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CONTRACTED, $\mathfrak{m}$-FULL AND RELATED CLASSES OF IDEALS IN LOCAL RINGS

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Abstract. The class of $\mathfrak{m}$-full and four related classes of ideals in a local ring $(R, \mathfrak{m})$ are extended by replacing $\mathfrak{m}$ with other ideals and the resulting classes of ideals are compared. It is shown that contracted ideals are $\mathfrak{m}$-full in a local ring with infinite residue field.

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1. Introduction. Throughout let $(R, \mathfrak{m})$ be a local (Noetherian) ring. An $\mathfrak{m}$-primary ideal I of R is said to be *contracted* if $IR[\frac{\mathfrak{m}}{x}] \cap R = I$ for some regular $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. An R-ideal I is said to be *full* if $I :_R \mathfrak{m} = I :_R x$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Contracted ideals and full ideals played important roles in Zariski's factorisation theorem for complete ideals in a two-dimensional regular local domains (see for example [**20**, Appendix 5] and [**8**]). An R-ideal I is said to be $\mathfrak{m}$-*full* if $\mathfrak{m}I :_R x = I$ for some $x \in \mathfrak{m}$. This class of ideals was first considered by D. Rees (unpublished) and has received substantial attention since the first papers [**4**, **18**] on this topic appeared. An R-ideal I is said to have the *Rees property* if $\mu(J) \leq \mu(I)$ for any ideal J containing I with finite colength, $\lambda(J/I) < \infty$, where $\mu(I)$ denotes the minimal number of generators, $\lambda(I/\mathfrak{m}I)$, of I. It was shown in [**4**] and [**18**] that if $R/\mathfrak{m}$ is infinite, then $\mathfrak{m}$-full ideals have the Rees property. A proper ideal I of R is said to be *basically full* if no minimal set of generators of I can be extended to a minimal set of generators of an R-ideal that properly contains I [**6**, Definition 2.1]. It is shown in [**6**, Theorem 2.1] that a basically full ideal of R is $\mathfrak{m}$-primary and in [**6**, Theorem 2.12] that an $\mathfrak{m}$-primary ideal is basically full if and only if $I = (I\mathfrak{m} :_R \mathfrak{m})$.

In a recent paper [**7**], these five classes of ideals were compared to each other and to the class of integrally closed ideals. In this paper, after developing a few basic facts about closure operations on the set of ideals of a ring, two particular closure operations are used to show that contracted ideals are $\mathfrak{m}$-full in any local ring $(R, \mathfrak{m})$ with $R/\mathfrak{m}$ infinite. This greatly simplifies the diagram of implications between the above classes of ideals, which was given in [**7**, p. 2628]. Further, it is shown that all of the implications in this diagram continue to hold if the above five definitions are generalised by replacing the maximal ideal $\mathfrak{m}$ of R by another ideal L of R. Usually when one considers ideals, which are contracted from $R[\mathfrak{m}/x]$, it is assumed that $x \in \mathfrak{m}$ is regular. Thus, when considering ideals, which are contracted from $R[L/x]$ for some ideal L, we will assume that $x \in L$ is regular.

This paper is organised as follows. In Section 1, we give the definitions of closure operations, semi-prime and prime operations and a few relevant examples. In Section 2, it is shown that for any regular ideal L of the local ring R, there is a semi-prime operation

$I \mapsto I_{\star_L}$ such that $I = I_{\star_L}$ if and only if there exists a regular element $x \in L$ such that $IR[L/x] \cap R = I$. Thus, $I_{\star_L}$ is the unique smallest ideal J of R such that $I \subseteq J$ and $JR[L/x] \cap R = J$ for some regular $x \in L$. Finally, in Section 3, after extending the notion of $\mathfrak{m}$-full to L-full for other ideals L as in [**15**, p. 42], we define a semi-prime operation $I \mapsto I_*$ with the property that I_* is L-full for each ideal I, and then use it and the L-contraction closure $I_{\star_L}$ to show that if $R/\mathfrak{m}$ is infinite, then the L-contracted ideals of $(R, \mathfrak{m})$ are L-full.

2. Closure operations. In this section, we give the definition and some examples of closure operations. For some further examples and applications of closure operations, see for example [**3**, **16**, **17**]. We use the following terminology from [**10**, **11**, **13**].

