SIMULATION OF GENERAL RELATIVISTIC CORRECTIONS IN LONG TERM NUMERICAL INTEGRATIONS OF PLANETARY ORBITS

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ABSTRACT. Long term numerical integrations of planetary orbits designed to study the stability of the Solar System over timescales comparable to its age have become very promising thanks to the availability of very powerful computers and to a substantial improvement in our methods of investigating the stability of hierarchical dynamical systems. The stability of such numerical integrations relies on the ability to control all possible sources of error. Among the errors caused by the inadequacy of the physical model are those due to the fact that Newton's theory of gravitation is used instead of general relativity. We show that the secular advance of perihelia predicted by general relativity can be simulated exactly by a $1/r^2$ perturbing potential with almost negligible additional cost in computer time.

1. LONG TERM NUMERICAL INTEGRATIONS.

The problem of the stability of the Solar System can be formulated in the following way: is the present arrangement of planetary orbits - nearly circular, nearly coplanar, non overlapping (apart from Pluto) - going to be preserved over timescales comparable to the age of the Solar System, ($\sim 4.5 \times 10^9$ yr)? Until the last decade or so the problem was tackled only with analytical methods (for a review see Message, 1984). Laplace and Lagrange (see Lagrange, 1781) first developed a secular perturbation theory to compute the long-period oscillations in the orbital elements of the planets. To first order in the masses μ of the perturbing planets (in units of the mass of the Sun) and after averaging over the fast variables, they found that the semi-major axes of the planetary orbits are subject to no secular variations. This is often regarded as a proof of the stability of the Solar System, but it is worth stressing that - being a first order theory in the μ's - it is valid only for timescales of the order of $1/\mu$, e.g. of thousands of years. Second order theories in the μ's have been developed (see e.g. Bretagnon, 1974) as well as a secular perturbation theory based on new smallness parameters exploiting the hierarchical structure of the system (Milani and Nobili, 1983); however, the problem of the stability of the Solar System for timescales of the order of 10^9 yr is still open.

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Nowadays very fast computers are available to produce numerically computed ephemerides; they can be analysed on the basis of new methods of stability analysis (Milani and Nobili, 1985) and if adequate filtering techniques are used it is possible to extract the long-term dynamical behaviour. The 5 outer planets have recently been integrated numerically for 5 × 10^6 year (Kinoshita and Nakai, 1985) in the pure Newtonian point mass approximation. An analysis of the output of this numerical integration by Milani and Nobili (1984, 1985) led to the discovery that the perihelion of Jupiter and the aphelion of Uranus are locked to one another within about ± 70^o with an oscillation period of ~ 1.1 × 10^6 year, this locking turns out to play a crucial role in ensuring the stability of the outer Solar System. Moreover, a very refined filtering shows that an exchange in energy occurs over the same period, between Uranus and Neptune (Fig 1.) (Carpino, Milani and Nobili, 1985). For the first time, and thanks to numerical integrations, we have information about the behaviour of planetary semi-major axes over such a long span of time.

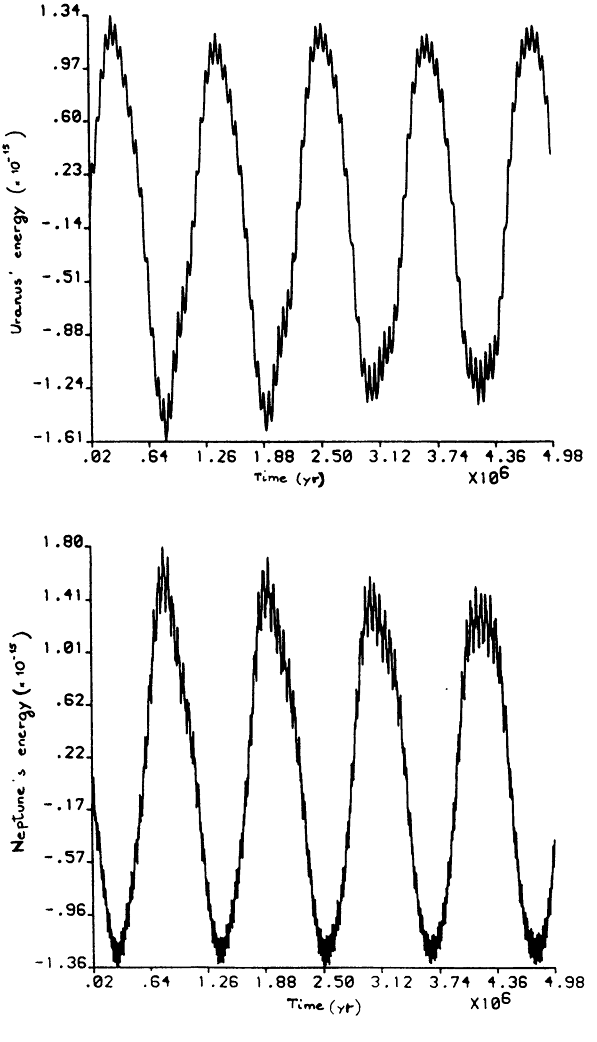

Fig 1. The energy of Uranus (top) and the energy of Neptune (bottom) as functions of time. They are obtained from the output of a 5 × 10^6 yr numerical integration (Kinoshita and Nakai, 1984) and after filtering out the short period terms (Carpino et al., 1985)

