping algebra of a finite dimensional Lie algebra. In these cases the non commutative ring R has an (increasing) filtration $(R_m)_{m\in\mathbb{N}}$ such that the graded

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associated ring is commutative, even more, is an affine commutative algebra. If we analyse the ring R we see that the filtration has the following properties:

- $R_0 = \mathbf{k}$.
- R_1 generates R as a $\mathbf{k}$-algebra and $\dim_{\mathbf{k}}(R_1) < +\infty$.
- $R_m = R_1^m$ for any m.
- The ring $gr(R)$ is commutative.

Our purpose is study such filtered algebras which will be called, after Duflo, almost commutative algebras, see [**10**, p. 299] for a precise definition.

Duflo proves that these algebras may be described exactly as homomorphic images of $U(\mathbf{g})$, where $\mathbf{g}$ is a finite dimensional Lie $\mathbf{k}$-algebra [**6**].

1.1. Let M be a finitely generated left R-module with a finite dimensional standard filtration $(M_m)_{m\in\mathbf{N}}$, see [**8**]. Following [**10**, p. 302], we consider the Hilbert-Samuel polynomial relative to this filtration

$$p_M^{HS}(m) := dim_{\mathbf{k}}(M_m) \quad \text{for } m \gg 0.$$

The degree of the Hilbert-Samuel polynomial p_M^{HS} is independent of the finite dimensional filtration on M ([**8**, p. 91]) and it is equal to $GK(M)$, the Gelfand-Kirillov dimension of M. On the other hand, the leading coefficient of p_M^{HS} does depend on the standard filtration considered on R [**8**, p. 92]. Let a be the leading coefficient of p_M^{HS}. The integer $e_M := a \cdot GK(M)!$ is called the *multiplicity* or the *Bernstein number* of M.

For completeness let us recall a link between Krull dimension and Gelfand-Kirillov dimension. Thus we have that if M is a finitely generated R-module (with a finite dimensional filtration), then $GK(M) = Kdim_{gr(R)}(gr(M))$.

2. The division theorem

As in the above section let R be an almost commutative algebra with standard filtration $(R_m)_{m\in\mathbf{N}}$. In order to study (left) ideals in R, using the Duflo theorem, we can build a finite dimensional Lie algebra $\mathbf{g}$ such that $R \cong U(\mathbf{g})/L$ with L a two-sided ideal in $U(\mathbf{g})$. Then, if I is a left ideal of R there exists a left ideal J of $U(\mathbf{g})$ such that $I \cong J/L$. Therefore for studying left ideals on R it is enough to study left ideals of the enveloping algebra of a finitely dimensional Lie algebra.

Throughout this paper $\mathbf{g}$ will be a finite dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$. Using the Poincaré-Birkhoff-Witt theorem a basis of $U := U(\mathbf{g})$, as a $\mathbf{k}$-vector space, is given by the set $\{x_1^{\alpha_1} \ldots x_n^{\alpha_n} : (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n\}$. For each element $P \in U$ we may write P, uniquely, as a finite sum $P = \sum_{\alpha\in\mathbf{N}^n} p_\alpha X^\alpha$, where $p_\alpha \in \mathbf{k}$ and X^α means $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The set of α such that $p_\alpha \neq 0$ is called the *Newton diagram* of P and is denoted by $\mathcal{N}(P)$. Using the standard filtration $(U_m)_{m\in\mathbf{N}}$ on $U(\mathbf{g})$, the associated graded ring is isomorphic to the polynomial ring in n indeterminates. For any element $P \in U$, we define the *principal symbol* of P (and we denote $\sigma(P)$) as the coset

$\sigma(P) = P + U_{m-1}$ where $P \in U_m \backslash U_{m-1}$. It is clear that $\sigma(P)$ is a homogeneous polynomial.

As in the case of a commutative polynomial ring, the above situation allows us to treat monomials in U as points in $\mathbf{N}^n$ and use combinatory methods in the study of (left) ideals on U.