DEFINITION 2.1. Let $I \mapsto I_c$ be an operation on the set of ideals I of a ring R, and consider the following rules, where I and J are ideals of R and b is a regular element in R: (a) $I \subseteq I_c$; (b) if $I \subseteq J$, then $I_c \subseteq J_c$; (c) $(I_c)_c = I_c$; (d) $IJ_c \subseteq (IJ)_c$ and (e) $(bI)_c = bI_c$. Then, $I \mapsto I_c$ is a *closure operation* if (a)–(c) hold for all ideals I and J in R, it is a *semi-prime operation* if (a)–(d) hold for all ideals I and J in R, and it is a *prime operation* if (a)–(e) hold for all ideals I and J and regular non-units b of R.

REMARK 2.2. The following are easily seen to hold for any semi-prime operation $I \mapsto I_c$ on the set of ideals I of R, where Γ is an index set: (1) $(I_cJ_c)_c = (IJ)_c$; (2) $(\sum_{i\in\Gamma}(I_i)_c)_c = (\sum_{i\in\Gamma} I_i)_c$ and (3) $(\cap_{i\in\Gamma}(I_i)_c)_c = \cap_{i\in\Gamma}(I_i)_c$.

There are many well-known examples of closure operations including integral closure and tight closure. In the following, we list a few others, which we will refer to later.

EXAMPLE 2.3. The Δ-closure is a semi-prime operation [**11**]. Let R be a commutative ring with identity and Δ a multiplicatively closed set of non-zero finitely generated ideals of R. If I is an ideal in R, then $\mathbf{D}(I) = \{IK :_R K \mid K \in \Delta\}$ is a directed set and $\cup\{IK :_R K \mid K \in \Delta\} = \sum_{K\in\Delta}(IK :_R K)$ is an ideal I_Δ called the *delta-closure of I*.

EXAMPLE 2.4. Let R be a ring, let $\{f_\lambda : R \to R_\lambda \mid \lambda \in \Lambda\}$ be a family of ring homomorphisms, and for each $\lambda \in \Lambda$, let $I \mapsto I_{c_\lambda}$ be a closure operation on R_λ. The closure operation induced from the family of closure operations $\{c_\lambda \mid \lambda \in \Lambda\}$ is defined by $I \mapsto \cap\{f_\lambda{}^{-1}((IR_\lambda)_{c_\lambda}) \mid \lambda \in \Lambda\}$.

EXAMPLE 2.5. Fix an R-module M and define $I^M = (IM :_R M)$. Then, $I \mapsto I^M$ is a semi-prime operation. In fact, the proof for the special case $M = \mathfrak{m}$ [**6**, Proof of Theorem 4.2] goes through without change in this case. It was shown by Yongwei Yao [**19**, Theorem 2.5(ii) and Remark 2.6] that under certain conditions on a ring R, there exists an R-module M such that the tight closure is given by $I^* = I^M$ for any ideal I (see also [**17**, p. 592]).

The Ratliff–Rush closure $I \mapsto \widetilde{I} = \cup_{n=1}^{\infty}(I^{n+1} :_R I^n)$ is not a closure operation as defined above because condition (b) of Definition 2.1 can fail [**5**, Example 1.11].

3. Two closure operations associated to contracted ideals. If $(R, \mathfrak{m})$ is a local (Noetherian) ring, then as defined above, an ideal I of R is said to be contracted if $I = IR[\frac{\mathfrak{m}}{x}] \cap R$ for some regular $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. In the following, we replace $\mathfrak{m}$ by an arbitrary regular ideal K and say that an ideal I of R is *K-contracted* if $IR[\frac{K}{x}] \cap R = I$ for some regular $x \in K$. We use the following lemma. For this lemma, observe that since the ideal $\cup_{k\geq 1}(IK^k :_R K^k)$ is a union of an ascending chain of ideals and R is Noetherian, this ideal is $(IK^k :_R K^k)$ for all large k. Similarly for $\cup_{k\geq 1}(IK^k :_R x^k)$. If K is an ideal of R, we let $\text{reg}(K)$ denote the set of regular elements in K.