However, 5 × 10^6 yr is still ~ 1 thousandth of the age of the Solar System, not to speak of the fact that the inner planets have not been included (the number of integration steps would increase because of the shorter orbital period). The research project LONGSTOP-an acronym for LONG term numerical STability analysis of the Outer Planets - is a joint project between British and Italian scientists. One of the goals of LONGSTOP is to investigate numerically the stability of the outer Solar System for 10^8 yr. The computer being used at present is the CRAY 1S of the University of London; support has recently been offered by IBM to use the LCAP supercomputer at Kingston U.S. in the near future. However, even though the amount of required CPU time is considerable, even on a fast computer, it is not by any means the main limitation. The limit is actually set by the requirement that the ephemerides after 10^8 yr are causally related to the initial status of the system, and not simply a result of the integration error. This can be achieved only by exploiting the present computer technology, as well as our capability to control the growth of the integration error to its limits.

The total integration error results from the interaction - most probably in a non linear way - of different error sources: i) truncation: the differential equations of motion are replaced by a finite difference scheme and therefore there is a remainder at each step; ii) rounding off: the computer works with a finite mantissa and arithmetic operations are not the ones abstractly defined between real numbers; iii) instability: nearby orbits can diverge exponentially if they belong to a chaotic region of the phase-space; iv) physical model: a pure Newtonian point mass 6-body problem is only an approximation to the real system. A full analysis of the integration error will be published elsewhere together with the main results of the analysis of the output. In this paper we discuss only the effects due to general relativity and the way in which they can be simulated in the actual numerical integration.

2. THE EFFECTS OF GENERAL RELATIVITY

According to general relativity the equation of planetary orbit in a 2-body point-mass approximation can be written as follows:

$$\frac{d^2u}{d\Phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2} u^2 \qquad (1)$$

where u is the inverse of the distance r between the reduced mass and the central body of mass M equal to the total mass of the system; Φ is the true anomaly; $h = (GMa\ (1 - e^2)\ )^{1/2}$ the angular momentum per unit mass (a = semi-major axis; e = eccentricity); G the gravitational constant and c the velocity of the light. The general relativistic correction appears through the term $\frac{3GM}{c^2} u^2$ without which eq.(1) gives the classical orbit differential equation of the 2-body point-mass Newtonian problem. Since the general relativistic correction is very small (of the order of $\frac{3GM}{c^2 r}$, e.g. ~ 6 × 10^{-9} in the case of Jupiter) the solution of eq.(1) is

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usually written as

$$u = u_0 + u_1 \qquad (2)$$

where

$$u_0 = \frac{GM}{h^2}(1 + e\cos\Phi) \qquad (3)$$

is the equation of the classical ellipse and u_1 can be regarded as a small correction to it when substituting (2) into (1).

The solution turns out to be:

$$u = \frac{GM}{h^2}(1 + e\cos\Phi) + \frac{3GM}{c^2}\left[\frac{GM}{h^2}\right]^2 \left\{ \left[1 + \frac{e^2}{2}\right] + e\Phi\sin\Phi - \frac{e^2}{6}\cos 2\Phi \right\} \qquad (4)$$

where the term multiplied by $\frac{3GM}{c^2}\left[\frac{GM}{h^2}\right]^2$ represents the correction to the classical ellipse. This correction consists of 3 parts. There is a constant part, causing a variation in the average size of the classical orbit of the order of

$$\left[\frac{\Delta a}{a}\right]_C \simeq \frac{3GM}{c^2 a}\left[1 + \frac{5}{2}e^2\right] \qquad (5)$$

e.g. ~ 6×10^{-9} for Jupiter and smaller for the outer planets; it is anyway smaller than the present accuracy in a and therefore can be neglected. The term containing $\cos 2\Phi$ represents an oscillation of the orbit predicted by general relativity around the classical ellipse, with half the orbital period of the planet and of the order of

$$\left[\frac{\Delta a}{a}\right]_P \simeq \frac{GM}{c^2 a}\,\frac{e^2}{2} \qquad (6)$$

e.g., ~ 4×10^{-12} for Jupiter with an eccentricity of 0.05. This can be neglected. The last term of the general relativity correction in eq.(4) contains $e\Phi\sin\Phi$ and therefore gives rise to a secular effect. Retaining this term only eq.(4) can be written as

$$u = \frac{GM}{h^2}\left\{ 1 + e\cos(1 - \delta)\Phi \right\} \qquad (7)$$

with $\delta = \frac{3GM}{c^2} \cdot \frac{GM}{h^2}$, a very small dimensionless parameter; u is periodic with period $2\pi/(1 - \delta)$ and therefore we get the well known result for the perihelion advance per revolution:

$$\delta\omega_{GR} = 2\pi\delta = 6\pi\frac{GM}{c^2}\frac{1}{a(1-e^2)} \qquad (8)$$