2.1. Good compatible orders

Let $<$ be a total order on $\mathbf{N}^n$, compatible with the addition, satisfying $0 < \alpha$ for any $0 \neq \alpha \in \mathbf{N}^n$. It is clear that if $<$ satisfies the above properties then it is a good order on $\mathbf{N}^n$ (i.e. any non empty subset $A \subset \mathbf{N}^n$ has a first element). Using this total order, we will define the principal exponent and monomial of an element of U.

Definition 2.1. Let $P \in U$ be a non zero element. The *principal exponent* of P is defined as the greatest element of $\mathcal{N}(\sigma(P))$ with respect to the order $<$, it is denoted by $\mathrm{pe}(P)$, where $\mathcal{N}(\sigma(P))$ is defined analogously to $\mathcal{N}(P)$. If α is the principal exponent of P then $p_\alpha X^\alpha$ is defined as the *principal monomial* of P and it is denoted by $\mathrm{pm}(P)$.

It is clear that $\mathrm{pm}(P \cdot Q) = \mathrm{pm}(P) \cdot \mathrm{pm}(Q)$. If J is a left ideal of U we define $PE(J)$ the set $\{\mathrm{pe}(P) : P \in J\}$. Therefore from the definition we have $PE(J) + \mathbf{N}^n = PE(J)$, and so $PE(J)$ is an ideal in $\mathbf{N}^n$.

2.2. Standard bases and the division theorem

Definition 2.2. Let J be a left ideal of U. A finite subset $\{P_1, \ldots, P_t\} \subseteq J$ is a *standard basis* of J if

$$PE(J) = \bigcup_{i=1}^{t} (\mathrm{pe}(P_i) + \mathbf{N}^n).$$

By Dickson's lemma, see [5], $PE(J)$ is a finitely generated ideal, i.e. there exists $\{\alpha^1, \ldots, \alpha^s\} \subset PE(J)$ such that

$$PE(J) = \bigcup_{i=1}^{s} (\alpha^i + \mathbf{N}^n).$$

As a consequence, if we take for any index i an element P_i in J such that $\mathrm{pe}(P_i) = \alpha^i$, the set $\{P_1, \ldots, P_s\}$ is a standard basis of J.

At this moment an algorithm to build a standard basis may be developed.

Let $(\alpha^1, \ldots, \alpha^s) \in (\mathbf{N}^n)^s$, we consider the associated partition of $\mathbf{N}^n$ defined by

- $\Delta_1 = \alpha^1 + \mathbf{N}^n$
- $\Delta_i = (\alpha^i + \mathbf{N}^n) \backslash (\cup_{j<i} \Delta_j)$
- $\overline{\Delta} = \mathbf{N}^n \backslash (\cup_{i=1}^{s} \Delta_i)$

Theorem 2.3. (Division Theorem). *Let* $(P_1, \ldots, P_s) \in U^s$ *with* $P_i \neq 0$ *and let* $\{\Delta_1, \ldots, \Delta_s, \overline{\Delta}\}$ *the partition associated to* $(\mathrm{pe}(P_1), \ldots, \mathrm{pe}(P_s))$. *Then, for each* P *in* U *there exists a unique* $(Q_1, \ldots, Q_s, R) \in U^{s+1}$ *satisfying:*

(1) $P = \sum_i Q_i P_i + R$.

(2) $\mathrm{pe}(P_i) + \mathcal{N}(Q_i) \subset \Delta_i$ *and* $\mathrm{pe}(Q_i P_1) \leq \mathrm{pe}(P)$ *for* $i = 1, \ldots, s$.

(3) $\mathcal{N}(R) \subset \overline{\Delta}$ *and* $\mathrm{pe}(R) \leq \mathrm{pe}(P)$.