LEMMA 3.1. *If $(R, \mathfrak{m})$ is a local ring and I, K are ideals of R with K regular, the following hold.*

(3.1.1) $IR[K/x] \cap R = \cup_{k\geq 1}(IK^k :_R x^k)$ *if $x \in K$ is regular.*
(3.1.2) $\cap_{i=1}^n IR[K/x_i] \cap R = \cap_{z\in \text{reg}(K)} IR[K/z] \cap R = \cup_{k\geq 1}(IK^k :_R K^k)$ *if $x_1, \dots, x_n$ is a set of regular generators for K.*
(3.1.3) *If $R/\mathfrak{m}$ is infinite, then there exists a regular element $x \in K$ such that* $\cup_{k\geq 1}(IK^k :_R K^k) = \cup_{k\geq 1}(IK^k :_R x^k) = IR[K/x] \cap R$.

Proof. For (3.1.1), let $y \in IR[K/x] \cap R$. Then, for all large k, $y = \sum_{t=0}^k i_t(f_t/x^t)$, where $i_t \in I$ and $f_t \in K^t$. So $x^k y = \sum_{t=0}^k i_t f_t x^{k-t} \in IK^k$ and $y \in (IK^k :_R x^k)$. Conversely, if $y \in (IK^k :_R x^k)$, $yx^k = \sum_{t=0}^k i_t(f_t)$, where $i_t \in I$ and $f_t \in K^k$. So $y = \sum_{t=0}^k i_t(f_t)/x^k$, where $i_t \in I$ and $f_t \in K^k$. It follows that $IR[K/x] \cap R = (IK^k :_R x^k)$ for all large k.

For (3.1.2), we will show that $(IK^k :_R K^k) \subseteq \cap_{z\in\text{reg}(K)} IR[K/z] \cap R \subseteq \cap_{i=1}^n IR[K/x_i] \cap R \subseteq (IK^k :_R K^k)$ for large k. Let $y \in (IK^k :_R K^k)$ and $z \in \text{reg}(K)$. Then, $yz^k \in yK^k \subseteq IK^k$, which implies that $y \in I(K^k/z^k) \subseteq IR[K/z]$. Thus, $(IK^k :_R K^k) \subseteq \cap_{z\in\text{reg}(K)} IR[K/z] \cap R$. Clearly, $\cap_{z\in\text{reg}(K)} IR[K/z] \cap R \subseteq \cap_{i=1}^n IR[K/x_i] \cap R$.

Let $y \in \cap_{i=1}^n IR[K/x_i] \cap R$. Then, for each j, there exists k_j such that $yx_j^{k_j} \in IK^{k_j}$. Then, for $k = \max\{k_j \mid j = 1, \dots, n\}$, we have $yx_j^k \in IK^k$ for every j. It follows that $y \in (IK^{nk} :_R K^{nk})$. Indeed each $z \in K^{nk}$ is an R linear combination of monomials $x_1^{t_1} \cdots x_n^{t_n}$, where $t_1 + \cdots + t_n = nk$. For each of these monomials, some $t_i \geq k$. If say $i = 1$, then $yx_1^{t_1} \cdots x_n^{t_n} = (yx_1^k)x_1^{t_1-k} \cdots x_n^{t_n} \in (IK^k)(K^{nk-k}) = IK^{nk}$. Therefore $y \in (IK^{nk} :_R K^{nk}) \subseteq \cup_{k\geq 1}(IK^k :_R K^k)$. This proves (3.1.2).

For (3.1.3), a special case of [**12**, Theorem 2.5.1] states that, if $R/\mathfrak{m}$ is infinite, there exists a regular $x \in K$ such that $IK^k :_R K^k = IR[K/x] \cap R = IK^k :_R x^k$ for all large k. □

DEFINITION 3.2. Let $(R, \mathfrak{m})$ be a local ring, let K be a regular ideal of R and let I be an ideal of R. The *K-contraction closure* $I_{\star_K}$ of I is defined by $I_{\star_K} = \cup_{k\geq 0}(IK^k :_R K^k) = I_\Delta$, where $\Delta = \{K^i \mid i \geq 0\}$. We write $I_\star$ for $I_{\star_\mathfrak{m}}$.