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As far as the outer planets are concerned the major effect is on Jupiter. Since we cover the 10^8yr span by integrating 50×10^6 yr forward and 50×10^6yr backward in time, the error in the perihelion of Jupiter were this correction neglected is

$$\Delta_{GR}\omega \simeq 8^o.6 \qquad (9)$$

3. SIMULATION WITH A DIPOLE-LIKE POTENTIAL

As suggested by B.Bertotti the perihelion advance (8) can be simulated with a dipole-like perturbing potential

$$R = k \frac{GM}{r^2} \qquad (10)$$

where the constant k is to be determined by requiring that the secular advance $\delta_s \omega$ caused by the simulating potential R is equal to (8). According to Lagrange's equations:

$$\delta_s\omega = \frac{2\pi}{n} \dot{\omega} = \frac{2\pi}{n} \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial \bar{R}}{\partial e} \qquad (11)$$

where n is the mean motion and $\bar{R}$ the secular part of the perturbing potential obtained by averaging out the mean anomaly ℓ:

$$\bar{R} = \frac{1}{2\pi} \int_0^{2\pi} R \, d\ell \qquad (12)$$

From eqn. (11), using (12) with $d\ell = dE\,(1-e\cos E)$ (E is the eccentric anomaly) and expanding in powers of e we get:

$$\delta_s\omega \;\; \frac{2\pi k}{a} (1 + e^2 + e^4 + e^6 + ...) \; ; \qquad (13)$$

since $\frac{1}{1-e^2} = 1 + e^2 + e^4 + e^6 + ...$, and choosing $k = 3GM/c^2$, this is the same as (8).

Why does the perturbing potential (10) produce, to any degree in e, the same secular advance of perihelion as predicted by general relativity? What do we know about constant or short period effects which it might produce as well? The differential equation of the orbit for a Newtonian 2-body point-mass problem with a small dipole-like perturbing potential given by (10) with $k = 3GM/c^2$ is:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{6GM}{c^2} \frac{GM}{h^2} u \quad , \qquad (14)$$

and it is important to notice that the term coming from the perturbing potential is proportional to u and not to u^2 (as in eq.(1)). Solving eqn.(14) as we did for eqn.(1) we get the solution:

$$u_s = \frac{GM}{h^2}(1 + e\cos\Phi) + \frac{3GM}{c^2}\left[\frac{GM}{h^2}\right]^2\left[2 + e\,\Phi\sin\Phi\right] \quad (15)$$

where subscript s refers to the fact that this is the solution when using the simulating potential. By comparing eq.(15) with eq.(4) we can see that the secular effect is exactly the same in both cases. However, the constant variation in a is twice as large as predicted by general relativity (see (5)), but even so it is within the error in the value of a. Also, the dipole potential does not give rise to any short period effect, but as estimated in (6) it is even smaller and doesn't really matter.

Finally the force due to the simulating potential discussed so far can be easily added to the force term of the Newtonian problem in the computer code and the additional cost in computing time is almost negligible.

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DISCUSSION

Question : do you intend to introduce circular or elliptic rings to simulate the effect of the inner planets ?

Nobili : yes, circular rings. This is the same as to give a zonal harmonic J_2 to the Sun different from zero. The value chosen is dictated by the actual masses of the planets spread around their semi-major axes.

Branham : which computer do you use and how much computer time do you need ?

Nobili : the hundred million year integration of the outer planets will be done on the CRAY- 1S of the University of London Computer Centre and we estimate that it will require about 40 CPU hours, may be less if it can be optimized. Subsequent experiments will be done on the IBM supercomputer LCAP, for which we have received support from IBM. The annoying point is that some errors are machine-dependent and moreover these very fast computers usually require an ad hoc program to be written.

Finkelstein : you said that relativistic effects are included by changing the Newtonian potential m/r to $m/r + A/r^2$ But it is known that one cannot obtain all relativistic effects by only changing the potential function.

Nobili : I did not mean to include all the relativistic effects. I am interested only in secular effects in planetary orbits and they can be obtained by this procedure, as it can be shown by direct calculations.