Proof. First we prove the existence by induction on $\mathrm{pe}(P)$. We may suppose $P \neq 0$. First we assume P is a constant. If there exists i such that $\mathrm{pe}(P_i) = (0, \ldots, 0)$ then we may write $P = (1/P_i)P \cdot P_i$. If, for all i, P_i is not a constant then $\mathcal{N}(P) \subset \overline{\Delta}$ and the division is given by $P = \sum_i 0 \cdot P_i + P$. On the other hand, if P is not constant, we may assume that P is a monomial i.e. $\mathrm{pm}(P) = P$. If there exists i such that $\mathrm{pe}(P) \in \Delta_i$ then $\mathrm{pm}(P) = \mathrm{pm}(M \cdot P_i)$ where M is a monomial. We may write $P' = P - M \cdot P_i$. By construction, we have $\mathrm{pe}(P') < \mathrm{pe}(P)$. Hence, by induction, we may write $P' = \sum_j Q'_j P_j + R'$ and finally $P = \sum_{j \neq i} Q'_j P_j + (M + Q'_i)P_i + R'$. By the contrary, if $\mathrm{pe}(P) \in \overline{\Delta}$ then $P = \sum_i 0 \cdot P_i + P$. This proves the existence.

For the uniqueness, let $P = \sum_i Q_i P_i + R = \sum_i Q'_i P_i + R'$ satisfy conditions (2) and (3) in the theorem. We have $\sum_i (Q_i - Q'_i)P_i + (R - R') = 0$ and $\mathrm{pe}((Q_i - Q'_i)P_i) \in \Delta_i$, $\mathrm{pe}(R - R') \in \overline{\Delta}$; on the other hand, $\{\Delta_1, \ldots, \Delta_s, \overline{\Delta}\}$ is a partition, hence $R = R'$ and for any i we have $Q_i = Q'_i$. □

Remark. This theorem was first proved by Castro [**3**, **4**] for the Weyl algebra and by Apel and Lassner [**1**] for $U(\mathbf{g})$.

Corollary 2.4. *Any standard basis of a left ideal* J *of* U *is a system of generators of* J.

Proof. Let $\{P_1, \ldots, P_t\}$ be a standard basis of J. We use the notation of the theorem. Then for any $P \in J$ we may write, by the Division Theorem, $P = \sum_i Q_i P_i + R$, with $\mathcal{N}(R) \subset \overline{\Delta}$. Hence $R \in J$ and $\mathrm{pe}(R) \in PE(J) = \cup_i \Delta_i$, which is a contradiction unless $R = 0$. □

2.3. The construction of standard bases

In this section, starting from a system of generators we will build a standard basis of a left ideal J of U.

For any pair (P, Q) of elements in U, let T be the monic least common multiple of $\mathrm{pm}(P)$ and $\mathrm{pm}(Q)$. We define the *semisyzygy* of (P, Q) as the element $S(P, Q) = M \cdot P - N \cdot Q$, where M and N are monomials such that $\mathrm{pm}(M \cdot \mathrm{pm}(P)) = \mathrm{pm}(N \cdot \mathrm{pm}(Q)) = T$. The following theorem is due to Buchberger [**2**] in the case of a commutative polynomial ring.

Theorem 2.5. *Let* $\mathcal{F} = \{P_1, \ldots, P_s\}$ *be a system of generators of* J. *If the remainder of the division of any semisyzygy* $S(P_i, P_j)$ *by* $(P_1, \ldots, P_s)$ *is zero, then* $\mathcal{F}$ *is a standard basis of* J.