PROPOSITION 3.3. *Let $(R, \mathfrak{m})$ be a local ring with $R/\mathfrak{m}$ infinite and let K be a regular ideal of R. Then, for each ideal I of R, the K-contraction closure $I_{\star_K} = \cup_{k\geq 0}(IK^k :_R K^k)$ of I is the smallest ideal J of R containing I, which is contracted from $R[K/b]$ for some $b \in \text{reg}(K)$.*

Proof. Since $I \mapsto I_{\star_K}$ is a semi-prime operation by Example 2.3, it suffices to show that I is a K-contracted ideal if and only if $I_{\star_K} = I$. If $I_{\star_K} = I$, Lemma (3.1.3) gives $I = I_{\star_K} = IR[K/b] \cap R$ for some regular $b \in K$. Conversely, if I is K-contracted, assume that $I = IR[K/b] \cap R$, $b \in \text{reg}(K)$. By Lemma (3.1.2), we have $I_{\star_K} = \cap_{z\in\text{reg}(K)} IR[K/z] \cap R \subseteq IR[K/b] \cap R = I \subseteq I_{\star_K}$. □

It is shown in [**2**, Lemma 3.3] that if $(R, \mathfrak{m})$ is a local domain with $R/\mathfrak{m}$ infinite and I is an integrally closed $\mathfrak{m}$-primary ideal, then I is $\mathfrak{m}$-contracted. The following is a generalisation and converse of [**2**, Lemma 3.3].

Corollary 3.4. *If* $(R, \mathfrak{m})$ *is a local domain with* $R/\mathfrak{m}$ *infinite and* I *is a regular ideal of* R*, then* I *is* K*-contracted for every regular ideal* K *of* R *if and only if* I *is integrally closed.*

Proof. Since the ideal K in Definition 3.2 is assumed to be regular, it is immediate that $I_{\star_K} \subseteq \overline{I}$, the integral closure of I (for example, see [**9**, Theorem 2.1]). Thus, if $I = \overline{I}$, then since $I \subseteq I_{\star_K} \subseteq \overline{I}$, it follows from Proposition 3.3 that I is K-contracted for each regular ideal K of R.

Conversely, assume that I is K-contracted for each regular ideal K of R and let $r \in \overline{I}$. Then, an equation of integral dependence of r over I of degree n gives $I(I, r)^{n-1} = (I, r)^n$ (for example, see [**14**, Proposition 1.1.7]). It follows that $r(I, r)^{n-1} \subseteq (I, r)^n = I(I, r)^{n-1}$. So if we take take $K = (I, r)$, we get $r \in (IK^{n-1} :_R K^{n-1}) \subseteq I_{\star_K} = I$, where the equality is by Proposition 3.3 and the fact that I is K contracted. □

4. L-contracted implies L-full. We use the following generalisation of $\mathfrak{m}$-full mentioned in [**15**, p. 42].

Definition 4.1. Let $(R, \mathfrak{m})$ be a local ring with $R/\mathfrak{m}$ infinite, let I and L be regular ideals and let $x \in L$. Then, I is said to be L-full with respect to x if $(LI :_R x) = I$.

By [**15**, p. 43], if $(R, \mathfrak{m})$ is local with $R/\mathfrak{m}$ infinite and L is a regular ideal of R, then for any ideal I of $(R, \mathfrak{m})$, there exists a smallest L-full ideal I_* containing I. In the following, we define a related closure operation, which exists even if $R/\mathfrak{m}$ is finite, and then use it and the L-contraction closure to show that if $R/\mathfrak{m}$ is infinite, then the L-contracted ideals of $(R, \mathfrak{m})$ are L-full.

Lemma 4.2. *Let* $(R, \mathfrak{m})$ *be a local ring, let* L *be a regular ideal and let* $x \in L$*. The map* $I \mapsto I_* = \cup_n(L^n I :_R x^n)$ *is a semi-prime operation and* I_* *is* L*-full with respect to* x *for each ideal* I *of* R*.*

Proof. This is straight forward from the definition of I_*, or one could use Lemma (3.1.1) and Example 2.4. □

The following result is new, even in the case $L = \mathfrak{m}$, although it is known in special cases, for example if R is a two-dimensional regular local ring, as noted in [**7**, pp. 2628–2829], or for homogeneous ideals I in a polynomial ring $K[X_1, \ldots, X_n]$ over a field K [**1**, Proposition 2.11]. In the case that $R/\mathfrak{m}$ is infinite, it also substantially strengthens [**7**, Theorem 1.2], which says that an $\mathfrak{m}$-contracted $\mathfrak{m}$-primary ideal is basically full (without assuming $R/\mathfrak{m}$ is infinite). A consequence is a significant simplification of the diagram of implications between the above properties, which was given in [**7**, p. 2628] (see the new diagram (1) at the end of this paper). It also strengthens [**4**, Theorem 2.4] and [**18**, Theorem 5], which give the same conclusion under the stronger hypothesis that I is integrally closed.