Proof. We follow [9, pp. 47–48]. It is enough to prove the inclusion $PE(J) \subset \cup_i(\mathrm{pe}(P_i) + \mathbf{N}^n) = \Delta$. Let $P = \sum_i Q_i P_i$ be an element in J. Let $\gamma^i = \mathrm{pe}(Q_i P_i)$ and γ the supremum of all γ^i. Let $\{i_0, \ldots, i_t\}$ the set of indexes i such that $\gamma^i = \gamma$. If $t = 0$ then $\gamma = \mathrm{pe}(P) = \mathrm{pe}(Q_{i_0}) + \mathrm{pe}(P_{i_0}) \in \Delta$. On the other hand, suppose $t > 0$. Denote $\mathrm{pe}(P_i) = \alpha^i$ and $\mathrm{pe}(Q_i) = \beta^i$. Let $S = S(P_{i_0}, P_{i_1}) = M_{i_0} P_{i_0} - M_{i_1} P_{i_1}$ be the corresponding semisyzygy. Let us denote by τ the principal exponent of $l.c.m.(\mathrm{pm}(P_{i_0}), \mathrm{pm}(P_{i_1}))$. Then there exists $\delta \in \mathbf{N}^n$ such that $\gamma = \tau + \delta$. If $\mu^{i_j} = \mathrm{pe}(M_{i_j})$ then $\tau = \mu^{i_0} + \alpha^{i_0} = \mu^{i_1} + \alpha^{i_1}$ and we have the following relation: $\beta^{i_j} + \alpha^{i_j} = \gamma = \mu^{i_j} + \alpha^{i_j} + \delta$. From this we obtain $\beta^{i_j} = \mu^{i_j} + \delta$. As a consequence there exists a monomial D in U such that $\mathrm{pm}(Q_{i_j}) = DM_{i_j} +$ other terms of lower degree. Put $H_{i_j} = \mathrm{pm}(Q_{i_j}) - DM_{i_j}$. Thus we have the following identity

$$\begin{aligned} Q_{i_0} P_{i_0} &= \mathrm{pm}(Q_{i_0}) P_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0})) P_{i_0} \\ &= DM_{i_0} P_{i_0} + H_{i_0} P_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0})) P_{i_0}. \end{aligned}$$

On the other hand we have $S = S(P_{i_0}, P_{i_1}) = \sum_i S_i P_i$ with $\mathrm{pe}(S_i P_i) \le \mathrm{pe}(S)$ by the Division Theorem, 2.3, also, by construction, we have $\mathrm{pe}(S) < \tau$. We proceed as follows:

$$\begin{aligned} P &= Q_{i_0} P_{i_0} + \sum_{j \ne i_0} Q_j P_j \\ &= DM_{i_0} P_{i_0} + H_{i_0} P_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0})) P_{i_0} + \sum_{j \ne i_0} Q_j P_j \\ &= DM_{i_1} P_{i_1} + DS + H_{i_0} P_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0})) P_{i_0} + \sum_{j \ne i_0} Q_j P_j \\ &= DM_{i_1} P_{i_1} + \sum_i DS_i P_i + H_{i_0} P_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0})) P_{i_0} + \sum_{j \ne i_0} Q_j P_j \\ &= (DS_{i_0} + H_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0}))) P_{i_0} + (DS_{i_1} DM_{i_1} + Q_{i_1}) P_{i_1} + \sum_{j \ne i_0, i_1} Q_j P_j + \sum_{j \ne i_0, i_1} DS_j P_j. \end{aligned}$$

Let us call

- $Q'_{i_0} = DS_{i_0} + H_{i_0} + (Q_{i_0} - \mathrm{pm}(Q_{i_0}))$.
- $Q'_{i_j} = DS_{i_1} + DM_{i_1} + Q_{i_1}$.
- $Q'_j = Q_j + DS_j$ for $j \ne i_0, i_1$.

Hence we have a new expression for $P, P = \sum Q'_j P_j$. Let us compute, for all j, the principal exponent of $Q'_j P_j$. If $j = i_0$ the principal exponent must be obtained from the product $DS_{i_0} P_{i_0}$, but it is bounded by $\delta + \mathrm{pe}(S_{i_0} P_{i_0}) \le \delta + \mathrm{pe}(S) < \delta + \tau = \gamma$. If $j = i_1$ the principal exponent of $Q'_{i_1} P_{i_1}$ is bounded by $max_{<}\{\mathrm{pe}(DS_{i_1} P_{i_1}), \mathrm{pe}(DM_{i_1} P_{i_1}), \mathrm{pe}(Q_{i_1} P_{i_1})\} \le \gamma$. Finally, it is clear that $\mathrm{pe}(Q'_j P_j) = \mathrm{pe}(Q_j P_j + DS_j P_j) \le \gamma$. Now, by induction on γ and t we finish the proof. □

Let $\mathcal{F} = \{P_1, \ldots, P_s\}$ be a system of generators of J. We say that $\mathcal{F}$ *has an extension* if there exists a semisyzygy $S(P_i, P_j)$ with non zero remainder $R_{i,j}$ with respect to

$(P_1, \dots, P_s)$. The system $\mathcal{F}_1 = \mathcal{F} \cup \{R_{i,j}\}$ is called an *extension* of $\mathcal{F}$. From the noetherian condition of $\mathbf{N}^n$ we can find for any system of generators $\mathcal{F}$ of J an extension $\hat{\mathcal{F}} = \mathcal{F}_r = \{P_1, \dots, P_s, P_{s+1}, \dots, P_{s+r}\}$ such that $\hat{\mathcal{F}}$ has no extension. Hence, by Theorem 2.5, $\hat{\mathcal{F}}$ is a standard basis of J.

3. Applications

In this section we apply the construction of a standard basis of the ideal J to compute the Hilbert-Samuel polynomial of the left U-module $M = U/J$. We denote by $(M_m)_{m \in \mathbf{N}}$ the induced filtration on M, i.e., $M_m = (U_m + J)/J$.

For $m \in \mathbf{N}$ we denote by $PE(J)_m$ the set $\{\alpha \in PE(J) : |\alpha| \le m\}$.

Proposition 3.1. *With the above notation, we have* $dim_{\mathbf{k}}(M_m) = cardinal(\mathbf{N}^n \backslash PE(J)_m)$.

Proof. By the division theorem, a $\mathbf{k}$-basis for $(U_m + J)/J$ is given by the set $\{X^\alpha + J : \alpha \notin PE(J), |\alpha| \le m\}$. □

Starting from a system of generators of J we may compute a standard basis $\{P_1, \dots, P_s\}$ of J. From this standard basis we may compute the set $PE(J)$ and for every $m \in \mathbf{N}$ the cardinal of $\mathbf{N}^n \backslash PE(J)_m$. Now, using E. R. Kolchin [7, p. 49], we may compute the Hilbert-Samuel polynomial of $PE(J)$ which coincides with p_M^{HS} by the above proposition. It is an easy exercise to prove that the degree of p_M^{HS}, which coincides with $GK(M)$, may be computed as the greatest dimension of the coordinate varieties contained in $\mathbf{N}^n \backslash PE(J)$, where a coordinate variety is a variety with equation $x_{i_1} = \dots = x_{i_r} = 0$.

In the same way that the dimension is calculated, the reader could compute the multiplicity (or the Bernstein number) of M (see 1.1), by following the next algorithm. Let W be a coordinate variety of maximal dimension contained in $\mathbf{N}^n \backslash PE(J)$ with equation $x_{i_1} = \dots = x_{i_r} = 0$. Let $\{j_1, \dots, j_{n-r}\}$ the complement of $\{i_1, \dots, i_r\}$ in $\{1, \dots, n\}$. By the maximality of W the set

$$\{\beta \in \mathbf{N}^n : \beta_{j_1} = \dots = \beta_{j_{n-r}} = 0, (\beta + W) \subset (\mathbf{N}^n \backslash PE(J))\}$$

is finite. If we call e_W its cardinal, the multiplicity of M is exactly the sum of the e_W for all the coordinate varieties W of maximal dimension contained in $\mathbf{N}^n \backslash PE(J)$ [3, 5].

Finally we remark that the results contained in this note may be extended to calculate the Gelfand-Kirillov dimension of a finitely generated $U(\mathbf{g})$-module given by a finite presentation. To do this it is necessary to prove a division theorem for the free module U^m.

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