Theorem 4.3. *If* $(R, \mathfrak{m})$ *is local with* $R/\mathfrak{m}$ *infinite and* L *is a regular ideal of* R*, then* L*-contracted ideals are* L*-full. In particular,* $\mathfrak{m}$*-contracted ideals are* $\mathfrak{m}$*-full.*

Proof. Assume the ideal I is contracted from $R[L/x]$. Then, by Proposition 3.3, $I = I_{\star_L} = \cup_{k \geq 1}(IL^k :_R L^k)$. But since $R/\mathfrak{m}$ is infinite, we get by Lemma 3.1.3 that there exists a regular $b \in L$ such that $I_{\star_L} = IR[L/b] \cap R = IL^k :_R b^k$ for all large k. Thus, $I = \cup_n(L^n I :_R b^n)$ is L-full with respect to b by Lemma 4.2. □

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EXAMPLE 4.4. It follows from [**1**, Example 2.12] that the converse of Theorem 4.3 does not hold for L equal to the maximal ideal $\mathfrak{m}$ in a three-dimensional regular local ring. Indeed in [**1**, Example 2.12], it is pointed out that the ideal $I = (X^3, Y^3, X^2Z)R + (X, Y, Z)^4R$ in the polynomial ring $R = K[X, Y, Z]$ over a field K is $\mathfrak{m}$-full but not contracted. Of course this ring R is not local, but since $I = (I\mathfrak{m} :_R f)$ for some linear form $f \in \mathfrak{m}$, then localising at $\mathfrak{m} = (X, Y, Z)$, we get that $IR_\mathfrak{m}$ is $\mathfrak{m}R_\mathfrak{m}$-full in the local ring $R_\mathfrak{m}$. Further, it is easily checked that $X^2Y \in (I\mathfrak{m}^2R_\mathfrak{m} :_{R_\mathfrak{m}} \mathfrak{m}^2R_\mathfrak{m}) \setminus IR_\mathfrak{m}$. Thus, by Proposition 3.3, $IR_\mathfrak{m}$ is not a contracted ideal in the regular local ring $R_\mathfrak{m}$.

Recall that an ideal I of a local ring $(R, \mathfrak{m})$ is said to have the Rees property if $\lambda_A(J/\mathfrak{m}J) \leq \lambda_A(I/\mathfrak{m}I)$ for any ideal $J \supseteq I$ with $\lambda_A(J/I)$ finite. In the spirit of Vasconcelos' definition of L-full, if L is an ideal of a local ring $(R, \mathfrak{m})$ with $\lambda_R(R/L) < \infty$, we say that an ideal I of R has the *L-Rees property* if $\lambda_A(J/LJ) \leq \lambda_A(I/LI)$ for any ideal $J \supseteq I$ with $\lambda_A(J/I)$ finite. The following proposition generalises the result that $\mathfrak{m}$-full ideals I have the Rees property [**18**, Theorem 3], [**4**, Lemma 2.2(2)] to the fact that *L-full ideals* have the *L-Rees property*.

PROPOSITION 4.5. *Let $(R, \mathfrak{m})$ be local, let L and I be ideals of R with $\lambda_R(R/L) < \infty$. If I is L-full and $J \supseteq I$ with $\lambda_A(J/I)$ finite, then $\lambda_A(J/LJ) \leq \lambda_A(I/LI)$.*

Proof. Let $x \in L$ be such that $(LI :_R x) = I$. Then, the sequence

$$0 \to I/LI \to J/LI \xrightarrow{x} J/LI \to J/(LI + xJ) \to 0$$

is exact. That $J/(LI + xJ)$ is the cokernel of the map m_x, which is multiplication by x is clear. Also $I/LI \subseteq \ker(m_x) \subseteq (LI :_R x)/LI = I/LI$. Since $LI + xJ \subseteq LJ$, then $\lambda_A(J/LJ) \leq \lambda_A(J/(LI + xJ)) = \lambda_A(I/LI)$, where the last equality is by the above exact sequence. So $\lambda_A(J/LJ) \leq \lambda_A(I/LI)$. □

The following corollary of Theorem 4.3 seems to be worth explicit mention.

COROLLARY 4.6. *Let $(R, \mathfrak{m})$ be local, let L and I be ideals of R with $\lambda_R(R/L) < \infty$. If I is L-contracted, then I has the L-Rees property. In particular, $\mathfrak{m}$-contracted ideals have the Rees property.*

As mentioned in the Introduction, a proper ideal I of R is said to be basically full if no minimal set of generators of I can be extended to a minimal set of generators of an R-ideal that properly contains I, and a basically full ideal of a local ring $(R, \mathfrak{m})$ is $\mathfrak{m}$-primary [**6**, Theorem 2.1]. Thus, it is immediate from the definitions that an $\mathfrak{m}$-primary ideal I is basically full if it has the Rees property. Further, by [**6**, Theorem 2.12], an $\mathfrak{m}$-primary ideal is basically full if and only if $I = (I\mathfrak{m} :_R \mathfrak{m})$. Again in the spirit of Vasconcelos's definition of L-full, we define an ideal I of $(R, \mathfrak{m})$ to be *L-basically full* if $I = (LI :_R L)$, which we have denoted I^L in Example 2.5. The following proposition generalises the result that $\mathfrak{m}$-primary ideals having the Rees property are basically full, to the result that ideals having the L-Rees property are L-basically full.

PROPOSITION 4.7. *If the ideal L of $(R, \mathfrak{m})$ is $\mathfrak{m}$-primary and $\lambda_R(J/LJ) \leq \lambda_R(I/LI)$ for each ideal $J \supseteq I$, with $\lambda_R(J/I)$ finite, then $IL :_R L = I$.*

Proof. We have $\lambda_R(I^L/I) < \infty$ since L is $\mathfrak{m}$-primary and I^L/I is an R/L-module. Then, by the L-Rees property, we have $\lambda_A(I^L/LI^L) \leq \lambda_A(I/LI)$, but $I^LL = IL$. Indeed $I^LL = (IL :_R L)L \subseteq IL$, but $I \subseteq I^L$ implies $IL \subseteq I^LL$. So $I^LL = IL$ and $\lambda_A(I^L/LI) \leq \lambda_A(I/LI)$. But since $I \subseteq I^L$, this implies $I = I^L$. □

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It remains to extend the notion of full ideal, which is defined in the Introduction. If I and L are ideals of $(R, \mathfrak{m})$, we say that I is *full for* L if $I :_R L = I :_R x$ for some $x \in L$. Then, L-full ideals I are full for L since if $IL :_R x = I$, then $I :_R x \subseteq LI :_R Lx = (LI :_R x) :_R L = I :_R L \subseteq I :_R x$.

We have the following diagram of implications, which simplifies the diagram given in [**7**, p. 2628]. Further, we may replace $\mathfrak{m}$ by any regular ideal L with $\lambda_A(A/L) < \infty$.

$$\begin{array}{ccccccccc} \text{Integrally closed} & \overset{(i)}{\Rightarrow} & L\text{-contracted} & \overset{(ii)}{\Rightarrow} & L\text{-full} & \overset{(iii)}{\Rightarrow} & L\text{-Rees property} & \overset{(iv)}{\Rightarrow} & L\text{-basically full} \\ & & & & \Downarrow \text{(v)} & & & & \\ & & & & \text{full for } L & & & & \end{array} \tag{1}$$

The implication (i) holds by Corollary 3.4 and this implication is not reversible, even for $L = \mathfrak{m}$ in a two-dimensional regular local ring $(R, \mathfrak{m})$, by an example in [**20**, p. 388] as was noted in [**7**, p. 2629]. The implication (ii) holds by Theorem 4.3 and by Example 4.4, it is not reversible, even for $L = \mathfrak{m}$ in a three-dimensional regular local ring. The implication (iii) holds by Proposition 4.5 and it is apparently not known if it is reversible in local rings with infinite residue field. The implication (iv) holds by Proposition 4.7 and, by [**6**, Example 9.1], it is not reversible even for $L = \mathfrak{m}$ in a two-dimensional regular local ring $(R, \mathfrak{m})$. The implication (v) holds by the above paragraph and by [**7**, Example 1.3], full for L does not imply L-basically full, even for $L = \mathfrak{m}$ in a three-dimensional regular local ring $(R, \mathfrak{m})$. Thus, full for $\mathfrak{m}$ does not imply any of the properties in the line above it in the case $L = \mathfrak{m}$.

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